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THE BRUNN–MINKOWSKI INEQUALITY AND NONTRIVIAL CYCLES IN THE DISCRETE TORUS*

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Abstract. Let $(C_m^d)_\infty$ denote the graph whose set of vertices is Z_m^d in which two distinct vertices are adjacent iff in each coordinate either they are equal or they differ, modulo m, by at most 1. Bollobás, Kindler, Leader, and O'Donnell proved that the minimum possible cardinality of a set of vertices of $(C_m^d)_\infty$ whose deletion destroys all topologically nontrivial cycles is $m^d - (m-1)^d$. We present a short proof of this result, using the Brunn–Minkowski inequality, and also show that the bound can be achieved only by selecting a value x_i in each coordinate i, $1 \le i \le d$, and by keeping only the vertices whose ith coordinate is not x_i for all i.

Key words. Brunn–Minkowski inequality, discrete torus, nontrivial cycles

AMS subject classifications. 05C35, 05C38, 05C40

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1. Introduction.

Let $(C_m^d)_\infty$ denote the graph whose set of vertices is Z_m^d in which two distinct vertices are adjacent iff in each coordinate either they are equal or they differ, modulo m, by at most 1. This graph is the product of d copies of the cycle of length m and can be viewed as the graph of the discrete torus. The problem of determining the minimum possible cardinality of a set of vertices of this graph that intersects all noncontractible cycles in it has been considered by Saks, Samorodnitsky, and Zosin in [4], motivated by the problem of exhibiting directed multicommodity problems that have a large integrality gap. Their estimate has been improved to a tight one, which is $m^d - (m-1)^d$, by Bollobás et al. in [2], where a connection to the parallel repetition of the odd cycle game is mentioned. In this note we describe a short intuitive proof of the same result based on the Brunn–Minkowski isoperimetric inequality. The proof also implies that equality is achieved only when the remaining $(m-1)^d$ vertices form the graph of a d-dimensional hypercube of edge length $m-1$, that is, the product of d paths, each having $m-1$ vertices.

It is worth noting that the problem of determining the minimum cardinality of a set of *edges* of the graph $(C_m^d)_\infty$ that intersects all nontrivial cycles, discussed in [3], [1], seems more difficult, and only an asymptotic estimate of this minimum is known.

2. The proof.

Let Z_m^d be the set of vertices of $(C_m^d)_\infty$, and consider them as points in $\mathbb{Z}^d$. It is convenient to view $\mathbb{Z}^d$ as an infinite graph in which two distinct vectors are adjacent iff they differ by at most 1 in each coordinate. For two vectors $\bar{a} = (a_1, a_2, \ldots, a_d)$ and $\bar{b} = (b_1, b_2, \ldots, b_d)$ in Z_m^d or in $\mathbb{Z}^d$ we write that $\bar{b} \nearrow \bar{a}$ iff $a_i - b_i \in \{0, 1\}$ for all i. Note that $\nearrow$ is a reflexive relation. Note also that the following holds.

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Observation 1. *If $\bar{b}_1, \bar{b}_2 \nearrow \bar{a}$, then $\bar{b}_1$ and $\bar{b}_2$, considered as vertices of $(C_m^d)_\infty$, are either equal or connected.*

Recall that the Brunn–Minkowski inequality, generalized by Lusternik (see, e.g., [5]), is the following.

Theorem 2.1 (Brunn–Minkowski inequality). *Let $n \geq 1$, and let μ be the Lebesgue measure on $\mathbb{R}^n$. Define $A + B := \{a + b \in \mathbb{R}^n | a \in A, b \in B\}$. Let A and B be two nonempty compact subsets of $\mathbb{R}^n$. The following inequality holds:*

$$[\mu(A + B)]^{1/n} \geq [\mu(A)]^{1/n} + [\mu(B)]^{1/n}.$$

Equality is achieved iff A and B are homothetic (that is, one is a rescaled version of the other).

Using Brunn–Minkowski we obtain the following useful lemma.

Lemma 2.2. *Let $S \subseteq \mathbb{Z}^d$. Suppose that $S^+ = \{\bar{a} | \exists \bar{b} \in S\ (\bar{b} \nearrow \bar{a})\}$; then $\sqrt[d]{|S^+|} \geq \sqrt[d]{|S|} + 1$, and equality holds iff S is a hypercube.*

Proof. Define $\widehat{S} = \bigcup_{\bar{a} \in S} \{\Pi_{i \in \{1,\ldots,d\}}\ [a_i - 1, a_i)\}$, and note that $|S| = \mu(\widehat{S})$. It is easy to check that $\widehat{S^+} = \widehat{S} + [0,1)^d$. Plugging this and the fact that $|S^+| = \mu(\widehat{S^+})$ into the Brunn–Minkowski inequality, the result follows. ∎

We can now state and prove the main theorem.

Theorem 2.3. *If $S \subset Z_m^d$ is a set of vertices of Z_m^d that does not contain any noncontractible cycle of the torus, then $|S| \leq (m-1)^d$. Equality holds iff S is a hypercube with edges of size $m - 1$.*

Proof. Striving for contradiction, suppose that either $|S| > (m-1)^d$ or $|S| = (m-1)^d$, but S is not a hypercube. Denote the connected components of S by $C_1, \ldots, C_k$. Pick a vertex representative for each component C_i, and denote it by $\bar{c}_i$. Let the natural projection from $\mathbb{Z}^d$ into Z_m^d be $\pi(\bar{x})$. Slightly abusing notation, denote by $\pi^{-1}(C_i)$ the connected component of $\bar{c}_i$ in $\pi^{-1}(S)$, regarding here $\bar{c}_i$ as an element of $\mathbb{Z}^d$. (This is instead of taking the whole π preimage of C_i.) As S contains no nontrivial cycle, $\pi^{-1}(C_i)$ must be finite for all i. We next show that there exist two distinct preimages of some vertex $\bar{a}$ in one of the connected components C_i of S, implying that it contains a nontrivial cycle, and thus contradicting the assumption.

Define $\tilde{S} = \bigcup_{i=1}^k \pi^{-1}(C_i)$. Since every vertex in S has a unique corresponding vertex in $\tilde{S}$, we deduce that $|S| = |\tilde{S}|$. Looking at $\tilde{S}^+ = \{\bar{a} | \exists \bar{b} \in \tilde{S}(\bar{b} \nearrow \bar{a})\}$ we can apply our assumption and Lemma 2.2 to conclude that $|\tilde{S}^+| > m^d$. By the pigeonhole principle we deduce the existence of $\bar{a}_1 \neq \bar{a}_2$ in $\tilde{S}^+$ such that $\pi(\bar{a}_1) = \pi(\bar{a}_2)$. By the definition of $\tilde{S}^+$ there must be two elements $\bar{b}_1, \bar{b}_2 \in \tilde{S}$ such that $\bar{b}_1 \nearrow \bar{a}_1$ and $\bar{b}_2 \nearrow \bar{a}_2$. By Observation 1 we know that $\pi(\bar{b}_1)$ and $\pi(\bar{b}_2)$ are connected in S, and thus $\bar{b}_1$ and $\bar{b}_2$ belong to the same connected component $\pi^{-1}(C_i)$ of $\tilde{S}$ for some i. Denote $\bar{b}'_1 = \bar{a}_2 - \bar{a}_1 + \bar{b}_1$. Note that $\bar{b}'_1 \neq \bar{b}_1$, $\pi(\bar{b}'_1) = \pi(\bar{b}_1)$, and $\bar{b}'_1 \nearrow \bar{a}_2$, since $\bar{a_2} - \bar{b'_1} = \bar{a_2} - (\bar{a_2} - \bar{a_1} + \bar{b_1}) = \bar{a_1} - \bar{b_1}$.

By Observation 1 we conclude that $\bar{b}'_1$ and $\bar{b}_2$ are either equal or connected. As $\bar{b}_2 \in \pi^{-1}(C_i)$ we conclude that $\bar{b}'_1 \in \pi^{-1}(C_i)$, which leads to a contradiction, since $\bar{b}_1$ also lies in C_i. Therefore, either $|S| = (m-1)^d$ and S is a hypercube, or $|S| < (m-1)^d$, completing the proof. ∎

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