EIGENVALUES, EXPANDERS AND SUPERCONCENTRATORS
(Extended Abstract)

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ABSTRACT

Explicit construction of families of linear expanders and superconcentrators is relevant to theoretical computer science in several ways. There is essentially only one known explicit construction. Here we show a correspondence between the eigenvalues of the adjacency matrix of a graph and its expansion properties, and combine it with results on Group Representations to obtain many new examples of families of linear expanders. We also obtain better expanders than those previously known and use them to construct explicitly n-superconcentrators with $\simeq 157.4\,n$ edges, much less than the previous most economical construction.

1. INTRODUCTION

A graph G is called (n, α, β)*-expanding*, where $0 < \alpha \leqslant \beta \leqslant n$, if it is a bipartite graph on the sets of vertices I(inputs) and O(outputs), where $|I| = |O| = n$, and every set of at least α inputs is joined by edges to at least β different outputs. An (n, k, d)-expander is a graph with $\leqslant k \cdot n$ edges which is $(n, \alpha, \alpha(1 + d(1 - \alpha/n)))$-expanding for all $\alpha \leqslant n/2$. A family of linear expanders of density k and expansion d is a set $\{G_i\}_{i=1}^{\infty}$, where G_i is an (n_i, k, d)-expander, $n_i \to \infty$ and $n_{i+1}/n_i \to 1$ as $i \to \infty$.

Such a family is the main component in the recent parallel sorting network of Ajtai, Komlós and Szemerédi[2]. It also forms the basic building block used in the construction of graphs with special connectivity properties and small number of edges (see, e.g., Chung[4]). An example of a graph of this type is an n-superconcentrator (s.c.), which is a directed acyclic graph with n inputs and n outputs such that for every $1 \leqslant r \leqslant n$ and every two sets A of r inputs and B of r outputs there are r vertex disjoint paths from the vertices of A to the vertices of B. A family of linear s.c.-s of density k is a set $\{G_n\}_{n=1}^{\infty}$, where G_n is an n-s.c. with $\leqslant (k + o(1))\,n$ edges. Superconcentrators, which are the subject of an extensive literature, are relevant to computer science in several ways. They have been used in the construction of graphs that are hard to pebble (see Lengauer and Tarjan[9], Pippenger[14] and Paul, Tarjan and Celoni[15]), in the study of lower bounds (see Valiant[20]) and in the establishment of time space tradeoffs for computing various functions (Abelson[1], Ja'Ja[6] and Tompa [19]).

It is not too difficult to prove the existence of a family of linear expanders (and hence a family of linear s.c.-s) using probabilistic arguments, (see, e.g. Chung[4], Pinsker[12] and Pippenger[13]). However, for applications an explicit construction is desirable. Such a construction is much more difficult and there is essentially only one known example, due to Margulis[10] (Angluin[3] and Gaber and Galil[5] gave slight modifications). Margulis gave an explicit family of linear expanders of density 5 and used several deep results from the theory of Group Representations to prove that it has expansion d for some $d > 0$. However, he was not able to bound d strictly away from 0. Gaber and Galil[5] modified Margulis' construction and were able to obtain, using Fourier Analysis, a family of linear expanders of density 7 and expansion $(2-\sqrt{3})/2$. They used this family to construct explicitly a family of linear s.c.-s of density $\simeq 271.8$.

Here we first show, in Section 2, a relation similar to the one shown by Tanner[18], between the eigenvalues of the adjacency matrix of a graph and its expansion properties. This suggests an efficient algorithm to prove that a given graph is an expander.

Combining this relation with results of Kazhdan on Group Representations, we obtain, in Section 3, many new examples of families of linear expanders. Our examples are related to the only known one given in [10], but our method is more general since it supplies an infinite number of examples. Roughly speaking, the graphs in our families are double covers of Cayley graphs of homomorphic images of lattices in certain Lie groups. More details appear in Section 3.

We conclude this summary in Section 4, where we combine our methods with those of Gaber and Galil to obtain a better family of linear expanders than the one given in [5]. Our expanders enable us to construct explicitly a family of linear s.c.-s of density $\simeq 157.4$. This improves the previous results of Gaber and Galil and

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Chung, who gave explicit families of s.c.-s of densities $\simeq 271.8$ and $\simeq 261.5$, respectively. We also construct a family of linear non acyclic s.c.-s of density $\simeq 64$, much better than the construction of Shamir[17].

We would like to thank D. Kazhdan for many fruitful discussions.

2. EIGENVALUES, ENLARGERS AND EXPANDERS

The *adjacency matrix* $A_G = (a_{uv})_{u \in V, v \in V}$ of a graph $G = (V,E)$ is a 0-1 matrix where $a_{uv} = 1$ iff $uv \in E$. Put $Q_G = \mathrm{diag}(\rho(v))_{v \in V} - A_G$, where $\rho(v)$ is the degree of the vertex $v \in V$, and let $\lambda_1(G)$ be the second smallest eigenvalue of Q_G. One can show that $\lambda_1(G) \geqslant 0$ with equality iff G is not connected. An *(n, k, ϵ)-enlarger* is a k-regular graph G on n vertices with $\lambda_1(G) \geqslant \epsilon$. The *(extended) double cover* of a graph $G = (V, E)$ where $V = \{v_1, v_2, ..., v_n\}$ is a bipartite graph H on the sets of inputs $X = \{x_1, ..., x_n\}$ and outputs $Y = \{y_1, ..., y_n\}$ in which $x_i \in X$ and $y_j \in Y$ are adjacent iff $i = j$ or $v_i v_j \in E$.

The following theorem is our basic tool for constructing linear expanders.

Theorem 2.1

The double cover of an (n, k, ϵ)-enlarger is an $(n, k + 1, d)$-expander, where $d = 4\epsilon/(k + 2\epsilon)$.

The proof uses elementary linear algebra (Courant-Fisher Inequality). We omit the details. Note that there are several efficient algorithms to compute eigenvalues (see, e.g., [16]), and thus one can check efficiently if a graph is an (n, k, ϵ)-enlarger. In contrast, there is no known efficient algorithm to decide if a given graph is an (n, k, d)-expander.

3. GROUP REPRESENTATIONS AND CAYLEY GRAPHS

Let H be a finite group with a generating set δ satisfying $\delta = \delta^{-1}$, $1 \notin \delta$. The *Cayley graph* $G = G(H,\delta)$ is a graph on the vertex set H in which u and v are adjacent iff $u = sv$ for some $s \in \delta$. Clearly G is $|\delta|$ regular. Combining Theorem 2.1 with results of Kazhdan[7] on property (T) we can show that double covers of certain families of Cayley graphs form families of linear expanders. To save space, we give only one infinite class of such families. For $n \geqslant 3$, let $SL(n,Z)$ denote the group of all $n \times n$ matrices over the integers Z with determinant 1. In [11] an explicit set B_n of two generators of $SL(n, Z)$ is given. Put $S_n = B_n \cup B_n^{-1}$, $(|S_n| = 4)$. Let $SL(n, Z_i)$ be the group of all $n \times n$ matrices over the ring of integers modulo i with determinant 1, and let $\phi_i^{(n)}: SL(n, Z) \to SL(n, Z_i)$ be the group homomorphism defined by $\phi_i^{(n)}((a_{rs})) = (a_{rs} (mod\ i))$.

Theorem 3.1

For every fixed $n \geqslant 3$ there is an $\epsilon > 0$ such that for every $i \geqslant 2$ the Cayley graph $G_i^{(n)} = G(SL(n, Z_i), \phi_i^{(n)}(S_n))$ is an $(|SL(n, Z_i)|, 4, \epsilon)$-enlarger. Thus the family $\{H_i^{(n)}\}_{i=1}^{\infty}$, where $H_i^{(n)}$ is the double cover of $G_i^{(n)}$ is a family of linear expanders of density 5. □

To prove Theorem 3.1 we use the fact, proved in [7], that the lattice $SL(n, Z)$ of the Lie Group $SL(n, \mathbb{R})$ has property (T) (see [7] for the definition), provided $n \geqslant 3$. One can check that the adjacency matrix of the Cayley graph $G_i^{(n)}$ is $\sum_{s \in S_n} \Pi \circ \phi_i^{(n)}(s)$, where Π is the left regular representation of $SL(n, Z_i)$. These two facts, together with elementary linear algebra, imply the desired assertion. We omit the details.

It is worth noting that we can obtain similarly, an infinite number of families of linear expanders of density 3. We can also show that double covers of families of Cayley graphs of commutative groups cannot yield families of linear expanders. This is related to some of the results of Klawe,[8] that imply the last assertion for cyclic groups.

4. BETTER EXPANDERS AND SUPERCONCENTRATORS

Let $n = m^2$ and let A_m be $\{0, 1, ..., m-1\} \times \{0, 1, ..., m-1\}$. Define the following 7 permutations on A_m.

$$\sigma_0(x,y) = (x,y),$$

$$\sigma_1(x,y) = (x, y+2x), \sigma_2(x,y) = (x, y+2x+1), \sigma_3(x,y) = (x, y+2x+2)$$

$$\sigma_4(x,y) = (x+2y, y), \sigma_5(x,y) = (x+2y+1, y), \sigma_6(x,y) = (x+2y+2, y).$$

Let G_n denote the bipartite graph with classes of vertices $X = A_m$, $Y = A_m$, where $(x, y) \in X$ is joined to $\sigma_i(x, y) \in Y$ for $0 \leqslant i \leqslant 6$.

Gaber and Galil[5] proved that G_n is an $(n, 7, d_0')$-expander, where $d_0' = (2 - \sqrt{3})/2 = 0.139....$ They used these expanders to construct a family of linear s.c.-s of density $\simeq 271.8$.

Let H_n denote the bipartite graph with classes of vetices $X = A_m$, $Y = A_m$, where $(x, y) \in X$ is joined to $\sigma_i(x, y) \in Y$ for $0 \leqslant i \leqslant 6$ and to $\sigma_i^{-1}(x, y) \in Y$ for $1 \leqslant i \leqslant 6$.

Combining Lemma 4 of [5] with the basic idea of the proof of Theorem 2.1 here, we can show that H_n is an $(n, 13, c)$-expander, where $c = \dfrac{8d_0'}{2d_0' + 1 + \sqrt{4d_0'^2 + 1}} = 0.465....$ (Actually we get a slightly stronger result, but we omit it to avoid too complicated statements.) The main difference between our proof and the one given in [5] is that our method supplies a lower bound to $|(\bigcup_{i=1}^{6} \sigma_i(A) \cup \sigma_i^{-1}(a)) \backslash A|$, for $A \subseteq X$, which is the actual quantity we are interested in, whereas the method of [5] estimates $\sum_{i=1}^{6} |\sigma_i(A) \backslash A|$, and uses this to bound $|\bigcup_{i=1}^{6} \sigma_i(A) \backslash A|$.

Our expanders supply easily a family of linear s.c.-s of density 175. One can further reduce the density using the idea of Appendix 1 of [5] to $\simeq 157.4$, as computed by Z. Galil. This improves the previous best known result, due to Chung ($\simeq 261.5$).

Shamir[17] constructed a family of nonacyclic directed s.c.-s of density $\simeq 204$ and of undirected s.c.-s of density $\simeq 118$. Our expanders enable us to improve these bounds to $\simeq 64$ and $\simeq 37$, repsectively.

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