An Abstract Path Ordering

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Abstract. Abstract combinatorial commutation properties for separating well-foundedness of unions of relations can be applied to generic path orderings used in termination proofs.

1 Introduction

Path orderings provide a convenient and popular method of proving termination, particularly of term-rewriting systems. Here, we set out to prove the well-foundedness of an abstract path ordering – in the style of [14,3] – which includes the usual path orderings on first-order terms as special cases. We will apply the commutation methods of [7] plus a strong variant of lifting.

2 The Selection Property

All relations herein are binary. Juxtaposition is used for composition of relations. We represent union by $+$, and denote the reflexive, transitive, and reflexive-transitive closures of relation E by E^ε, E^+, and E^*, respectively. We will use E^∞ to represent the set of "immortal" elements s for which there is an infinite E-chain $s\ E\ s'\ E\ s''\ E \cdots$ of elements of the underlying set.

Definition 1 (Selection [7]). *Relation B* selects *relation A if*

$$BA^+ \subseteq A(A+B)^* + B^+ \ .$$

In other words, if one can get from an element s to an element t by one B-step followed by one or more A-steps, then one can also get from s to t by first taking an A-step and then some combination of A- and B-steps, or else one can get there by one or more B-steps alone. This is a weaker requirement than the "local" condition explored in [10,11] and called "lazy commutation" in [7].

Theorem 2 ([7, Theorem 72][1]). *If relation B selects relation A, then*

$$(A+B)^\infty = A^*B^*(A^\infty + B^\infty) \ .$$

[1] The proof in [7] relies on a more general claim (Theorem 39) about "constriction". The latter, however, is phrased there too broadly. Nevertheless, it does apply to the case in hand. I am grateful to Ori Brost for pointing this out.

This notation is meant to convey that one can get from any element that is immortal in the union $A + B$ to an element that is immortal in one of the two component relations by taking some number of A-steps followed by some number of B-steps. This implies, of course, that the union is well-founded whenever both A and B are.

When A is transitive, as it will be in the cases of interest here, selection is the same as the following local condition:

Definition 3 (Jumping). *Relation A* jumps over *relation B if*

$$BA \subseteq A(A+B)^* + B^+ \ .$$

This is noticeably weaker than lazy commutation [10,11,7], which allows only one B rather than B^+.

Corollary 4. *If transitive relation A jumps over relation B, then*

$$(A+B)^\infty = A^\varepsilon B^* (A^\infty + B^\infty) \ .$$

This, too, implies "separation" of termination of the union $A + B$.

3 The Abstract Path Ordering

We propose the following generic definition of path orderings:

Definition 5 (Abstract Path Ordering). *The* abstract path ordering *is a relation $>$ (not necessarily transitive) on some set T, parameterized by two other abstract relations, $\gg$ and well-founded $\rhd$, and by arbitrary binary conditions C and D, defined as follows:*

$$t > s \quad \text{if} \quad \begin{cases} t \succ s \text{ and } t\ C\ s & \text{(a)} \\ \text{or} & \\ t \gg s \text{ and } t\ (\succ + >)/\rhd\ s \text{ and } t\ D\ s\ , & \text{(b)} \end{cases}$$

where $\succ$ is short for $\rhd^+ >^*$ (or just $\rhd >^*$, in the transitive $\rhd$ case), and the "division" operator is defined by $B/A = \{\langle x, y\rangle : \forall z.\ yAz \Rightarrow xBz\}$. In other words, in case (b), $t \succ u$ or $t > u$ for all $\rhd$-neighbors u of s.

This is a generalization of the abstract ordering given in [14].

Let $\sqsupset$ be short for $\gg \cap (\succ + >)/\rhd$. By the cases of the definition, we have

$$> \subseteq \succ + \sqsupset \ .$$

Lemma 6. *For the above abstract path ordering, relation $\sqsupset$ selects $\rhd$.*

Proof. By the terms of the second case (b), one has $\sqsupset \rhd \subseteq \succ + >$. Also, the recursive definition of $>$ must expand so that $> \subseteq (\rhd + \sqsupset)^+$. Pasting the various facts together, we get

$$\sqsupset \rhd \;\subseteq\; \rhd^+ >^* + > \;\subseteq\; \rhd^+ >^* + \rhd^+ >^* + \sqsupset \;\subseteq\; \rhd\,(\rhd + \sqsupset)^* + \sqsupset \;.$$

So, in fact, $\rhd$ commutes lazily over $\sqsupset$, which implies selection (by an easy induction). □

It follows from Theorem 2 that $>$ is well-founded if $\sqsupset$ is. Of course, $\sqsupset$ is well-founded if $\gg$ is. So:

Proposition 7. *An abstract path ordering is well-founded whenever its component relation* $\gg$ *is.*

This works, as is, for some interpretation-based termination orderings.

4 Lifting and Escaping

The problem is that, for path orderings, $\gg$ is normally defined in terms of $>$ applied to subterms.

Theorem 8. *An abstract path ordering is well-founded if, for all subsets* S *of* T*, well-foundedness of* $>$ *on the* $\rhd$*-neighbors of elements of* S *implies well-foundedness of* $\gg$ *on* S*.*

Definition 9 (Lifting). *Relation* A lifts to *relation* B *if*

$$B^\infty \subseteq A(A+B)^\infty \;.$$

Theorem 10 ([7]). *If relation* B *selects relation* A *and* A *lifts to* B*, then*

$$(A+B)^\infty = (A+B)^* A^\infty \;.$$

Corollary 11. *An abstract path ordering* $>$ *is well-founded if* $\rhd$ *lifts to* $\sqsupset$*.*

This applies to the nested multiset ordering [9], where $\gg$ is the multiset ordering, and to lexicographic orderings. The general case of such "lifted" definitions was first studied in [16] and was pursued further in [13,14].

It turns out, however, that oftentimes we need a weaker alternative to lifting, in which the A-step need only take place *eventually.* Borrowing modal-logic notation, this is captured by the next definition.

Definition 12 (Escaping). *Relation* A escapes from *relation* B *if*

$$B^\infty \models \Diamond [\![A(A+B)^\infty]\!] B^\infty \;.$$

Here, B^∞ is being used to denote the set of all infinite B-chains. The double-bracket notation turns the set (of sequences) $A(A+B)^\infty$ into the relation between those elements having immortal A-neighbors and everything. Accordingly, the definition means that there is a point in every infinite B-chain such that an A-step out of that point leads to a potentially "immortal" element in the union. Escaping is somewhat reminiscent of the "bar induction" criterion in [14].

It follows from the definition that

Proposition 13. *If relation A escapes from relation B, then*

$$B^\infty \subseteq B^* A^\infty \; .$$

Theorem 14. *An abstract path ordering $>$ is well-founded if $\rhd$ escapes from $\sqsupset$.*

The multiset path ordering [4], lexicographic path ordering [16], and recursive path ordering [17,5] are all special cases, where $\rhd$ is the proper subterm relation (so, $\rhd^+ = \rhd$), C and D are always true, and $\gg$ is a recursive lifting of $>$ to multisets, precedence (first) and (then) multisets, precedence and tuples lexicographically, and a mixture thereof, respectively.

5 Discussion

We are optimistic that the commutation-based approach taken here will likewise help for advanced path orderings, like the general path ordering [8] and higher-order recursive-path-ordering [12,15,2], without recourse to reducibility/computability predicates, because (as pointed out in [6]) there is an analogy between the use of reducibility predicates and the use in proofs of well-foundedness of the "constricting" derivations used in the proof of Theorem 2 cited above.

We can apply this commutation method to analyze the dependency-pair method of proving termination. (See [1]; compare [6].) We also hope to analyze minimal bad sequence arguments for well-quasi-orderings in a similar fashion. (See [18]; compare [14].)

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