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A NOTE ON SUBDIGRAPHS OF DIGRAPHS WITH LARGE OUTDEGREES

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In his survey article [3] Nash Williams gives a list of unsolved problems. The last problem is the following.

Let an $(n, \geq q)$-digraph denote a digraph without loops and parallel directed edges on a set of n vertices such that the outdegree of every vertex is at least q. If D is an $(m+n, \geq q+r)$-digraph, must there be some subdigraph of D which is an $(m, \geq q)$ or an $(n, \geq r)$ digraph?

The following proposition shows that the answer is "No" even if we allow a somewhat weaker conclusion.

Proposition. *For every $k>0$ and every prime $p \equiv 3 \pmod 4$ that satisfies $p > k^2 \cdot 2^{2k-2}$ there exists a $(p, \geq \frac{1}{2}(p-1))$-digraph that contains neither $(k, \geq \lceil \frac{1}{2}k \rceil)$- nor $(p-k, \frac{1}{2}(p-1)-k+1)$-subdigraphs.*

Proof. Let $T = T_p = (V, E)$ denote the tournament whose vertices are the elements of Z_p, where $(i, i+s)$ is a directed edge iff s is a quadratic residue ($\neq 0$) modulo p. Clearly T is a $(p, \geq \frac{1}{2}(p-1))$-digraph. Every subdigraph of T on k vertices has exactly $\binom{k}{2}$ edges and thus cannot be a $(k, \geq \lceil \frac{1}{2}k \rceil)$-digraph. Consider a subdigraph D of T on a set W of $p-k$ vertices. By a theorem of Graham and Spencer [1] there exists some $v \in W$ that dominates all vertices of $V-W$ and thus in D the outdegree of v is $\frac{1}{2}(p-1)-k < \frac{1}{2}(p-1)-k+1$. □

The assumption $p > k^2 \cdot 2^{2k-2}$ is not the best possible. Indeed, for $k=2$ we can take $p=7$. The tournament T_7 is a $(7, \geq 3)$-digraph with neither $(2, \geq 1)$- nor $(5, \geq 2)$-subdigraph.

It is worth noting that using Difference Sets we can construct many examples of $(n+2, \geq q+1)$-digraphs that are not tournaments and contain neither $(n, \geq q)$- nor

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$(2, \geq 1)$-subdigraphs. As an illustration we use the difference set $A = \{2, 3, 8, 10, 20\}$ in Z_{21} (see [2]) to construct a $(21, \geq 5)$-digraph with neither $(19, \geq 4)$- nor $(2, \geq 1)$-subdigraphs. Let $D = (V, E)$ be the digraph whose vertices are the elements of Z_{21}, where $(i, i+s)$ is a directed edge iff $s \in A$. Clearly D is a $(21, \geq 5)$-digraph and since $A \cap (-A) = \emptyset$ (in Z_{21}) we conclude that it contains no $(2, \geq 1)$-subdigraph. Consider a subdigraph H of D on a set W of 19 vertices and put $V - W = \{i, j\}$. Since A is a difference set there exist $a, b \in A$ such that $a - b = i - j$, i.e., $i - a = j - b$. Define $k = i - a = j - b$. By the definition of D, (k, i) and (k, j) are edges of D and thus the outdegree of k in H is $3 < 4$.

References

[1] R.L. Graham and J.H. Spencer, A constructive solution to a tournament problem, Canad. Math. Bull. 14 (1971) 45–48.

[2] M. Hall, Jr., Combinatorial Theory (Wiley, New York, 1967) 131.

[3] C.St.J.A. Nash Williams, A glance at graph theory — Part II, Bull. London Math. Soc. 14 (1982) 294–328.