Modular Confluence Revisited

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1 Introduction

In [6], Toyama proved that the union of two confluent (Church-Rosser) term-rewriting systems that share absolutely no function symbols or constants is likewise confluent. The proof of this beautiful modularity result was later substantially simplified in [3]. Unfortunately, confluence is not, in general, preserved when the two systems have constructors in common.

When, however, the union of the two systems is terminating, confluence is preserved by constructor-sharing unions, as an immediate consequence of Knuth's Critical Pair Lemma [4]. Better yet, Ohlebusch [5] showed that confluence is preserved when each system is (weakly) normalizing. We are looking for weaker conditions that imply modularity.

In this paper we look at a further simplification of the proof of Toyama's result for confluence, which shows that the crux of the problem lies in two different properties: a cleaning lemma, whose goal is to anticipate the application of collapsing and constructor-lifting reductions; a modularity property of ordered completion, namely, that ordered completion commutes with unions. And we show that both properties are satisfied by constructor-sharing unions under Ohlebush's assumption that constructor-lifting derivations are weakly-terminating. In conclusion, we show by means of an example that this assumption can be relaxed, provided the modularity of completion remains satisfied, as well as a weaker form of the cleaning lemma.

2 Preliminaries

We assume given a *signature* (or *vocabulary*) $\mathcal{F}$ of *function symbols* together with their arity, and a set $\mathcal{X}$ of *variables*. Let $T(\mathcal{F},\mathcal{X})$ and $T(\mathcal{F})$ denote respectively the set of *terms* built up from $\mathcal{F}$ and $\mathcal{X}$, and the set of *ground terms* built up from $\mathcal{F}$ only. We assume familiarity with the basic concepts and notations of term rewriting systems. We refer to [2] for supplementary definitions, notations and examples.

Terms are identified with finite labelled trees as usual. *Positions* are strings of positive integers, the root position corresponding to the empty string Λ. Concatenation of positions u and v is written $u \cdot v$. Positions can be compared in the prefix ordering as follows: $p \leq q$ iff $q = p \cdot q'$ for some q'. We use $\mathcal{P}os(t)$ (resp. $\mathcal{FP}os(t)$) to denote the set of positions (resp. non-variable positions) of t, and $t(p)$ to denote the symbol at position p in t (when $p = \Lambda$ we also use the notation $root(t)$). The *subterm* of t at position p is denoted by $t|_p$ and the result of replacing $t|_p$ with u at position p in t is denoted by $t[u]_p$. This notation is also used to indicate that u is a subterm of t. $\mathcal{V}ar(t)$ denotes the set of variables occuring in t. A term t such that $\mathcal{V}ar(t) = \{x_1, \ldots, x_n\}$ is *linear* if each x_i occur exactly once in t, in whicvh case we write $t[x_1, \ldots, x_n]$.

Substitutions are written as in $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ where t_i is assumed different from x_i. $\mathcal{D}om(\sigma) = \{x_1, \ldots, x_n\}$ is the domain of σ. We use greek letters for substitutions and postfix notation for their application. We say that s *subsumes* t or that t is an *instance* of s if $t = s\sigma$ for some substitution σ, and write $s \lessdot t$. The relation $s \doteq t$ iff $s = t\xi$ for some bijective mapping $\xi : \mathcal{X} \mapsto \mathcal{X}$ called a *variable renaming* is the equivalence associated to subsumption.

A *rewrite rule* is a pair of terms, written $l \to r$, such that $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. A *term rewriting system* is a set of rewrite rules $R = \{l_i \to r_i\}_i$. A term t rewrites to a term u at position p with the rule $l \to r$ and the substitution σ, written $t \longrightarrow^{p}_{l \to r} u$, if $t|_p = l\sigma$ and $u = t[r\sigma]_p$. A term t as above is called *reducible*. An irreducible term is called a *normal form*. The reflexive transitive closure of the relation $\to$, denoted by $\to^*$, is called *derivation*, while its symmetric, reflexive, transitive closure, denoted by $\leftrightarrow^*$ is called *conversion*. A term rewriting system R is *confluent* if $t \to^* u$ and $t \to^* v$ implies $u \to^* s$ and $v \to^* s$ for some s, *weakly normalizing* if every term s has a normal form t, *terminating* (or *strongly normalizing*) if all reduction sequences are finite, and *convergent* if it is confluent and terminating.

Given a term rewriting system R, its vocabulary $\mathcal{F}$ can be partitioned into a set $\mathcal{D} = \{f \in \mathcal{F} \mid f = root(l), l \to r \in R\}$ of *defined symbols* and a set $\mathcal{C} = \mathcal{F} - \mathcal{D}$ of *constructors*. We will often write $\mathcal{D} \cup \mathcal{C}$ instead of $\mathcal{F}$.

Let now R and S be two rewrite systems over the respective vocabularies $\mathcal{F}_R = \mathcal{C}_R \cup \mathcal{D}_R$ and $\mathcal{F}_S = \mathcal{C}_S \cup \mathcal{D}_S$ such that $\mathcal{F}_R \cap \mathcal{F}_S = \mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}$. The system $R \cup S$ is a *combined* or *constructor-sharing* union. A *constructor context* is a linear term $C[x_1, \ldots, x_n] \in T(\mathcal{C}, \mathcal{X})$. We say that a rewrite rule $l \to r$ is *constructor-lifting* if $root(r) \in \mathcal{C}$, *constructing* if $r \in T(\mathcal{C}, \mathcal{X})$, and *collapsing* if $r \in \mathcal{X}$. We say that a derivation $C[v_1, \ldots, v_n] \longrightarrow^* C[v'_1, \ldots, v'_n]$ is constructor-lifting if some v'_i is constructor-headed. We denote by $R_{\mathcal{C}lift}$ the infinite set $\{u \longrightarrow^*_R v\}$ of constructor-lifting R-derivations, similarly for $S_{\mathcal{C}lift}$.

From now on, we assume given two rewrite systems R and S sharing a set $\mathcal{C}$ of constructors as above.

3 Slicing Terms

Definition 1 *Let R and S be two rewrite systems over the respective vocabularies $\mathcal{F}_R = \mathcal{C}_R \cup \mathcal{D}_R$ and $\mathcal{F}_S = \mathcal{C}_S \cup \mathcal{D}_S$ such that $\mathcal{F}_R \cap \mathcal{F}_S = \mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}$. A term in the union $\mathcal{T}(\mathcal{F}_R \cup \mathcal{F}_S, \mathcal{X})$ is* homogeneous *if it uses symbols of $\mathcal{F}_R$ or $\mathcal{F}_S$ exclusively.*

A non-homogeneous term can be (recursively) decomposed into a topmost maximal homogeneous part, its *cap*, and a multiset of remaining subterms, its *aliens*. The cap of a (shared) constructor-headed term may not be unique. Since R and S play symmetric roles, we give the definitions for R only.

Definition 2 (Cap, Aliens, and Equalizers) *Given a term t in a combined union, a subterm $t = t|_q$ is called an* alien *with respect to $\mathcal{F}_R$ if $t(q) \in \mathcal{F}_S \setminus \mathcal{F}_R$ and $\forall p < q,\ t(p) \in \mathcal{F}_R$ and $\exists p < q$ such that $t(p) \in \mathcal{F}_R \setminus \mathcal{F}_S$.*

*Let ξ be a one to one mapping from a denumerable set $\mathcal{Y} \supseteq \mathcal{X}$ to the set of terms $\mathcal{T}(\mathcal{F}_R \cup \mathcal{F}_S, \mathcal{X})$, such that $\xi(x) = x$ for all $x \in \mathcal{X}$. If t is homogeneous with respect to $\mathcal{F}_R$, then the (unique) R-*cap *of t is the term $\hat{t}_R = t$, and its R-*aliens substitution *γ^R_t is the identity. Otherwise, $t = \hat{t}_R \gamma^R_t$, where $\hat{t}_R \in \mathcal{T}(\mathcal{F}_R \cup \mathcal{F}_S, \mathcal{Y})$ is homogeneous and maximal with respect to subsumption, and $\forall y \in \mathcal{V}ar(\hat{t}), y\gamma^R_t = \xi(y)$. The multiset of R-*aliens *of t is the multiset $\{y\gamma^R_t \mid y \in \mathcal{V}ar(\hat{t}_R) \cap (\mathcal{Y} \setminus \mathcal{X})\}$, in which each term $y\gamma^R_t$ appears as many times as the number of occurrences of y in $\hat{t}_R$.*

We say that γ is an equalizer substitution *if the instance $C[x_1, \ldots, x_n]\gamma$ of some arbitrary constructor context $C[x_1, \ldots, x_n]$ such that $\mathcal{D}om(\gamma) = \{x_1, \ldots, x_n\}$ is an equalizer, and that t is an* equalizer *iff $\forall x, y \in \mathcal{D}om(\gamma_t)$, $x = y$ iff $\xi(x) \leftrightarrow^*_{R \cup S} \xi(y)$, and γ_t is an equalizer substitution.*

The rank *of an homogeneous term is defined to be 0. For an heterogeneous term, its rank is* 1+ *the maximal rank of its aliens.*

Example 3 *Let* $\mathcal{F}_R = \{f, c\}, \mathcal{F}_S = \{g, c\}, \mathcal{X} = \{x\}, \mathcal{Y} \supset \{x, y\}$. *The cap of the term* $c(f(g(x)), g(x))$ *with respect to* $\mathcal{F}_R$ *is the homogeneous term* $c(f(y), y)$, *and there are two aliens, the occurrences of* $g(x)$. *Let now* $S = \{g(x) \to x\}$. *Then, the term* $c(f(g(x)), g(x))$ *is an equalizer, while the term* $c(f(g(x)), x)$ *is not.*

In practice, we will usually leave implicit the signature $\mathcal{F}_R$ or $\mathcal{F}_S$ with respect to which the cap of a term originating a derivation is computed. More precisely, the cap $\hat{s}$ of the first term in a derivation is chosen to be $\hat{s}_R$ or $\hat{s}_S$ if both exist. Then, the caps of its successive reducts are chosen accordingly, as explained next.

Given a choice for the cap of s, we will write $s \longrightarrow_C t$ (called a *cap reduction*) if $s \longrightarrow^p_{R \cup S} t$ with $p \in \mathcal{FP}os(\hat{s})$, and $s \longrightarrow_A t$ (*alien reduction*) if $s \longrightarrow^p_{R \cup S} t$ with $p \notin \mathcal{FP}os(\hat{s})$.

Lemma 4 *Assume that* $s \longrightarrow^*_A t$ *with* $\hat{s} = \hat{s}_R$. *Then,* $\hat{t}_R$ *exists, and we define* $\hat{t} = \hat{t}_R$. *Moreover,* $\hat{t} = \hat{s}$ *if* s *and* t *are equalizers.*

Proof: Because $\mathcal{FP}os(\hat{s}_R) \subseteq \mathcal{FP}os(\hat{t})$, and $\hat{t}_R(p) = \hat{s}_R(p)$ for all $p \in \mathcal{FP}os(\hat{s}_R)$. □

Lemma 5 *Assume that* $s = l\sigma$ *for some homogeneous term* l *such that* $\mathcal{D}om(l) \subseteq \mathcal{D}om(\hat{s})$ *and some substitution* σ. *Then,* $\sigma = \delta\gamma_s$ *for some substitution* δ.

Note that the cap of s is unique here, if l is a lefthand side of rule, hence headed by a defined symbol in either R or S.

Proof: Assume that $l \in T(\mathcal{F}, \mathcal{X})$. Let $x \in \mathcal{V}ar(l)$, and $l|_p = l|_q = x$. Then, $p, q \in \mathcal{D}om(\hat{s}_R)$, therefore $x\sigma = s|_p = \hat{s}_R\gamma^R_s|_p = \hat{s}_R|_p\gamma^R_s = \hat{s}_R|_p\xi$. Similarly, $x\sigma = s|_q = \hat{s}_R\gamma^R_s|_q = \hat{s}_R|_q\gamma^R_s = \hat{s}_R|_q\xi$. Since ξ is a bijection, $\hat{s}_R|_p\xi = \hat{s}_R|_q\xi$ implies $\hat{s}_R|_p = \hat{s}_R|_q$. Therefore, $\delta(x) = \hat{s}_R|_p$ is a well-defined substitution, and $\sigma = \delta\gamma^R_s$. □

Lemma 6 *Assume that* $s \longrightarrow_C t$ *with the rule* $l \to r \in R$, *hence* $\hat{s} = \hat{s}_R$ *and substitution* σ. *Then, either*

(i) $\hat{s}_R \xrightarrow[l \to r]{p} u$, $u = \hat{t}_R \notin T(\mathcal{C}, \mathcal{Y}) \setminus T(\mathcal{C}, \mathcal{X})$ *and* $t = \hat{t}_R\gamma_s$, *in which case we let* $\hat{t} = \hat{t}_R$ *or*

(ii) $\hat{s} \xrightarrow[l \to r]{p} u \in T(\mathcal{C}, \mathcal{Y}) \setminus T(\mathcal{C}, \mathcal{X})$, *hence* $r \in T(\mathcal{C}, \mathcal{V}ar(l))$, $t = u\gamma_s$ *and* $\hat{t} = \hat{t}_S$.

In both cases, t *is an equalizer if* s *is an equalizer.*

Here, the cap of s is determined by our choice that the rule $l \to r$ rewrites in its cap. It is important to notice that, in case (i), t may be a term built from shared constructors and variables in $\mathcal{V}ar(s)$. Such a term is obtained iff $s[x]_p \in T(\mathcal{C}, \mathcal{X})$, the rule $l \to r$ is constructing or collapsing, and all aliens are erased, that is, for all $x \in \mathcal{V}ar(r)$, $x\sigma$ is a subterm of $\hat{s}$. In case (ii), there must exist some variable $x \in \mathcal{V}ar(r)$ such that $x\sigma$ is a subterm of s but not of $\hat{s}$, that is, contains an alien of s. As a consequence, the cap of this alien gets glued to u and the cap of t has to be in the other signature with respect to the cap of s.

Proof: By lemma 5, $\sigma = \delta\gamma_{s|_p}$ for some δ. Therefore, $s = s[s|_p]_p = \hat{s}\gamma_s[l\delta\gamma_{s|_p}]_p = (\hat{s}[l\delta]_p)\gamma_s$ since $p \in \mathcal{FP}os(\hat{s})$, and $t = (\hat{s}[r\delta]_p)\gamma_s$. There are two cases:

1. $\hat{s}[r\delta]_p \notin T(\mathcal{C}, \mathcal{Y}) \setminus T(\mathcal{C}, \mathcal{X})$, that is, either $\hat{s}[r\delta]_p$ contains a defined symbol, or it does not contain a variable from $\mathcal{Y} \setminus \mathcal{X}$. In both cases, $\hat{s} \longrightarrow_{l \to r} \hat{t} = \hat{s}[r\delta]_p$ and $t = \hat{t}\gamma_s$. Assuming s is an equalizer, t is therefore an equalizer as well.

2. $\hat{s}[r\delta]_p \in T(\mathcal{C}, \mathcal{Y} \setminus T(\mathcal{C}, \mathcal{X})$. Necessarily, $r \in T(\mathcal{C}, \mathcal{V}ar(l))$ and $t = u\gamma_s$ is the instance of a constructor term by aliens of γ_s. Therefore, there is no choice for the cap of t, it has to be in the signature of the cap of the aliens of s. Assuming now that s is an equalizer, then γ_s is an equalizer substitution, hence $t = u\gamma_s$ is an equalizer as well.

□

From now on, we will not need to subscript the cap of a term by the signature R or S, once the choice of a cap for a term originating a derivation is made, which we will always assume.

4 Stable equalizers

Due to the possible action of constructor lifting or collapsing reductions, the cap and the aliens may grow or disappear along derivations. In particular, the cap may disappear if the term is equivalent to one of its maximal aliens, up to some constructor context.

Definition 7 *[Stability] A term s is* cap-collapsing *if there exist aliens $u_1, \ldots, u_n$ ($n > 0$) of s, a constructor context $C[x_1, \ldots, x_n]$ and a term t such that $s \longrightarrow^*_{R \cup S} t \longleftarrow^*_{R \cup S} C[u_1, \ldots, u_n]$.*

A term s headed by a defined symbol is constructor-lifting *if there exists a constructor headed term t such that $s \longrightarrow^* t$.*

A term s is cap-stable *if it is neither cap-collapsing nor constructor-lifting,* stable *if it is cap-stable and its aliens are themselves stable, and* alien-stable *if its aliens are stable.*

Collapsing a cap arises because of the application of a cap rewrite as described in Lemma 6 (ii). Note that we cannot simply take $t_i = u_i$ as shown by the following example: $R = \{f(x,x) \to c(x)\}$ and $S = \{a \to c(d),\ b \to c(d)\}$. Then, we have $f(a,b) \longrightarrow f(c(d),b) \longrightarrow f(c(d),c(d)) \longrightarrow c(c(d)) \longleftarrow c(a)$. Note that the alien a of the original term has to be rewritten to $c(d)$. On the other hand, the original term is not an equalizer. For equalizers, we could adopt the simpler definition $s \longrightarrow^*_{R\cup S} C[u_1,\ldots,u_n]$. Similarly, we cannot take an empty context in general because of the constructor-lifting rules, as shown by the same example.

Lemma 8 *Assume that v is a stable term such that $v \longrightarrow^* t$. Then v and t have their cap in the same signature and t is stable.*

Proof: By (outer) induction on the rank, the property is true of the aliens of v. Therefore, it suffices to prove that the cap of v does not collapse, and that t is cap-stable. We do it by induction on the length of the derivation from v to t. The case $v = t$ being trivial, let $v \longrightarrow s \longrightarrow^* t$. Using the inner induction on the derivation $s \longrightarrow^* t$, we simply need to prove the property for s. We distinguish two cases.

1. Assume that $v \longrightarrow_C s$. Since v is not cap-collapsing, case (i) of Lemma 6 applies, therefore, both caps are in the same signature and the aliens of s are aliens of v (hence stable by assumption).

2. Assume now that $v \longrightarrow_A s$. Since v is stable, so are its aliens, and, by outer induction hypothesis, its reducts. This shows that the aliens of s are stable. Besides, by Lemma 4, $\hat{s}$ and $\hat{v}$ are in the same signature.

We are left to show that s is neither cap-collapsing nor constructor-lifting in case it is headed by a defined symbol. If it were cap-collapsing, then, using the notations of Definition 7, there exist aliens $u_1,\ldots,u_n$ of s, a constructor context $C[y_1,\ldots,y_n]$ and a term t such that $s \longrightarrow^*_{R\cup S} t \longleftarrow^*_{R\cup S} C[u_1,\ldots,u_n]$. Putting the derivations together yields $v \longrightarrow^*_{R\cup S} t \longleftarrow^*_{R\cup S} C[u_1,\ldots,u_n]$. Since v is stable, the aliens $u_1,\ldots,u_n$ of s, are reducts of aliens $v_1,\ldots,v_n$ of v, and we obtain therefore $v \longrightarrow^*_{R\cup S} t \longleftarrow^*_{R\cup S} C[v_1,\ldots,v_n]$, contradicting our assumption that v is stable. □

Lemma 9 *Assume that* $s \longrightarrow_A^* t$*, where* s *is an alien-stable equalizer. Then, there exists a substitution* θ *from* $\mathcal{V}ar(\hat{t}) \cap (\mathcal{Y} \setminus \mathcal{X})$ *to* $\mathcal{V}ar(\hat{s}) \cap (\mathcal{Y} \setminus \mathcal{X})$ *such that* $\hat{s} = \hat{t}\theta$ *and* $\theta\gamma_s \longrightarrow^* \gamma_t$*. Moreover,* θ *is a bijection if* t *is also an equalizer.*

Proof: By Lemma 4, $\hat{s}$ and $\hat{t}$ are in the same signature, and by Lemma 8 t is alien-stable, therefore its aliens are stable. Hence, $\mathcal{FP}os(\hat{s}) = \mathcal{FP}os(\hat{t})$, $\forall p \in \mathcal{FP}os(s)$, $s(p) = t(p)$, and $\hat{s}$ and $\hat{t}$ may only differ by the names of their variables. Now, if $\hat{t}|_p = \hat{t}|_q = x$ then $t|_p = t|_q$, therefore $s|_p \leftrightarrow^* s|_q$ since $s \longrightarrow_A^* t$. Hence $s|_p = s|_q$ since s is an equalizer. Therefore $\hat{s}|_p = \hat{s}|_q$, and $\hat{s} = \hat{t}\theta$ for some θ from $\mathcal{V}ar(\hat{t}) \cap (\mathcal{Y} \setminus \mathcal{X})$ to $\mathcal{V}ar(\hat{s}) \cap (\mathcal{Y} \setminus \mathcal{X})$. Also $\theta\gamma_s \longrightarrow^* \gamma_t$ since $s \longrightarrow_A^* t$.

Moreover, if t is also an equalizer, $\hat{s}|_p = \hat{s}|_q = x$ implies $t|_p = t|_q$, therefore $\hat{t}|_p = \hat{t}|_q$, and θ is a bijection. □

5 Structure Lemma

The goal of this section is to show that proofs between equivalent non-homogenous alien-stable terms can be decomposed into a proof between their caps, and a proof between their aliens.

Due to collapsing reductions, the transitive closure $\longrightarrow_C^*$ of the relation $\longrightarrow_C$ may actually end up rewriting in a descendant of an alien of the term *originating* the derivation. To cope with this problem, abbreviating "original cap" by OC and "original aliens" by OA, we define the ternary relation $s \longrightarrow_{OC}^* v \longrightarrow_{OA}^* t$, for a given alien-stable term s by induction on derivations as follows:

1. $s = v \longrightarrow_A^* t$,
2. $s \longrightarrow_C v = C[v_1, \ldots, v_n] \longrightarrow^* t$, where $C[x_1, \ldots, x_n]$ is a constructor context and $v_1, \ldots, v_n$ are aliens of s,
3. $s \longrightarrow_C u \longrightarrow_{OC}^* v \longrightarrow_{OA}^* t$, where u is not of the form $C[v_1, \ldots, v_n]$, with $C[x_1, \ldots, x_n]$ a constructor context and $v_1, \ldots, v_n$ aliens of s.

Similarly, we define the number n of rewrite steps in the original cap of the alien-stable term s originating the derivation $s \longrightarrow^* t$, called its *index*, as follows:

1. if $s = t$, then $n = 0$,
2. if $s \longrightarrow_A u$ and $u \longrightarrow^* t$ is a derivation of index m, then $n = m$,

3. if $s \longrightarrow_C v$ where $v = C[v_1, \ldots, v_n]$ for some constructor context C and aliens $v_1, \ldots, v_n$ of s, then $n = 1$,

4. otherwise $s \longrightarrow_C u$ and $u \neq C[v_1, \ldots, v_n]$, and $u \longrightarrow^* t$ is a derivation of index m, then $n = m + 1$,

A key property of alien-stable equalizers is the following:

Lemma 10 (Commutation) *Let s be an alien-stable equalizer such that $s \longrightarrow^* v$. Then, $s \longrightarrow^*_{OC} w \longrightarrow^*_{OA} v$, where w is an alien-stable equalizer.*

Proof: By induction on the index n of the derivation $s \longrightarrow^* v$. For the base case ($n = 0$), $w = s$. For the induction step, let $s \longrightarrow^*_A u \longrightarrow_C t \longrightarrow^* v$. By Lemma 9, $\hat{s} = \hat{u}\theta$ for some θ such that $\theta\gamma_s \longrightarrow^* \gamma_u$. By Lemma 6, we have either (i) $\hat{u} \longrightarrow \hat{t}$ and $\gamma_u = \gamma_t$ or (ii) $\hat{u} \longrightarrow u' \in T(\mathcal{C}, \mathcal{Y}) \setminus T(\mathcal{C}, \mathcal{X})$, and $t = u'\gamma_u$.

In case (i), $s = \hat{s}\gamma_s = \hat{u}\theta\gamma_s \longrightarrow_{OC} \hat{t}\theta\gamma_s \longrightarrow^*_A \hat{t}\gamma_u = \hat{t}\gamma_t = t \longrightarrow^* v$. By Lemma 6, $\hat{t}\theta\gamma_s$ is an equalizer (of a rank smaller or equal to that of s), which is alien-stable by Lemma 8, therefore the index of the derivation $\hat{t}\theta\gamma_s \longrightarrow^*_A t \longrightarrow^* v$ is equal to $n-1$. We conclude by the induction hypothesis.

In case (ii), $s = \hat{s}\gamma_s = \hat{u}\theta\gamma_s \longrightarrow_{OC} u'\theta\gamma_s \longrightarrow^*_{OA} u'\gamma_u = t \longrightarrow^*_{OA} v$, and we are done since $u'\theta\gamma_s$ is an alien-stable equalizer (γ_s is an equalizer). □

Lemma 11 (Cap-collapsing) *Let e be an alien-stable equalizer. Then, e is* cap-collapsing *iff $\hat{e} \longrightarrow^*_{R \cup S} C[y_1, \ldots, y_n]$ for some constructor context $C[y_1, \ldots, y_n] \in T(\mathcal{C}, \mathcal{Y} \setminus \mathcal{X})$.*

Proof: Using the notations of Definition 7, let $e \longrightarrow^* t \longleftarrow^* C[u_1, \ldots, u_n]$ for some aliens $u_1, \ldots, u_n$ of e. We write $e \longrightarrow^*_{OC} e' \longrightarrow^*_{OA} t$ by Lemma 10, where e' is an alien-stable equalizer.

Since $u_1, \ldots, u_n$ are aliens of e, the caps of e and $u_1, \ldots, u_n$ are not in the same signature. Since e is alien-stable, the caps of $u_1, \ldots, u_n$ do not change by rewriting, therefore the caps of e and t are not in the same signature. By definition of $e \longrightarrow^*_{OC} e'$, the cap cannot change along that derivation except, possibly, during the last step. Assume now that e' and e have their cap in the same signature. Then, $e' \longrightarrow^*_{OA} t$ must occur in the aliens of e', which does not change the cap by Lemma 4, a contradiction. Therefore, the caps of e and e' must be in different signatures, hence the derivation from e to e' must have the form $e \underset{OC}{\overset{*}{\longrightarrow}} e'' \underset{l \to C'[x_1, \ldots, x_m]}{\longrightarrow} e' = C''[v_1, \ldots, v_p]$, where $C''[y_1, \ldots, y_p] \in T(\mathcal{C}, \mathcal{Y}) \setminus T(\mathcal{C}, \mathcal{X})$, and $v_1, \ldots, v_p$ are aliens of e''. By repeated applications of Lemma 6, $\hat{e} \longrightarrow^* \hat{e''}$, e'' is an alien-stable equalizer, and $\hat{e''} \longrightarrow C''[y_1, \ldots, y_p] \in T(\mathcal{C}, \mathcal{Y}) \setminus T(\mathcal{C}, \mathcal{X})$, hence $\hat{e} \longrightarrow^* C''[y_1, \ldots, y_p]$. □

Lemma 12 (Cleaning) *Assume that every term in $T(\mathcal{F}_R, \mathcal{Y})$ (resp., $T(\mathcal{F}_S, \mathcal{Y})$) has a (non necessarily unique) normal form with respect to $R_{\mathcal{C}lift}$ (resp., $S_{\mathcal{C}lift}$). Let t be a term whose aliens are confluent in $R \cup S$. Then, there exists a stable equalizer e such that $t \longrightarrow^*_{R \cup S} e$.*

Proof: By induction on the rank of $t = \hat{t}\gamma_t$. By confluence assumption, $\gamma_t \longrightarrow^* \gamma'_t$ such that $\gamma_t(x) \leftrightarrow^* \gamma_t(y)$ iff $\gamma'_t(x) = \gamma'_t(y)$. By induction hypothesis, $\gamma'_t \longrightarrow^* \gamma''_t$, a stable equalizer substitution. Let now $s = \hat{t}\gamma''_t$, hence $t \longrightarrow^*_A s$. We now compute $\hat{s}$ and γ_s, show that γ_s is a stable equalizer substitution, and that s rewrites to a stable equalizer e.

Let $y \in \mathcal{V}ar(\hat{t}) \setminus \mathcal{V}ar(t)$, and assume without loss of generality that the cap of t belongs to the signature R. Let $\theta(y) = \hat{s}_R$. By construction, $s = \hat{t}\theta\xi$, $\hat{t}\theta = \hat{s}$, and $\xi|_{\mathcal{V}ar(\hat{s}) \setminus \mathcal{V}ar(s)} = \gamma_s$. Now, since γ_s is made of subterms of γ''_t or alien subterms of γ''_t, it is a stable equalizer.

If s is cap-collapsing, then $\hat{s} \longrightarrow^* C[y_1, \ldots, y_n]$, by Lemma 11, hence $s \longrightarrow^* C[y_1\gamma_s, \ldots, y_n\gamma_s]$, which is a stable equalizer and we are done.

If s is not cap-collapsing, we take a normal form under the constructor-lifting derivations and we are done again. □

We finally come to the structure lemma. For this, we will use a tool called ordered paramodulation, or, in the context of equations, ordered completion. Given a set E of equations and a rewrite ordering $\succ$ total on ground terms, ordered completion computes a (possibly infinite) set of equations E^∞ which is ground confluent in the following sense: $s \leftrightarrow^*_E t$ iff $s \longrightarrow^*_{E^\infty} u$ and $t \longrightarrow^*_{E^\infty} u$, where $v \longrightarrow_{E^\infty} w$ iff $v|_p = l\sigma, w = v[r\sigma]_p$ and $v \succ w$ for some $l = r \in E^\infty$. The main observation is that in the absence of shared constructors, then $(R \cup S)^\infty = R^\infty \cup S^\infty$ for any rewrite ordering $\succ$ total on ground terms. Note that the result of completion is not changed by adding a set of free variables provided the ordering is extended so as to remain a total rewrite ordering. This is possible with, e.g., the lexicographic path ordering. We will show in Section 7 under which condition this observation remains true when there are shared constructors. Saying this, we assume that R^∞ and R have the same set of constructors, that is, symbols in $\mathcal{C}$ never appear at the head of a lefthand side of a rule in R^∞.

Lemma 13 *Assume that the alien-stable equalizer u is cap-collapsing with respect to $R^\infty \cup S^\infty$. Then, u is cap-collapsing with respect to $R \cup S$.*

Proof: By Lemma 11, $\hat{u} \longrightarrow^*_{R^\infty \cup S^\infty} C[x_1, \ldots, x_n]$. Since $\hat{u}$ is homogeneous, we may assume without loss of generality that $\hat{u} \longrightarrow^*_{R^\infty} C[x_1, \ldots, x_n]$, hence $\hat{u} \leftrightarrow^*_{R^\infty} C[x_1, \ldots, x_n]$ and therefore $\hat{u} \leftrightarrow^*_R C[x_1, \ldots, x_n]$, and by confluence of R, $\hat{u} \longrightarrow^*_R C[x_1, \ldots, x_n]$, and therefore $\hat{u} \longrightarrow^*_{R \cup S} C[x_1, \ldots, x_n]$. We conclude by Lemma 11 again. $\square$

Lemma 14 (Structure) *Let $R \cup S$ be a combined union for which ordered completion is modular. Let v and w be two equalizers whose aliens are confluent, and such that*

(i) v and w are stable with respect to $\longrightarrow_{R \cup S}$,

(ii) $v \leftrightarrow^*_{R \cup S} w$.

Then,

(i) $\hat{v} \leftrightarrow^*_{R \cup S} \hat{w}\eta$ and

(ii) $\gamma_v \leftrightarrow^*_{R \cup S} \eta^{-1}\gamma_w$ for some bijection $\eta : \mathcal{V}ar(\hat{v}) \cap \mathcal{Y} \mapsto \mathcal{V}ar(\hat{w}) \cap \mathcal{Y}$.

Proof: By assumption, $\leftrightarrow^*_{R \cup S} = \leftrightarrow^*_{R^\infty \cup S^\infty}$, and $R^\infty \cup S^\infty$ is confluent. Therefore, $v \longrightarrow^*_{R^\infty \cup S^\infty} s$ and $w \longrightarrow^*_{R^\infty \cup S^\infty} s$ for some s. By Lemma 13, v and w must be stable with respect to $R^\infty \cup S^\infty$. By Lemmas 10 and 8, there exist two stable equalizers v' and w' such that $v \longrightarrow^*_{OC, R^\infty \cup S^\infty} v' \longrightarrow^*_{OA, R^\infty \cup S^\infty} s$ and $w \longrightarrow^*_{OC, R^\infty \cup S^\infty} w' \longrightarrow^*_{OA, R^\infty \cup S^\infty} s$. Since aliens of v and w are confluent, the aliens of s are confluent as well, hence, by Lemma 6, s can be assumed without loss of generality to be an equalizer. By Lemma 6 again (since v, w are stable), $\hat{v} \longrightarrow^* \hat{v'}$ and $\hat{w} \longrightarrow^* \hat{w'}$. Since $v' \longrightarrow^*_A s$, by Lemma 9, $\hat{v'} = \hat{s}\mu$, and $\mu\gamma_{v'} \longrightarrow^* \gamma_s$ for some bijection μ. Similarly $\hat{w'} = \hat{s}\nu$ and $\nu\gamma_{w'} \longrightarrow^* \gamma_s$ for some bijection ν. Let η be the bijection $\nu^{-1}\mu$. Then, $\hat{v'} = \hat{w'}\eta$, yielding (i), and $\mu\gamma_{v'} \longrightarrow^* \gamma_s \longleftarrow^* \nu\gamma_{w'}$, yielding (ii). $\square$

6 Modularity of Confluence

We can now conclude:

Theorem 15 *Assume ordered completion is modular. Then, the union of two confluent rewrite systems R, S with shared constructors $\mathcal{C}$, such that $R_{\mathcal{C}lift}, S_{\mathcal{C}lift}$ are weakly-terminating, is confluent.*

Proof: We show the Church-Rosser property: $v \leftrightarrow^*_{R \cup S} w$ implies $v \longrightarrow^* \longleftarrow^* w$ by induction on the maximal rank of the pair (v, w). See the figure.

There is a first difficulty, that we must choose the cap of v and w accordingly. We can actually choose the cap of all term in the proof $v \leftrightarrow^* w$

non-deterministically, and verify that the rules which define the cap of a reduct are satisfied. If there is a proof, it is easy to see that there must be a consistent choice, which we now assume.

By the cleaning Lemma 12, $v \longrightarrow^*_{R\cup S} v'$ and $w \longrightarrow^* w'$, v' and w' being stable equalizers.

By the structure Lemma 14, $\hat{v'} \leftrightarrow^*_{R\cup S} \hat{w'}\eta$ and $\gamma_{v'} \leftrightarrow^*_{R\cup S} \eta^{-1}\gamma_{w'}$.

By confluence assumptions on R and S, $\hat{v'} \longrightarrow^* s \longleftarrow^* \hat{w'}\eta$.

By induction hypothesis applied to $\gamma_{v'}$ and $\eta^{-1}\gamma_{w'}$ whose ranks are strictly smaller than the maximal rank of v, w, $\gamma_{v'} \longrightarrow^* \sigma \longleftarrow^* \eta^{-1}\gamma_{w'}$.

Therefore, $v' = \hat{v'}\gamma_{v'} \longrightarrow^* s\gamma_{v'} \longrightarrow^* s\sigma \longleftarrow^* s\eta^{-1}\gamma_{w'} \longleftarrow^* \hat{w'}\eta\eta^{-1}\gamma_{w'} = w'$, and we are done. □

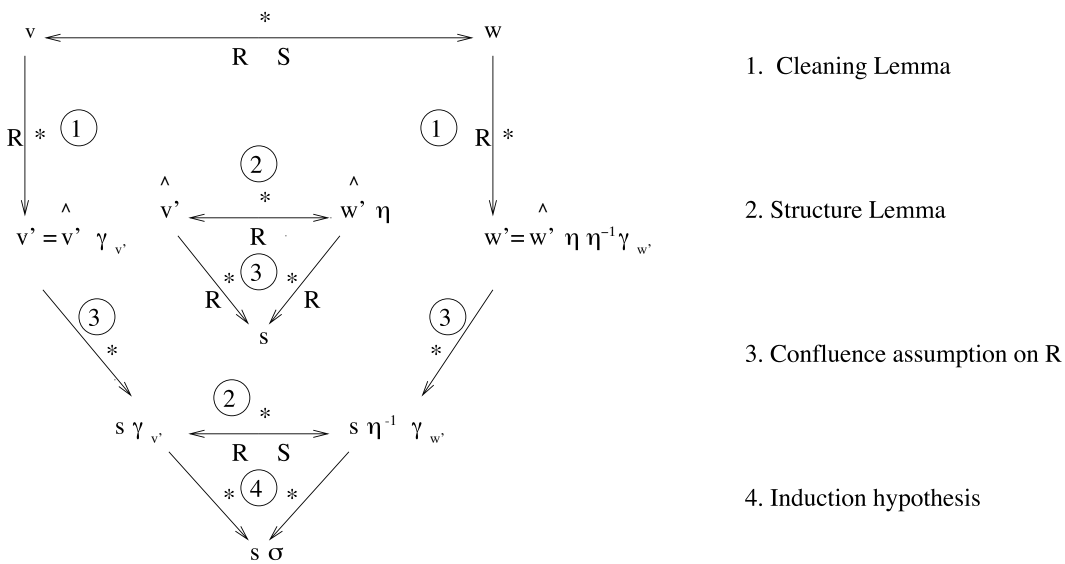

Figure 1: Proof of the modularity theorem

We are now left with the modularity property of ordered completion, a straightforward property when there are no shared constructors.

7 Modularity of Ordered Completion

We now want to complete R (and S) in such a way that the shared constructors are still constructors in the completed sets.

Theorem 16 (Modularity of Ordered Completion) *Assume that $R_{\mathcal{C}lift}$ and $S_{\mathcal{C}lift}$ are weakly terminating. Then $(R \cup S)^\infty = R^\infty \cup S^\infty$ for some well-founded, stable ordering $\succ$ which is total on ground terms.*

Proof: Since $R_{\mathcal{C}lift}$ is weakly-terminating, each term of $\mathcal{T}(\mathcal{F}, \mathcal{X})$ has a normal-form with respect to $R_{\mathcal{C}lift}$. But since R is confluent and $R_{\mathcal{C}lift} \subseteq \longrightarrow_R^*$, this normal form must be unique. Let $\succ_R$ be the relation on ground terms containing: the pairs (u, v) if u and v are different ground terms such that v is the constructor-lifting normal form of the ground term u; the pairs $(u\sigma, v\sigma$ if u and v are different terms such that v is the constructor-lifting normal form of u, and $u\sigma$ and $v\sigma$ are ground. Let $\succ = \succ_R \cup \succ_S$. $\succ$ is transitive and well-founded since all pairs are of the form $f(\overrightarrow{u}), C[\overrightarrow{v}]$, where $f \in \mathcal{F} \setminus C$ and $C[\ldots]$ is a non-empty constructor context. By Zorn's lemma, $\succ$ can be extended into a total, well-founded ordering on ground terms. The ordering is then extended to terms with variables by stability: $s \succ t$ iff $s\sigma \succ t\sigma$ for all substitutions such that $s\sigma$ and $t\sigma$ are ground. The obtained ordering is again well-founded, total on ground terms, and stable, hence it allows to perform ordered completion, as shown in [1].

Our completion process will generate two kinds of critical pairs, made of homogeneous terms of the same signature, say $\mathcal{F}_R$: pairs whose terms are both headed by defined symbols. These pairs are normally incomparable, hence are treated as equations, whose instances are ordered according to the ordering; and pairs of the form $(p = f(\overrightarrow{u}), q = C[v_1, \ldots, v_n])$, where $f \in \mathcal{F}_R \setminus \mathcal{C}$ and $C[x_1, \ldots, x_n]$ is a constructor context. Let w be the unique normal form of q with respect to $R_{\mathcal{C}lift}$. Since a lefthand side of rule in R, hence in $R_{\mathcal{C}lift}$ cannot be headed by a constructor, then $w = C[w_1, \ldots, w_n]$, and $v_i \longrightarrow_{R_{\mathcal{C}lift}}^* w_i$ if w_i is constructor-headed. There are two cases: if w_i is not constructor-headed, then the pair (v_i, w_i) is left as an equation; if w_i is constructor headed, then it is the constructor-lifting normal form of v_i, hence $v_i \succ w_i$ by construction and that pair can be added as a constructor-lifting rule. We are left to show that $p \succ w$. Since R is confluent, $p \longrightarrow_R^* w'$ for some w' such that $w \longrightarrow_R^* w'$. But the derivation $q \longrightarrow_R^* w'$ is constructor-lifting, hence $w' = w$, and $p \succ w$ by construction again.

So, all pairs constructor-lifting pairs become rules in our completion process, which ensures modularity. □

Toyama's theorem and Ohlebusch extension are corollaries of Theorem 15 using Theorem 16.

8 Conclusion

We have given a really simple proof of Ohlebush generalization of Toyama's theorem that enabled us to better understand the difficulties of modularity in presence of shared constructors. Besides, our aqpproach can easily be tailored so as to cover new cases that are not covered by Ohlebush's result. Indeed, the key to our approach is the structure Lemma. In turn, this Lemma relies on the modularity of confluence, which cannot be so easily relaxed, and on the cleaning Lemma, which can be relaxed. Indeed, what is needed in Lemma 15 is the existence of terms v' and w' to which the structure lemma can be applied. The cleaning Lemma says that v' and w' can be chosen to be constructor-lifting normal forms of v and w. If such normal forms do not exists, we need to chose some other terms instead. In general, there is actually no need to go until a normal form is obtained. We can simply rewrite v and w until enough constructors have been poped up, so as to apply the structure Lemma. Here is an example where this idea applies:

Example 17 *$R = \{g \to c(g)\}$, and $S = \{f(x, x) \to x, f(x, cx) \to cx\}$. Ordered completion generates $\{c(g) \to g, f(x, x) \to x, f(x, cx) \to cx\}$. And the union is indeed confluent.*

We have not yet explored this direction, but we believe that it can be carried out with the same kind of tools. We also believe that there are a number of modularity results in the literature which could benefit of a similar treatment.

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