Families in Which Disjoint Sets Have Large Union

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INTRODUCTION AND STATEMENT OF THE RESULTS

Let $X = \{1, 2, \ldots, n\}$ and F be a family of subsets of X, that is $F \subset 2^X$. For $1 \le i \le j \le n$ set $[i, j] = \{i, \ldots, j\}$. For integers k, m with $k \ge 2$, $0 \le m \le n$, we say that F has property $P(k, m)$ if any k pairwise disjoint members of F have union of size greater than m. Thus $P(k, n)$ means simply that F contains no k pairwise disjoint sets.

Let us write m in the form $m = kt - r$, where $1 \le r \le k$. Define

$$F(n, k, m) = \{\mathbf{F} \subseteq X : |\mathbf{F}| + |\mathbf{F} \cap [1, r-1]| \ge t\}.$$

It is easy to check that $F(n, m, k)$ has property $P(k, n)$. In fact, if $\mathbf{F}_1, \ldots, \mathbf{F}_k$ are pairwise disjoint members of F, then

$$|\mathbf{F}_1 \cup \cdots \cup \mathbf{F}_k| = |\mathbf{F}_1| + \cdots + |\mathbf{F}_k| \ge kt - \sum_{1 \le i \le k} |\mathbf{F}_i \cap [1, r-1]| \ge kt - (r-1)$$

holds.

Note that for $m = kt - 1$ one has simply $F(n, k, m) = \{\mathbf{F} \subseteq X : |\mathbf{F}| \ge t\}$.

THEOREM 1: Suppose $F \subset 2^X$, F has $P(k, m)$. Then $|F| \le |F(n, k, m)|$ holds in each of the following cases.

(a) $m = kt - 1$.
(b) $k = 2$, $m = 2t - 2$,
(c) k, r arbitrary, $n > 2m^3$. Moreover, $|F| = |F(n, k, m)|$ is possible only if F is isomorphic to $F(n, k, m)$.

Let us mention that the condition $n > n_0(m)$ cannot be completely removed in (c). In fact, Kleitman [8] proved that for $n = m = kt - k$ the maximum size of a family having $P(n, k, n)$ is attained by $F = \{\mathbf{F} \subseteq X : |\mathbf{F} \cap \{1, 2, \ldots, n-1\}| \ge t - 1\}$.

Let us also note that if (c) holds for some triple (n, k, m), then it also holds for all (n', k, m) with $n' > n$—this will be clear from the inductive proof of (a) and (b).

The following old conjecture of Erdös is related to our problem.

CONJECTURE 1 [4]: Suppose $G \subset \binom{X}{t}$, $|X| \geq rt$, and G contain no r pairwise disjoint sets. Then

$$|G| \leq \max\left\{\binom{n}{t} - \binom{n-r+1}{t}, \binom{rt-1}{t}\right\}. \tag{1}$$

The case $t = 2$ of the preceding conjecture is a theorem of Erdös and Gallai [5]. Erdös [4] proved that for $n > n_0(r, t)$ (1) holds; moreover, if $|G|$ is maximal, then for some R, $|R| = r - 1$ one has

$$G = \left\{\mathbf{G} \in \binom{X}{t}: \mathbf{G} \cap R \neq \varnothing\right\}.$$

The case $r = 2$ is covered by the Erdös–Ko–Rado theorem (see the next section).

In the case $n = rt$ the inequality

$$|G| \leq \frac{r-1}{r}\binom{rt}{t} = \binom{rt-1}{t}$$

is easy to prove.

For the proof of (c) we need a strengthening of (1), which was obtained by Bollobás *et al.* [3]. First let us define the family

$$E_t(n, r) = \left\{\mathbf{E} \in \binom{X}{t}: \mathbf{E} \cap [1, r-2] \neq \varnothing\right\}$$
$$\cup \left\{\mathbf{E} \in \binom{X}{t}: (r-1) \in \mathbf{E}, \mathbf{E} \cap [r, r+t-1] \neq \varnothing\right\} \cup \{[r, r+t-1]\}.$$

It is not hard to check that $E_t(n, r)$ contains no r pairwise disjoint members,

$$|E_t(n, r)| = \binom{n}{t} - \binom{n-r+1}{t} - \binom{n-r-t+1}{t-1} + 1.$$

THEOREM 2 [3]: Suppose $F \subset \binom{X}{t}$, F contains no r pairwise disjoint members, $|F| > |E_t(n, r)|$ and $n > 2t^3(r-1)$, then for some $R \subset \binom{X}{r-1}$ one has $F \cap R \neq \varnothing$ for all $\mathbf{F} \in F$.

Let us call a family F k-times dense on Y if for all $Y_0 \subset Y$ there exist $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_k \in F$, so that $\mathbf{F}_i \cap Y = Y_0$ for $1 \leq i \leq k$, $\mathbf{F}_1 - Y, \ldots, \mathbf{F}_k - Y$ partition $X - Y$. For $0 \leq s \leq n$ let $d(n, k, s)$ denote the maximum size of F subject to the assumption that there is no s-element set Y on which F is k-times dense.

Also, set $f(n, k, m) = \max\{|F|: F \subset 2^X, F \text{ has } P(k, m)\}$.

THEOREM 3:

$$d(n, k, s) = f(n, k, n-s).$$

Remember that 2-times dense families were used by Alon and Milman [2] in connection with embedding problems of Banach spaces. They proved the case $k = 2$ of our theorems. Actually Theorem 3 will be easily derived using the compression techniques of [1] and [7].

SHIFTING AND SOME RELATED RESULTS

One of the basic results in extremal set theory is the following.

ERDÖS–KO–RADO THEOREM [6]: Suppose $F \subset \binom{X}{l}$, $n \geq 2l$ and $\mathbf{F} \cap \mathbf{F}' \neq \emptyset$ holds for all $\mathbf{F}, \mathbf{F}' \in F$. Then

$$|F| \leq \binom{n-1}{l-1}.$$

For the proof of this result Erdös, Ko, and Rado introduced an important operation; the (i, j)-shift S_{ij}

$$S_{ij}(F) = \{S_{ij}(\mathbf{F})\colon \mathbf{F} \in F\},$$

where

$$S_{ij}(\mathbf{F}) = \begin{cases} (\mathbf{F} - \{j\}) \cup \{i\}\colon i \notin \mathbf{F}, & j \in \mathbf{F} \quad \text{and} \quad ((\mathbf{F} - \{j\}) \cup \{i\}) \notin F \\ \mathbf{F}, & \text{otherwise.} \end{cases}$$

Note that the (i, j)-shift just replaces element j by i in those sets that contain j but not i and for which the new set was not already in the family.

The importance of this operation lies in the following.

PROPOSITION 1: Suppose F has property $P(k, m)$. Then $|S_{ij}(F)| = |F|$ and $S_{ij}(F)$ has $P(k, m)$, too.

Proof: Take pairwise disjoint sets $A_1, A_2, \ldots, A_k \in S_{ij}(F)$ and suppose for contradiction $|A_1 \cup \cdots \cup A_k| \leq m$. Let B_l be the inverse image of A_l, that is $S_{ij}(B_l) = A_l$. Since F has $P(k, m)$, we may assume $B_1 \neq A_1$, and consequently $i \in A_1, j \notin A_1$, $i \notin B_1, j \in B_1$. Since the A_l are pairwise disjoint, $i \notin A_l$ for $l \geq 2$. Then $j \in B_2$, $i \notin B_2$ and $S_{ij}(B_2) = A_2$. Why? The only possibility is that $B_2^* = (B_2 - \{j\}) \cup \{i\}$ is in F. However, $B_1, B_2^*, B_3, \ldots, B_k$ are pairwise disjoint sets in F with

$$|B_1 \cup B_2^* \cup B_3 \cup \cdots \cup B_k| = |A_1 \cup \cdots \cup A_k| \leq m,$$

a contradiction. □

During his recent visit to Japan, Erdös suggested that the following might be true.

THEOREM 4: Suppose $n \geq (r + 1)k - 1$, $\binom{X}{r} = F_1 \cup F_2$. Then either F_1 or F_2 contains k pairwise disjoint sets.

Proof: Suppose for contradiction neither F_1 nor F_2 contains k pairwise disjoint members. We may assume $F_2 = \binom{X}{r} - F_1$. Applying the (i, j)-shift to F_1 means to apply the (j, i)-shift to $\binom{X}{r} - F_1$. Thus, by Proposition 1, we may apply the (i, j)-shift repeatedly to F_1 for all $1 \le i < j \le n$. Since $\sum_{F \in F_1} \sum_{i \in F} i$ is not increasing during this process and is strictly decreasing with each nontrivial shift, after finite time we get a family G_1 (and thus $G_2 = \binom{X}{r} - G_1$), which is *shifted*, that is, $S_{ij}(G_1) = G_1$ holds for all $1 \le i < j \le n$. Moreover, neither G_1 nor G_2 contains k pairwise disjoint sets. However, the set $A_0 = \{k, 2k, \ldots, k\}$ must be either in G_1 or in G_2. Consider now the possible cases:

(a) $A_0 \in G_1$. Since G_1 is shifted $A_i = \{k - i, \ldots, rk - i\} \in G_1$ follows for $i = 0, 1, \ldots, k - 1$ (because $S_{lk-i,\, lk}(\mathbf{G}) = \mathbf{G}$ for all $\mathbf{G} \in G_1$, $l = 1, \ldots, r$). However, A_0, $A_1, \ldots, A_{k-1}$ are pairwise disjoint, a contradiction.

(b) $A_0 \in G_2$. Since $\mathbf{G}_2$ is shifted $A_i = \{k + i, 2k + i, \ldots, rk + i\} \in G_2$ for $i = 0, 1, \ldots, k - 1$. However, $A_0, \ldots, A_{k-1}$ are pairwise disjoint, a contradiction. □

REMARK 1: Note that for $n = (r + 1)k - 2$, neither

$$F_1 = \binom{\{1, \ldots, kr - 1\}}{r} \quad \text{nor} \quad F_2 = \binom{X}{r} - F_1$$

contain k pairwise disjoint sets. Thus Theorem 4 is the best possible.

PROOF OF THEOREM 1(A) AND (B)

We apply induction on n. The case $m = n$ was proved by Kleitman [8]. Thus, assume $m < n$. Suppose F has $P(k, m)$. In view of Proposition 1, just as in the proof of Theorem 4, we may suppose that F is shifted, that is, $S_{ij}(F) = F$ holds for all $1 \le i < j \le n$. Define

$$F(n) = \{\mathbf{F} \subset \{1, 2, \ldots, n-1\} : (\mathbf{F} \cup \{n\}) \in F\},$$

$$F(\bar{n}) = \{\mathbf{F} \subset \{1, 2, \ldots, n-1\} : \mathbf{F} \in F\}.$$

Clearly $|F| = |F(n)| + |F(\bar{n})|$. Suppose $m = kt - 1$ or $k = 2$ and $m = 2t - 2$. Note that if $F = F(n, k, m)$, $n > m \ge k$, then $F(n) = F(n - 1, k, n - k)$, $F(\bar{n}) = F(n - 1, k, m)$ hold. Thus the statement will follow from the induction hypothesis as soon as we show $F(\bar{n})$ has $P(k, m)$ and $F(n)$ has $P(k, m - k)$. The first is obvious. To prove the second, suppose for contradiction $A_1, \ldots, A_k$ are pairwise disjoint sets in $F(n)$ with $|A_1 \cup \cdots \cup A_k| \le m - k$. Since $n > m$, we can find elements $i_1, \ldots, i_k$ such that $(A_1 \cup \cdots \cup A_k) \cap \{i_1, \ldots, i_k\} = \varnothing$. Since $S_l(F) = F$ for $l = 1, \ldots, k$, $B_l = (A_l \cup \{i_l\}) \in F$ follows. However, $B_1, \ldots, B_k$ are pairwise disjoint and $|B_1 \cup \cdots \cup B_k| = |A_1 \cup \cdots \cup A_k| + k \le m$, a contradiction.

To prove the uniqueness of the extremal families apply induction again. From the proof we know $|F(n)| = |F(n - 1, k, m - k)|$, $n - 1 > m - k$; thus, $F(n) = F(n - 1,$

$k, m - k)$. Since F is shifted, $(\mathbf{F} \cup \{j\}) \in F$ for all $\mathbf{F} \in F(n)$ and $1 \leq j \leq n$. This gives $F \supseteq F(n, k, m)$.

Thus we proved that for $n > m$, $|F| = |F(n, k, m)|$ implies $F = F(n, k, m)$ if F is shifted.

To conclude the proof of the uniqueness we must show that if G has $P(k, m)$, and for some $1 \leq i < j \leq n$ one has $S_{ij}(G) = F(n, k, m)$, then G is isomorphic to $F(n, k, m)$.

As for all

$$\mathbf{G} \in G,\ |S_{ij}(\mathbf{G})| = |\mathbf{G}|,\ \binom{X}{l} \subset G \text{ follows for } t \leq l \leq n.$$

This concludes the proof for the case (a). In the case (b) we have to deal with $G' = \{\mathbf{G} \in G: |\mathbf{G}| = t - 1\}$.

Again $S_{ij}(G) = F(n, k, m)$ implies

$$|G'| = \binom{n-1}{t-2}.$$

As G has $P(2, 2t - 2)$, G' contains no two disjoint sets. That is, $\mathbf{G}'$ is an extremal family for the Erdös–Ko–Rado theorem ($l = t - 1$, $n > 2l$). Consequently, for some $x \in X$ one has

$$G' = \left\{\mathbf{G} \in \binom{X}{t-1}: x \in \mathbf{G}\right\},$$

concluding the proof. □

Remark 2: Actually, the same proof would work word for word in case (c) as well, except that the starting case ($m = n$) of the induction is missing.

Remark 3: We outline here an alternative proof of Theorem 1 for the case $m = kt - 1$, which does not use shifting. Suppose F has property $P(k, m)$. If $A \in F$ is of size $j \leq m$, then A is contained in $\binom{n-j}{m-j}$ m-subsets of X. It is easy to check that if $|F| > |F(n, k, t)|$ and $m = kt - 1$, this implies that there is an m-subset of X containing more than $|F(m, k, t)|$ members of F. The result now follows from the starting case of the induction: $n = m$.

PROOF OF THEOREM 1(C)

We suppose again that F is shifted, $|F|$ is maximal, and F has property $P(k, m)$. Apply induction on m. Suppose $r < k$.

Claim 1. F has $P(r, rt - r)$.

Suppose for contradiction $A_1, \ldots, A_r \in F$, $A_i \cap A_j = \emptyset$ and $|A_1 \cup \cdots \cup A_r| \leq rt - r$. Using shiftedness and the maximality of $|F|$, we may assume $A_1 \cup \cdots \cup A_r = [1, rt - r]$. Define $F^* = \{\mathbf{F} \in F: \mathbf{F} \cap [1, rt - r] = \emptyset\}$. Then F^* has property

$P(k-r,(k-r)(t+1)-(k-r))$. By the induction hypothesis

$$|F^*| \leq |F(n-(rt-r),(k-r),(k-r)(t+1)-(k-r))|$$
$$< 2^{n-rt+r} - \binom{n-rt+r-(k-r)}{t}.$$

Consequently,

$$|F| < 2^n - \binom{n-r(t-2)-k}{t} < 2^n - \binom{n}{t-1} < |F(n,k,m)|,$$

a contradiction for, for example, $n > 2mt$.

Thus F has $P(r, rt-r)$ and by the induction assumption $|F| \leq |F(n, r, rt-r)| = |F(n,k,kt-r)|$ follows, together with the uniqueness of the extremal configurations.

Finally we have to consider the case $r = k$, that is $m = kt - k$.

CLAIM 2: F has $P(n, k-j, (k-j)(t-1)-j)$ for all $1 \leq j < k$.

Proof: Suppose for contradiction $A_1, \ldots, A_{k-j} \in F$ are pairwise disjoint and A is a set of size $(k-j)(t-1)-j$ containing $A_1 \cup \cdots \cup A_{k-j}$. Define

$$F^* = \{\mathbf{F} \in F\colon \mathbf{F} \cap A = \emptyset\}.$$

Then F^* has $P(n-|A|, j, j(t+1)-j)$, and this leads to a contradiction in the same way as in the case of Claim 1. □

Let us define

$$F^{(i)} = \{\mathbf{F} \in F\colon |\mathbf{F}| = i\}, f^{(i)} = |F^{(i)}|.$$

In view of Claim 2 there are no $k-1$ pairwise disjoint members in $F^{(i)}$ for $i < t-1$. This yields

$$f^{(i)} < (k-2)\binom{n-1}{i-1}.$$

And $|F| \geq |F(n,k,kt-k)|$ implies

$$\sum_{i<t} f^{(i)} \geq \binom{n}{t-1} - \binom{n-k+1}{t-1}.$$

These two inequalities lead to

$$f^{(t-1)} > \left(k - \frac{3}{2}\right)\binom{n-1}{t-1}$$

for, for example, $n > 2k(t-1)^3$. Since $F^{(t-1)}$ contains no k pairwise disjoint sets, from Theorem 2 and $n > 2m^3 > 2k(t-1)^3$, it follows that there exists $T \subset X$, $|T| = k-1$, so that

$$F^{(t-1)} \subset \left\{\mathbf{F} \in \binom{X}{t-1}\colon \mathbf{F} \cap T \neq \emptyset\right\}.$$

Consequently, for each $x \in T$ there are at least $\frac{1}{2}\binom{n-1}{t-1}$ sets $\mathbf{F} \in F^{(t-1)}$ with $\mathbf{F} \cap T = \{x\}$. In particular, there exist kt sets $\mathbf{F}_x^1, \ldots, \mathbf{F}_x^{kt}$, so that $\mathbf{F}_x^i \cap T = \{x\}$ and $\mathbf{F}_x^i \cap \mathbf{F}_x^j = \{x\}$ for $1 \leq i \neq j \leq kt$.

CLAIM 3: For all $G \in F^{(t-i)}$ one has $|G \cap T| \geq i$.

Proof: The statement holds voidly for $i \leq 0$, and we just proved it for $i = 1$. For $i \geq k$ Claim 2 implies $F^{(t-i)} = \varnothing$. Thus we may assume $2 \leq i < k$. Suppose for contradiction $\mathbf{G} \in F^{(t-i)}$, $|\mathbf{G} \cap T| < i$. Let $x_1, \ldots, x_{k-i}$ be distinct elements of $T - \mathbf{G}$. We want to find successively sets $\mathbf{F}_1, \ldots, \mathbf{F}_{k-i}$ so that $\mathbf{G}, \mathbf{F}_1, \ldots, \mathbf{F}_{k-i}$ are pairwise disjoint, $\mathbf{F}_j \in F^{(t-1)}$, $\mathbf{F}_j \cap T = \{x_j\}, j = 1, \ldots, k-i$.

Suppose $\mathbf{F}_1, \ldots, \mathbf{F}_{j-1}$ were already chosen, $j \leq k-i$. Then $|\mathbf{G} \cup \mathbf{F}_1 \cup \cdots \cup \mathbf{F}_{j-1}| < jt$; therefore, we can choose one out of the kt sets $\mathbf{F}_{xj}^1, \ldots, \mathbf{F}_{xj}^{kt}$ so that it is disjoint from $\mathbf{G} \cup \mathbf{F}_1 \cup \cdots \cup \mathbf{F}_{j-1}$.

However, $|\mathbf{G} \cup \mathbf{F}_1 \cup \cdots \cup \mathbf{F}_{k-i}| = (t-i) + (k-i)(t-1) = (k-i+1)t - k$, contradicting Claim 2.

Now the proof is finished because, by maximality, we must have

$$F = \{\mathbf{F} \subseteq X : |\mathbf{F} \cap T| + |\mathbf{F}| \geq t\}. \quad \square$$

A REDUCTION LEMMA FOR k-TIMES DENSE FAMILIES

For $F \subset 2^X$ and $i \in X$, let us define the following shifting-type operation C_i:

$$C_i(F) = \{C_i(\mathbf{F}) : \mathbf{F} \in F\}$$

where

$$C_i(F) = \begin{cases} \mathbf{F} \cup \{i\}, & \text{if } i \in \mathbf{F}, (\mathbf{F} \cup \{i\}) \notin F \\ \mathbf{F}, & \text{otherwise.} \end{cases}$$

LEMMA 1: Suppose F is a family that is not k-times dense on any s-element subset of X. Then $C_i(F)$ has the same property as well.

Proof: Suppose for contradiction that $C_i(F)$ is k-times dense on $S \in \binom{X}{s}$. Let T be an arbitrary subset of S. We want to show that there exist $\mathbf{F}_1(T), \ldots, \mathbf{F}_k(T) \in F$ so that $\mathbf{F}_j(T) \cap S = T$ and the sets $\mathbf{F}_j(T) - S, j = 1, \ldots, k$ partition $X - S$.

Suppose first $i \notin S$ and let $\mathbf{G}_1(T), \ldots, \mathbf{G}_k(T) \in C_i(F)$ satisfy the preceding assumptions. If $\mathbf{G}_j(T) \in F$ for $j = 1, \ldots, k$, then we have nothing to prove. Suppose $\mathbf{G}_1(T) \notin F$. Then $i \in \mathbf{G}_1(T)$, $F_1(T) = \mathbf{G}_1(T) - \{i\}$ is in F. Consider $\mathbf{G}_2(T) \in C_i(F)$. How could it happen that $i \notin \mathbf{G}_2(T)$? The only explanation is that $\mathbf{F}_2(T) = \mathbf{G}_2(T) \cup \{i\}$ is also in F. Now choosing $\mathbf{F}_j(T) = \mathbf{G}_j(T)$ for the remaining values $j = 3, \ldots, k$ we are done.

Suppose next $i \in S$ and set $\tilde{T} = T - \{i\}$.

As $C_i(F)$ is k-times dense on S, there exist $\mathbf{G}_j(\tilde{T}) \in C_i(F)$, $j = 1, \ldots, k$, with $\mathbf{G}_j(\tilde{T}) \cap S = \tilde{T}$ and the sets $\mathbf{G}_j(\tilde{T}) - S$ forming a partition of $X - S$.

Since $i \notin \tilde{T}$, we infer that both $\mathbf{G}_j(\tilde{T})$ and $\mathbf{G}_j(\tilde{T}) \cup \{i\}$ are in F for $j = 1, \ldots, k$. This completes the proof of the lemma. □

Proof of Theorem 3: Suppose $F \subset 2^X$, F is not k-times dense on any $S \in \binom{X}{s}$, $|F| = d(n, k, s)$. Repeatedly applying the operation C_i, for $i = 1, \ldots, n$, to F, leads to a family G that is not k-times dense on any $S \in \binom{X}{s}$ either and that satisfies $C_i(G) = G$, that is, G is a *monotone* family ($\mathbf{G} \in G$, $\mathbf{G} \subset H \subseteq X$, imply $H \in G$). We claim that G has $P(n, k, n - s)$. Suppose the contrary, that is, there exist pairwise disjoint sets $\mathbf{G}_1, \ldots, \mathbf{G}_k \in G$ with $|\mathbf{G}_1 \cup \cdots \cup \mathbf{G}_k| \leq n - s$. Let S be an arbitrary s-element subset of $X - (\mathbf{G}_1 \cup \cdots \cup \mathbf{G}_k)$.

Since G is monotone, for every $T \subseteq S$, the k-sets $\mathbf{G}_1 \cup T$, $\mathbf{G}_2 \cup T$, $\ldots$, $\mathbf{G}_{k-1} \cup T$ and $(X - (\mathbf{G}_1 \cup \cdots \cup \mathbf{G}_{k-1})) \cup T$ are in G, showing that G is k-times dense on S.

As $|G| = |F|$, $|F| = d(n, k, s) \leq f(n, k, n - s)$ follows. The opposite inequality is trivial; if F has $P(n, k, n - s)$, $S \in \binom{X}{s}$, then consider $T = \varnothing$ to show that F is not k-times dense on S. □

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