Notations for Rewriting

Nachum Dershowitz
Department of Computer Science
University of Illinois
1304 West Springfield Ave.
Urbana, IL 61801 U.S.A.

Jean-Pierre Jouannaud
LRI
Université de Paris 11
Bat. 490
91405 Orsay FRANCE

January 15, 1990

The best notation is no notation.
—Paul Halmos

Appropriate notations are important for stating complex results in a way that can be easily understood. Oftentimes, notation is crucial to carrying out correct and simple proofs.[1] Our purpose here is to contribute to the development of good notations for term rewriting and related areas. To that end, we have engaged in many discussions with numerous colleagues,[2] including algebraicists.[3] The term-rewriting community has already reached agreement on many of them;[4] some others are controversial or new. Freese, McKenzie, McNulty and Taylor use (or will use) similar notations (with only minor variations) in their series of books.[5] We do not, of course, expect that everybody will agree with all our suggestions, nor that they will all become a standard that must be followed to have a paper accepted for an EATCS conference. But we do hope that everyone will consider them (and the justifications we give for our choices) and compare them with what he or she is accustomed to.[6] We, ourselves, enjoy using them, as do our students,[7] and have adopted them in our recent work.[8]

[1] Linear algebra is an illuminating example.

[2] Particulary those in "book-writing mode".

[3] This is the place to express our appreciation to Leo Bachmair, Claude Kirchner, Pierre Lescanne, George McNulty, David Plaisted, Wayne Snyder, and all the others for their many constructive suggestions. (This is not to imply that they necessarily concur with our decisions.)

[4] As evidenced by the papers in N. Dershowitz, ed., *Third International Conference on Rewriting Techniques and Applications*, Lecture Notes in Computer Science **355**, Springer, Berlin, 1989.

[5] R. McKenzie, G. F. McNulty and W. Taylor, *Algebras, Lattices, Varieties*, Vol. I, Wadsworth, Monterey, CA, 1987; Vol. II with R. Freese, to appear.

[6] To keep this note reasonably self-contained, we include inobvious definitions in footnotes.

[7] At least those who have not been scared away.

[8] N. Dershowitz and J.-P. Jouannaud, "Rewrite Systems", in *Handbook of Theoretical Computer Science*, Vol. B: Formal Methods and Semantics (J. van Leeuwen, ed.), North-Holland, Amsterdam, to appear 1990.

In-line Version	**Display Version**	**LaTeX Definition**	**Meaning**
$\leftarrow$	$\leftarrow$	`\leftarrow`	inverse of any arrow-like binary relation $\rightarrow$
$\not\rightarrow$	$\not\rightarrow$	`\not\rightarrow`	complement of any arrow-like binary relation $\rightarrow$
$<$	$<$	`<`	inverse of any greater-like ordering $>$
$\geq$	$\geq$	`\geq`	reflexive closure of any greater-like ordering $>$
$\rightarrow^n$	$\stackrel{n}{\rightarrow}$	`\mathop{\rightarrow}^n`	n-fold composition of any binary relation $\rightarrow$
$\rightarrow^+$	$\stackrel{+}{\rightarrow}$	`\mathop{\rightarrow}^+`	transitive closure of any binary relation $\rightarrow$
$\rightarrow^*$	$\stackrel{*}{\rightarrow}$	`\mathop{\rightarrow}^*`	reflexive-transitive closure of any binary relation $\rightarrow$[1]
$\rightarrow^=$	$\stackrel{=}{\rightarrow}$	`\mathop{\rightarrow}^=`	reflexive closure of any arrow-like binary relation $\rightarrow$[2]
$\rightarrow^!$	$\stackrel{!}{\rightarrow}$	`\mathop{\rightarrow}^!`	normalization[3] for any binary relation $\rightarrow$
$\leftrightarrow$	$\leftrightarrow$	`\leftrightarrow`	symmetric closure of any binary relation $\rightarrow$
$\leftrightarrow^*$	$\stackrel{*}{\leftrightarrow}$	`\mathop{` `\leftrightarrow}^*`	reflexive-symmetric-transitive closure of any arrow-like binary relation $\rightarrow$[4]

[1]Called *derivability* for arrow-like relations and *reachability*, in general. This notation is more adaptable than $\twoheadrightarrow$. A sequence $s_0 \to s_1 \to \cdots \to s_i \cdots$ is called a *derivation* of $\to$ *issuing* from s_0.

[2]This seems more intuitive than $\rightarrow^{\epsilon}$.

[3]That is, $s \rightarrow^! t$ if $s \rightarrow^* t$, but $t \not\rightarrow u$ for any u.

[4]I.e. the smallest equivalence relation containing $\rightarrow$, called *convertibility*.

$\uparrow$	$\uparrow$	`\uparrow`	common ancestor relation[5]
$\downarrow$	$\downarrow$	`\downarrow`	common descendent relation[6]
$R(t)$	$R(t)$	`R(t)`	set of normal forms[7] of t for binary relation R
$R(T)$	$R(T)$	`R(T)`	normal forms of set T for binary relation R
$\mathcal{T}(\mathcal{F},\mathcal{X})$	$\mathcal{T}(\mathcal{F},\mathcal{X})$	`{\cal T(F,X)}`	set of (finite, first-order) terms with function symbols $\mathcal{F}$[8]and variables $\mathcal{X}$
$\mathcal{T}$	$\mathcal{T}$	`{\cal T}`	... for short
$\mathcal{T}^\infty(\mathcal{F},\mathcal{X})$	$\mathcal{T}^\infty(\mathcal{F},\mathcal{X})$	`{\cal T}^\infty{\cal(F,X)}`	set of finite and infinite terms with function symbols $\mathcal{F}$ and variables $\mathcal{X}$
$\mathcal{T}^\infty$	$\mathcal{T}^\infty$	`{\cal T}^\infty`	... for short
$\mathcal{T}^{\mathbf{Q}}(\mathcal{F},\mathcal{X})$	$\mathcal{T}^{\mathbf{Q}}(\mathcal{F},\mathcal{X})$	`{\cal T^{\bf Q}(F,X)}`	set of rational terms[9] with function symbols $\mathcal{F}$ and variables $\mathcal{X}$
$\mathcal{T}^{\mathbf{Q}}$	$\mathcal{T}^{\mathbf{Q}}$	`{\cal T}^{\bf Q}`	... for short

[5]I.e. the composed relation $\leftarrow^* \circ \rightarrow^*$, also called "meetability".

[6]I.e. the composed relation $\rightarrow^* \circ \leftarrow^*$, also called "joinability". A binary relation $\rightarrow$ is said to be *confluent* if any two elements with a common ancestor have a common descendent. This is equivalent to the *Church-Rosser property*: any two convertible elements have a common descendent.

[7]A *normal form* for t is any element s such that $t \rightarrow^! s$. A relation $\rightarrow$ is said to be *normalizing* if every element has at least one normal form, *terminating* if its graph has no infinite chains, and *convergent* if it is both terminating and confluent. Several alternatives to "terminating" have been used in the literature, including "Noetherian" (after the algebraicist Emily Noether), "Artinian" (after E. Artin), "finitely terminating", "uniformly terminating", and "well-founded". As there is a potential source of confusion between terms used for "no infinite ascending chain" and for "no infinite descending chain", we chose the more meaningful "terminating", and reserve well-founded for orderings. Poorer alternatives to "convergent" include "canonical" (cf. note 50) and "complete".

[8]It is convenient to write $\mathcal{F} = \cup_n \mathcal{F}_n$, where $\mathcal{F}_n$ is the set of symbols of *arity* (or "rank") n.

[9]*Rational terms* are possibly infinite terms with finitely many different subterms ($\{t \in T^\infty : |\{t|_{p \in \mathcal{P}os(t)}\}| < \infty\}$). They are solutions of equations between terms in the sense of PROLOG II.

<table>
<tr><th></th><th></th><th></th><th></th></tr>
<tr><td>$\mathcal{G}(\mathcal{F})$</td><td>$\mathcal{G}(\mathcal{F})$</td><td><code>{\cal G(F)}</code></td><td>set of ground terms[10] with function symbols $\mathcal{F}$[11]</td></tr>
<tr><td>$\mathcal{G}$</td><td>$\mathcal{G}$</td><td><code>{\cal G}</code></td><td>... for short[11]</td></tr>
<tr><td>$\mathcal{H}ead(t)$</td><td>$\mathcal{H}ead(t)$</td><td><code>{\cal H}ead(t)</code></td><td>function symbol heading term t</td></tr>
<tr><td>$\mathcal{V}ar(t)$</td><td>$\mathcal{V}ar(t)$</td><td><code>{\cal V}ar(t)</code></td><td>set of variables occurring in a term t</td></tr>
<tr><td>$\mathcal{P}os(t)$</td><td>$\mathcal{P}os(t)$</td><td><code>{\cal P}os(t)</code></td><td>set of positions in a term t[12]</td></tr>
<tr><td>$\mathcal{FP}os(t)$</td><td>$\mathcal{FP}os(t)$</td><td><code>{\cal FP}os(t)</code></td><td>set of non-variable positions in a term t[13]</td></tr>
<tr><td>$\mathcal{VP}os(t)$</td><td>$\mathcal{VP}os(t)$</td><td><code>{\cal VP}os(t)</code></td><td>set of variable positions in a term t[14]</td></tr>
<tr><td>Λ</td><td>Λ</td><td><code>\mbox{\footnotesize Λ}</code></td><td>top-most position[15]</td></tr>
<tr><td>$p \leq q$</td><td>$p \leq q$</td><td><code>p \leq q</code></td><td>position p is above q[16]</td></tr>
<tr><td>$p \parallel q$</td><td>$p \parallel q$</td><td><code>p \parallel q</code></td><td>disjoint positions[17]</td></tr>
<tr><td>$p \not\parallel q$</td><td>$p \not\parallel q$</td><td><code>p \not\parallel q</code></td><td>non-disjoint positions[18]</td></tr>
<tr><td>$t|_p$</td><td>$t|_p$</td><td><code>t|_p</code></td><td>subterm of t at position p[19]</td></tr>
</table>

[10] *Ground terms* are terms without variables.

[11] $\mathcal{G}$ can be superscripted to designate infinite, or rational, terms.

[12] A *position* may be represented as a sequence of positive integers pointing to a particular subterm in a term (seen as an ordered labeled tree). Positions have often been called "occurrences" in the literature, thereby confusing a position with the subterm it designates. We prefer to restrict the use of *occurrence* to the latter. This suggests the use of $\mathcal{P}os$ for the set of all positions in a term, rather than *Dom*.

[13] I.e. $\{p : t|_p \notin \mathcal{X}\}$ in the notation below.

[14] I.e. $\{p : t|_p \in \mathcal{X}\}$ in the notation below.

[15] The position of the root or outermost symbol. Alternate symbols, e.g. ϵ, are overused.

[16] A position p is *above* position q (and q is below p), if p is a prefix of q.

[17] Two positions are *disjoint*, or *parallel*, if neither is above the other.

[18] Two positions are *non-disjoint* if one is above the other. That is, $\not\parallel = (\leq \cup \geq)$.

[19] This notation for subterm possesses a nice symmetry with the notations that follow: $t[t|_p]_p = t = s[t]_p|_p$.

<table>
<tr><td>$t[\cdot]_p$</td><td>$t[\cdot]_p$</td><td><code>t[\cdot]_p</code></td><td>context t with designated position p[20]</td></tr>
<tr><td>$t[s]_p$</td><td>$t[s]_p$</td><td><code>t[s]_p</code></td><td>subterm of t at position p is replaced by s[21]</td></tr>
<tr><td>$t[s]$</td><td>$t[s]$</td><td><code>t[s]</code></td><td>... for short</td></tr>
<tr><td>$t[s_1,\ldots,s_n]_{p_1,\ldots,p_n}$</td><td>$t[s_1,\ldots,s_n]_{p_1,\ldots,p_n}$</td><td><code>t[s_1,\ldots,s_n]
_{p_1,\ldots,p_n}</code></td><td>subterms of t at disjoint positions p_i are replaced by s_i</td></tr>
<tr><td>$t[s]_\Pi$</td><td>$t[s]_\Pi$</td><td><code>t[s]_\Pi</code></td><td>subterms of t at set Π of disjoint positions are replaced by s</td></tr>
<tr><td>${\cal D}om(\sigma)$</td><td>${\cal D}om(\sigma)$</td><td><code>{\cal D}om(\sigma)</code></td><td>variable-domain of substitution σ[22]</td></tr>
<tr><td>${\cal VR}an(\sigma)$</td><td>${\cal VR}an(\sigma)$</td><td><code>{\cal VR}an(\sigma)</code></td><td>variable-range of substitution σ[23]</td></tr>
<tr><td>$\{\ldots x_i \mapsto s_i \ldots\}$</td><td>$\{\ldots x_i \mapsto s_i \ldots\}$</td><td><code>\{\ldots x_i \mapsto
s_i \ldots\}</code></td><td>substitution with finite domain $\{\ldots x_i \ldots\}$[24]</td></tr>
<tr><td>$\sigma_{|V}$</td><td>$\sigma_{|V}$</td><td><code>\sigma_{|V}</code></td><td>substitution σ restricted to variables in set V</td></tr>
<tr><td>$t\sigma$</td><td>$t\sigma$</td><td><code>t\sigma</code></td><td>application of substitution σ to term t[25]</td></tr>
<tr><td>$\sigma\rho$</td><td>$\sigma\rho$</td><td><code>\sigma\rho</code></td><td>composition of substitutions σ and ρ[26]</td></tr>
</table>

[20] A *context* is a term with a bound variable. The above notation is preferable to the more precise lambda-notation $\lambda x.t[x]_p$.

[21] This notation avoids extra arrows and allows for convenient abbreviation.

[22] By *variable-domain*, or just *domain*, is meant the variables in $\mathcal{X}$ for which something else is substituted, i.e. $\{x \in \mathcal{X} : x\sigma \neq x\}$.

[23] By *variable-range*, or just *range*, is meant the variables in $\mathcal{X}$ that are introduced by the substitution, i.e. ${\cal VR}an(\sigma) = \cup_{x \in {\cal VD}om(\sigma)} {\cal V}ar(x\sigma)$.

[24] The symbol $\mapsto$ is exactly what's called for here. A substitution is called a *renaming* ("conversion" is also used) when all s_i are distinct variables.

[25] Many authors prefer prefix notation. Sometimes, as here, intuitiveness should take precedence over prevalent mathematical usage.

[26] That is, σ followed by ρ.

$s = t$	$s = t$	`s = t`	syntactic equality of s and t[27]
$P^?(\cdots)$	$P^?(\cdots)$	`P^?(\cdots)`	satisfiability of a predicate $P(\cdots)$
$s =^? t$	$s \overset{?}{=} t$	`s \mathop{=}^? t`	syntactic unifiability of s and t[28]
$l \approx r$	$l \approx r$	`l \approx r`	equational axiom with left-hand side l and right-hand side r[29]
$l \simeq r$	$l \simeq r$	`l \simeq r`	equational axiom with one side l and other side r[30]
$s \leftrightarrow_e^p t$	$s \overset{p}{\underset{e}{\leftrightarrow}} t$	`s \mathop{` `\leftrightarrow}_e^p t`	application of axiom e at position p[31]
$s \leftrightarrow_E t$	$s \underset{E}{\leftrightarrow} t$	`s \mathop{` `\leftrightarrow}_E t`	... for short
$s =_E t$	$s \underset{E}{=} t$	`s \mathop{=}_E t`	equality of s and t in the models of the axioms E[32]
$\vdash_I$	$\underset{I}{\vdash}$	`\mathop{\vdash}_I`	inference using system I
$s \leftrightarrow_E^* t$	$s \overset{*}{\underset{E}{\leftrightarrow}} t$	`s \mathop{` `\leftrightarrow}_E^* t`	provability of equality of s and t with equational axioms E[33]

---

[27] I.e. s and t are identical terms in $\mathcal{T}$.

[28] I.e. $\exists\sigma\ s\sigma = t\sigma$. This notation treats unification as satisfiability of the equality predicate.

[29] Here, equational axioms are ordered pairs; l and r do not play the same role. Their usual application, however, is symmetric: the equational theory associated with a set of axioms is the reflexive, symmetric, transitive closure of the relation $s[l\sigma]_p \approx s[r\sigma]_p$, for all contexts $s[\cdot]_p$, substitutions σ and axioms $l \approx r$.

[30] Here, the axioms are viewed as unordered pairs, which is also very useful. In this case, the equational theory is simply the reflexive-transitive closure of the same relation as in the previous footnote. Accordingly, we use $\simeq$ (and not $\approx$) when we wish to stress that l and r play the same role.

[31] If $e = l \approx r$, then $s|_p = l\sigma$ and $t = s[r\sigma]_p$, for some substitution σ. If $e = l \simeq r$, then either $s|_p = l\sigma$ and $t = s[r\sigma]_p$ or $s|_p = r\sigma$ and $t = s[l\sigma]_p$, for some substitution σ.

[32] I.e. $\models_E s \approx t$.

[33] I.e. $\vdash_E s \approx t$ ($s \approx t$ is in the theory of E). By Birkhoff's Completeness Theorem for equational logic, $=_E$ and $\leftrightarrow_E^*$ coincide.

$s_0 \leftrightarrow_{e_0}^{p_0} s_1 \cdots s_n$	$s_0 \mathop{\leftrightarrow}^{p_0}_{e_0} s_1 \cdots s_n$	`s_0 \mathop{ \leftrightarrow}^{ p_0}_{e_0} s_1 \cdots s_n`	equational proof of $s_0 \simeq s_n$[34]
$\mathcal{I}(E)$	$\mathcal{I}(E)$	`{\cal I}(E)`	inductive theory of E[35]
$s =_{\mathcal{I}(E)} t$	$s \mathop{=}_{\mathcal{I}(E)} t$	`s \mathop{=}_{{\cal I}(E)} t`	equality of s and t in the initial model of E
$s =_E^? t$	$s \mathop{=}_E^? t$	`s \mathop{=}_E^? t`	semantic unifiability of s and t in the models of E[36]
$l \rightarrow r$	$l \rightarrow r$	`l \rightarrow r`	rewrite rule with left-hand side l and right-hand side r
$l \leftrightarrow r$	$l \leftrightarrow r$	`l \leftrightarrow r`	two-way rewrite rule with one side l and other r
$l \rightleftharpoons r$	$l \rightleftharpoons r$	`l\rightleftharpoons r`	either-way rewrite rule with one side l and other r[37]
$c \mid l \rightarrow r$	$c \mid l \rightarrow r$	`c \mid l \rightarrow r`	conditional rewrite rule with condition c, left-hand side l and right-hand side r
$\rightarrow_r^p$	$\mathop{\rightarrow}_r^p$	`\mathop{ \rightarrow}_r^p`	rewriting with rule r at position p[38]

[34] That is, a *proof* is a finite derivation of the relation $\leftrightarrow$, "justified" by axiom and position. When necessary, the specific *instance* $e_i\sigma_i$ of an axiom can be indicated.

[35] By definition, an equation is true in $\mathcal{I}(E)$ if all its ground instances are true in E.

[36] That is, satisfiability of $s =_E t$, or $\exists\sigma\ s\sigma =_E t\sigma$.

[37] Ordered rewriting uses equational axioms in one direction or the other, depending on the instance under consideration. For example, one may use the associativity axiom $(x+y)+z \approx x+(y+z)$ from left to right to rewrite $(c+b)+a$ to $c+(b+a)$, or from right-to-left to rewrite $a+(a+b)$ to $(a+a)+b$. See note 43.

[38] By definition, $s \rightarrow_{l\rightarrow r}^{p} t$ if $s|_p = l\sigma$ and $t = s[r\sigma]_p$, for some substitution σ. The term $s|_p$ is the *redex*. This notation adapts conveniently to give the relative position of a redex: $\rightarrow_r^{\neq p}$ if the redex is other than p; $\rightarrow_r^{<p}$ if it's strictly above p; $\rightarrow_r^{>p}$ if it's strictly below; $\rightarrow_r^{\perp p}$ if it's either above or below; $\rightarrow_r^{\| p}$ if it's neither. To specify the substitution σ, called the *match* of *pattern* l to the redex,

we subscript by rule instance instead: $s \rightarrow^{p}_{r\sigma} t$. For two-way rules, $s \rightarrow^{p}_{l \leftrightarrow r} t$ if $s \rightarrow^{p}_{l \rightarrow r} t$ or $s \rightarrow^{p}_{r \rightarrow l} t$.

<table>
<tr><td>$\rightarrow_R$</td><td>$\underset{R}{\rightarrow}$</td><td><code>\mathop{\rightarrow}_R</code></td><td>rewrite closure of binary relation R on terms[39]</td></tr>
<tr><td>$\rightarrow_R^{p_1,\ldots,p_n}$</td><td>$\xrightarrow[R]{p_1,\ldots,p_n}$</td><td><code>\mathop{\mbox {\rightarrowfill}} _R^{p_1,\ldots,p_n}</code>[40]</td><td>rewriting with set of rules R at n successive positions $p_1 \ldots p_n$[41]</td></tr>
<tr><td>$\rightarrow_R^\Pi$</td><td>$\overset{\Pi}{\underset{R}{\rightarrow}}$</td><td><code>\mathop{\rightarrow }_R^\Pi</code></td><td>rewriting with set of rules R at set of disjoint positions Π</td></tr>
<tr><td>$\rightarrow_R^{\|}$</td><td>$\overset{\|}{\underset{R}{\rightarrow}}$</td><td><code>\mathop{\rightarrow }_R^\|</code></td><td>parallel rewriting with set of rules R[42] (without specifying positions)</td></tr>
<tr><td>$\rightarrow_{r^\succ}^p$</td><td>$\overset{p}{\underset{r^\succ}{\longrightarrow}}$</td><td><code>\mathop{\longrightarrow}_{ r^\succ}^p</code></td><td>ordered rewrite relation with either-way rewrite rule r at position p and ordering $\succ$[43]</td></tr>
<tr><td>$\rightarrow_{E^\succ}$</td><td>$\underset{E^\succ}{\longrightarrow}$</td><td><code>\mathop{\longrightarrow }_{E^\succ}</code></td><td>ordered rewrite relation with either-way rewrite rules E and ordering $\succ$</td></tr>
<tr><td>$\rightarrow_\succ$</td><td>$\underset{\succ}{\rightarrow}$</td><td><code>\mathop{\rightarrow}_\succ</code></td><td>... for short</td></tr>
<tr><td>$\rightarrow_{R/S}$</td><td>$\underset{R/S}{\longrightarrow}$</td><td><code>\mathop{\longrightarrow }_{R/S}</code></td><td>rewrite relation with rules R modulo equations S[44]</td></tr>
</table>

[39]The *rewrite closure* $\rightarrow_R$ is the smallest relation containing R having the *replacement property* ($s \rightarrow t$ implies $u[s]_p \rightarrow u[t]_p$, for all terms s and t and contexts $u[\cdot]_p$) and *full invariance property* ($s \rightarrow t$ implies $s\sigma \rightarrow t\sigma$, for all terms s and t and for all substitutions σ). There is no reason to use the overabused adjective "monotonic" for relations with these closure properties. A term t is *reducible* (with respect to a system R) if a subterm of t is a redex, and is *ground-reducible* if every ground instance is reducible.

[40]In in-line usage, we use the regular-length arrow; in displayed equations, we usually use long ones or variable length ones as here.

[41]The notions of *derivation* and *proof* as justified derivation carry over to sequences of rewrites.

[42]In parallel rewriting, zero or more disjoint redexes may be rewritten in one step. The *Parallel Moves Lemma* can accordingly be phrased: $\leftarrow_r^{\|} \circ \rightarrow_s^{\|} \subseteq \rightarrow_s^{\|} \circ \leftarrow_r^{\|}$ when r and s are orthogonal rules. Rules are *orthogonal* (formerly called "regular") if they are left-linear and their left-hand sides do not overlap.

[43]By definition, $s \rightarrow_{r^\succ}^p t$ if $s \leftrightarrow_r^p t$ and $s \succ t$. (This is the prefered definition.) We're tempted to suggest the symbol $\rightarrowtail_r^p$ instead, but this arrow requires a complicated definition in LaTeX.

[44]By definition, $s \rightarrow_{R/S} t$ if $s \leftrightarrow_S^* \circ \rightarrow_R \circ \leftrightarrow_S^* t$. This is *(congruence-) class rewriting.*

$\rightarrow_{S\backslash R}$	$\mathop{\longrightarrow}_{S\backslash R}$	`\mathop{\longrightarrow}_{S \backslash R}`	extended rewrite relation with rules R and equations S[45]
$\leadsto^p_r$	$\mathop{\leadsto}_r^p$	`\mathop{\leadsto}_r^p`	narrowing with rewrite rule r at position p[46]
$\rhd$	$\rhd$	`\rhd`	proper subterm ordering
$\stackrel{\bullet}{\geq}$	$\stackrel{\bullet}{\geq}$	`\stackrel{\scriptscriptstyle \bullet}{}\!\!\!\geq` [47]	subsumption ordering on terms or substitutions[48]
$\stackrel{\bullet}{=}$	$\stackrel{\bullet}{=}$	`\stackrel{\scriptscriptstyle \bullet}{}{=}`	literal similarity of terms or substitutions[49]
$\stackrel{\bullet}{\unrhd}$	$\stackrel{\bullet}{\unrhd}$	`\stackrel{\scriptscriptstyle \bullet}{}\!\!\!\!\!\!\unrhd`	encompassment ordering on terms[50]
$\stackrel{\bullet}{\geq}_E$	$\stackrel{\bullet}{\geq}_E$	`\stackrel{\scriptscriptstyle \bullet}{}\!\!\!\geq_E`	subsumption ordering modulo E[51]

[45] By definition, $s \rightarrow^p_{S\backslash R} t$ if $s\ (\xleftrightarrow[S]{\geq p})^* \circ \xrightarrow[R]{p} t$. The suggested notation is meant to draw attention to the stipulation that all S-steps take place before the rewrite and at positions at or below it; it is more meaningful than $\rightarrow_{R,S}$.

[46] By definition, $s \leadsto^p_{l\rightarrow r} t$, if $t = s\rho[r\rho]_p$, where ρ is the most general unifier of $s|_p$ and l, with $p \in \mathcal{FP}os(s)$. Many of the same variations as used for $\rightarrow$ apply to $\leadsto$; e.g. $s \leadsto_{(l\rho\rightarrow r\rho)/S} t$ means that ρ is the most general unifier of l and $s'|_p$, for some $s' =_S s$, $p \in \mathcal{P}os(s')$, and $t =_S s'\rho[r\rho]_p$. A *normal* narrowing step is $\leadsto_R \circ \rightarrow^!_R$. A *basic* narrowing step operates on pairs of terms and substitutions: $s\sigma \leadsto^p_{l\rightarrow r} t\tau$, if $t = s[r]_p$ and $\tau = \sigma\rho$, for ρ as above.

[47] Backward motions `\!` and sizes may need fine-tuning in size-changing situations, like footnotes.

[48] By definition, $t \stackrel{\bullet}{\geq} s$ if $t = s\sigma$, for some σ. Subsumption is extended to a quasi-ordering on substitutions: $\sigma \stackrel{\bullet}{\geq} \tau$ if $x\sigma \stackrel{\bullet}{\geq} x\tau$, for all $x \in \mathcal{VD}om(\sigma) \cup \mathcal{VD}om(\tau)$.

[49] *Literal similarity* is the equivalence associated with $\stackrel{\bullet}{\geq}$. That is $s \stackrel{\bullet}{=} t$, sometimes read as s is a "variant" of t, if $s = t\sigma$, for some renaming σ. This explains the dot the two notations share.

[50] The *encompassment ordering* is defined as the subterm ordering composed with the subsumption ordering, i.e. $t \stackrel{\bullet}{\unrhd} s$ if $t|_p = s\sigma$, for some position p and substitution σ. "Encompassment" conveys the notion that s "appears" in t, but with context $t[\cdot]_p$ "above" and substitution σ "below" (and is therefore a better term than "containment" or "specialization"). This is why the notation for encompassment combines those for subterm and subsumption. Encompassment is used to define *(inter-) reduced* sets of rules, that is sets R of rules such that for every rule $l \rightarrow r$, r is in normal form, as are all terms smaller than l in the encompassment ordering. (This coincides with the usual definition of "reduced" when R is convergent.) Since reduced convergent sets of rules enjoy

$(\succsim_1,\ldots,\succsim_n)$	$(\succsim_1,\ldots,\succsim_n)$	`({\mathop{\succ }\limits_ \sim}_1,\ldots,{\mathop{ \succ}\limits_\sim}_n)`	component-wise extension of orderings $\succsim_1,\ldots,\succsim_n$ [52]
$(\succsim_1,\ldots,\succsim_n)_{lex}$	$(\succsim_1,\ldots,\succsim_n)_{lex}$	`({\mathop{\succ }\limits_ \sim}_1,\ldots,{\mathop{ \succ}\limits_\sim}_n)_{lex}`	lexicographic extension of orderings $\succsim_1,\ldots,\succsim_n$ [53]
$\succsim^n_{lex}$	$\succsim^n_{lex}$	`{\mathop{\succ} \limits_\sim}^n_{lex}`	lexicographic extension of ordering $\succsim$ to n-tuples[54]
$\succsim^*_{lex}$	$\succsim^*_{lex}$	`{\mathop{\succ} \limits_\sim}^*_{lex}`	lexicographic extension of ordering $\succsim$ to arbitrary sequences[55]
$\succ_{mul}$	$\mathop{\succ}\limits_{mul}$	`\mathop{\succ}_{mul}`	extension of ordering $\succ$ to multisets[56]
$\precsim_{emb}$	$\precsim_{emb}$	`{\mathop{\prec }\limits_\sim}_{emb}`	homeomorphic embedding relation for well-quasi ordering $\precsim$ on $\mathcal{F}$[57]

a uniqueness property, we term them *canonical.*

[51]By definition, $t \gtrdot_E s$ if $t =_E s\sigma$, for some σ. This ordering, too, extends to an ordering on substitutions. So, the subsumption ordering modulo E on substitutions restricted to variables in some set V amounts to $\sigma_{|V} \gtrdot_E \tau_{|V}$.

[52]By definition, $(s_1,\ldots,s_n)$ $(\succsim_1,\ldots,\succsim_n)$ $(t_1,\ldots,t_n)$ if $s_i \succsim_i t_i$ for all $i = 1,\ldots,n$.

[53] Let $\sim_i$ and $\succ_i$ be the equivalence and (strict) partial ordering associated with the quasi-ordering $\succsim_i$, respectively. By definition, $(s_1,\ldots,s_n)$ $(\succsim_1,\ldots,\succsim_n)_{lex}$ $(t_1,\ldots,t_n)$ if $s_1 \succ_1 t_1$, or $s_1 \sim_1 t_1$ and $(s_2,\ldots,s_n)$ $(\succsim_2,\ldots,\succsim_n)_{lex}$ $(t_2,\ldots,t_n)$. If the $\succ_i$ are well-founded, then so is the strict part of $(\succsim_1,\ldots,\succsim_n)_{lex}$—denoted $(\succ_1,\ldots,\succ_n)_{lex}$.

[54]An abbreviation of $(\succsim,\ldots,\succsim)_{lex}$. We write $\succ^n_{lex}$ for the associated strict partial-ordering.

[55]The strict part $\succ^*_{lex}$ of this ordering is not well-founded, since sequences may be of arbitrary length.

[56]By definition, $\succ_{mul}$ is the smallest partial ordering containing the following relation between multisets: $S \cup \{s\} \succ_{mul} S \cup \{t_1,\ldots,t_n\}$ for $s \succ t_1,\ldots t_n$ $(n \geq 0)$. If $\succ$ is well-founded, so is $\succ_{mul}$.

[57]By definition, $s \precsim_{emb} t$ if $s \precsim_{emb} t|_{p \in \mathcal{P}os(t)}$ or if $\mathcal{H}ead(s) \precsim \mathcal{H}ead(t)$, and there exist $1 \leq j_1 < \cdots < j_{arity(\mathcal{H}ead(s))} \leq arity(\mathcal{H}ead(t))$ such that $s|_i \precsim_{emb} t|_{j_i}$. By Kruskal's Tree Theorem, homeomorphic embedding is a well-quasi ordering of $\mathcal{T}$ if $\precsim$ is a well-quasi-ordering of $\mathcal{F}$.

$\succ_{lpo}$	$\mathop{\succ}_{lpo}$	`\mathop{\succ}_{lpo}`	lexicographic path ordering with precedence $\succ$[58]
$\succ_{mpo}$	$\mathop{\succ}_{mpo}$	`\mathop{\succ}_{mpo}`	multiset path ordering with precedence $\succ$[59]
$\succ_{rpo}$	$\mathop{\succ}_{rpo}$	`\mathop{\succ}_{rpo}`	recursive path ordering with precedence $\succ$[60]

[58]By definition, $s \underset{\sim lpo}{\succ} t$ if $s \underset{\sim lpo}{\succ} t|_{p \in \mathcal{P}os(t)}$ and one of the following holds: $\mathcal{H}ead(s) \succ \mathcal{H}ead(t)$, or $s|_{p \in \mathcal{P}os(s)} \underset{\sim lpo}{\succ} t$, or $\mathcal{H}ead(s) \sim \mathcal{H}ead(t)$ and $(s|_1, \ldots, s|_n)\ (\underset{\sim lpo}{\succ})^n_{lex}\ (t|_1, \ldots, t|_n)$, where $n = arity(\mathcal{H}ead(s)) = arity(\mathcal{H}ead(t))$.

[59]This is the original "recursive path ordering", with multiset "status" of operators, for which the last case in note 58 is replaced by a comparison of multisets: $\{s|_1, \ldots, s|_{arity(\mathcal{H}ead(s))}\}\ (\underset{\sim mpo}{\succ})_{mul}\ \{t|_1, \ldots, t|_{arity(\mathcal{H}ead(t))}\}$. Equivalence of multisets means that they are the same up to equivalence of elements under $\sim_{mpo}$ (that is, under the intersection of $\underset{\sim mpo}{\succ}$ and its inverse $\underset{\sim mpo}{\prec}$).

[60]With multiset and/or lexicographic "status" of operators.