On a Conjecture of Erdös, Simonovits, and Sós Concerning Anti-Ramsey Theorems*

Noga Alon
TEL AVIV UNIVERSITY, TEL AVIV, ISRAEL

ABSTRACT

For $n \geq k \geq 3$, let $f(n, C^k)$ denote the maximum number m for which it is possible to color the edges of the complete graph K^n with m colors in such a way that each k-cycle C^k in K^n has at least two edges of the same color. Erdös, Simonovits, and Sós conjectured that for every fixed $k \geq 3$,

$$f(n, C^k) = n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1),$$

and proved it only for $k = 3$. It is shown that $f(n, C^4) = n + [\tfrac{1}{3}n] - 1$, and the conjecture thus proved for $k = 4$. Some upper bounds are also obtained for $f(n, C^k)$, $k \geq 5$.

In this paper we use the notation of [1]. All graphs considered here are finite and have no loops and no multiple edges. $E(G)$ is the set of edges of a graph G. G^m is a graph with m vertices. K^k and C^k are the complete graph and the cycle with k vertices. If K^n is edge-colored in a given way and a subgraph H contains no two edges of the same color, then H will be called a totally multicolored (TMC) subgraph of K^n and we shall say that K^n contains a TMC H. For a graph H and an integer n, let $f(n, H)$ denote the maximum number of colors in any edge-coloring of K^n with no TMC H (i.e., with no TMC subgraph isomorphic to H).

Erdös, Simonovits, and Sós in [1] made the following conjecture.

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Conjecture 1. For every fixed $k \geq 3$,

$$f(n, C^k) = n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1).$$

They exhibited an edge-coloring of K^n showing that if $n = (k-1)q + r$, where $0 < r \leq k-1$, then

$$f(n, C^k) \geq n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) - r\left(\frac{k-1-r}{2} + \frac{1}{k-1}\right). \quad (1)$$

No general upper bound for $f(n, C^k)$ is given in [1]. However, the authors showed that

$$f(n, C^3) = n - 1,$$

and thus proved their conjecture for $k = 3$.

The following theorem establishes Conjecture 1 for $k = 4$.

Theorem 1.

(i) For every $n \geq 4$,

$$f(n, C^4) = n + [\tfrac{1}{3}n] - 1. \quad (2)$$

(ii) for every $n \geq k \geq 5$,

$$f(n, C^k) \leq g(k, n), \quad (3)$$

where

$$g(k,n) = \begin{cases} (k-2)n - (k-2)^2 & \text{for } n \geq 2k-4 \text{ and } k = 5 \text{ or } 6, \\ (k-2)n - \binom{k-1}{2} & \text{otherwise.} \end{cases} \quad (4)$$

We shall need one of the results of [1], and some more notation. For a family M of finite graphs, let $\mathrm{ext}(n, M)$ be the maximum number of edges a graph G^n can have if it has no subgraph isomorphic to any member of M. For a graph H let $L(H)$ be the family of graphs G having the property that any edge-colored complete graph containing a TMC G contains also a TMC H.

Lemma 1 of [1] asserts that if $L \subset L(H)$, then

$$f(n, H) \leq \mathrm{ext}(n, L). \quad (5)$$

Let F denote the family of all even cycles and all the graphs consisting of two odd cycles with one common vertex. For $k \geq 3$ let F_k denote the family of all cycles of length $\equiv 2[\mathrm{mod}(k-2)]$.

Our results follow from the following three lemmas.

Lemma 1. Let G, H be graphs. Suppose that adjoining a new edge e to G results in a graph which is the union of two members A, B of $L(H)$ which are edge-disjoint, except for the common new edge e. Then $G \in L(H)$.

Proof. Let K^n be an edge-colored complete graph that contains a TMC G. The color of the edge e' of K^n that corresponds to e can occur in at most one of the graphs $A - e'$, $B - e'$. Thus, K^n contains either a TMC $A \in L(H)$ or a TMC $B \in L(H)$. In both cases K^n contains a TMC H. Thus $G \in L(H)$. ∎

Lemma 2.

(i) For every $k \geq 3$, $F_k \subset L(C^k)$.

(ii) $F \subset L(C^4)$.

Proof. (i) Suppose $k \geq 3$, $t = 2 + q(k-2)$, $q \geq 1$. We use induction on q. If $q = 1$, then $C^t = C^k \in L(C^k)$. If $q > 1$, let a, b be two vertices of C^t that separate it into two paths P_1, P_2 of lengths $k-1$, $t-k+1$, respectively. Since $P_1 \cup ab = C^k \in L(C^k)$ and $P_2 \cup ab = C^{t-k+2} \in L(C^k)$ (by the induction hypothesis), Lemma 1 implies $C^t \in L(C^k)$. (ii) By part (i) of the Lemma every even cycle belongs to $L(C^4)$. Let G be a graph consisting of two odd cycles $C^q = (c, a_2, a_3, \ldots, a_q, c)$ and $C^r = (c, b_2, b_3, \ldots, b_r, c)$ with a common vertex c. G is the edge-disjoint union of the two odd paths $P_1 = (a_2, \ldots, a_q, c, b_2)$ and $P_2 = (b_2, \ldots, b_r, c, a_2)$. Since the even cycles $P_1 \cup a_2b_2$ and $P_2 \cup a_2b_2$ belong to $L(C^4)$, we conclude that $G \in L(C^4)$, by Lemma 1. ∎

Lemma 3.

(i) for every $n \geq k \geq 5$,

$$\mathrm{ext}(n, F_k) \leq g(k, n), \tag{6}$$

where $g(k, n)$ is defined in (4).

(ii)

$$\mathrm{ext}(n, F) = n + [\tfrac{1}{3}n] - 1,$$

for all $n \geq 1$.

Proof. (i) Suppose $n \geq k \geq 5$ and let G^n be a graph with more than $g(k, n)$ edges. By a simple result proved in [2, pp. 386, 387], G contains a cycle with $k-3$ diagonals emanating from one vertex. Thus there is a path P in G and a vertex x not on P that is joined by $k-1$ edges to $k-1$ vertices of P, say

$y_1, y_2, \ldots, y_{k-1}$. One can easily verify that there exist vertices y_i, y_j, $1 \leq i \leq j \leq k-1$, whose distance on P is 0 modulo $k-2$. The cycle formed by the edges xy_i, xy_j, and the segment of P from y_i to y_j belongs to F_k, which proves (6). (ii) For every n there is a connected graph G^n with $n + [\frac{1}{3}n] - 1$ edges, whose blocks are triangles and single edges, with no two triangles having a common vertex. Thus

$$\text{ext}(n, F) \geq n + [\tfrac{1}{3}n] - 1. \qquad (7)$$

The opposite inequality is proved by induction on n. For $n \leq 3$ it is trivial. Now let $G = G^n$ $(n > 3)$ be a graph with no subgraph in F. We must show that $|E(G)| \leq n + [\frac{1}{3}n] - 1$. If G is not connected this follows by applying the induction hypothesis to each component of G. Assume, therefore, that G is connected. If G has a bridge (i.e., a separating edge) u, apply the induction hypothesis to each component of $G - u$; this works even in the case in which one of these components is an isolated vertex. Since G contains no even cycle, it is easily checked that every block of G is either a single edge or an odd cycle. Therefore, if G contains no bridge, then G must be 2-connected, since otherwise the union of any two adjacent blocks of G would belong to F. However, if G is 2-connected then G is an odd cycle and $|E(G)| = n \leq n + [\frac{1}{3}n] - 1$. ■.

The proof of Theorem 1 is now trivial.

Proof of Theorem 1. The case $k = 4$ of inequality (1) (due to Erdös, Simonovits, and Sós) implies

$$f(n, C^4) \geq n + [\tfrac{1}{3}n] - 1.$$

The opposite inequality follows from (5), part (ii) of Lemma 2, and part (ii) of Lemma 3. This establishes (2). (3) follows from (5), part (i) of Lemma 2, and part (i) of Lemma 3. ■

References

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[2] B. Bollobás, *Extremal Graph Theory.* Academic, New York (1978).