TERMINATION OF LINEAR REWRITING SYSTEMS*
- preliminary version -

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ABSTRACT

Limitations, such as right-linearity, on the form of rules in a term-rewriting system are shown to restrict the class of derivations that must be considered when determining whether or not the system terminates for all inputs. These restricted derivations, termed "chains", are obtained by attempting to apply rules to the final terms of derivations that issue from the left-hand side of rules. Similar limitations are shown to guarantee that combining two terminating systems yields a terminating system.

I. INTRODUCTION

A term-rewriting system S over a set of terms T is a program expressed as a finite set of rewrite rules, each of the form $\ell_i\lceil\overline{\alpha}\rceil \rightarrow r_i\lceil\overline{\alpha}\rceil$, where the left-hand side $\ell_i\lceil\overline{\alpha}\rceil$ and the right-hand side $r_i\lceil\overline{\alpha}\rceil$ are open terms, i.e. terms constructed from operators (function symbols) and from variables $\overline{\alpha}$ ranging over T. For example,

$$\{\ (\alpha+\beta)+\gamma \rightarrow \alpha+(\beta+\gamma)\ \}$$

is a one-rule term-rewriting system over the set of terms constructed from natural numbers and the binary operator "+". The variables α, β, and γ represent arbitrary subterms.

The program is executed by repeatedly applying rules to some given initial term. A rule of the form $\ell_i\lceil\overline{\alpha}\rceil \rightarrow r_i\lceil\overline{\alpha}\rceil$ may be applied to a (closed or open) term t of T if t contains a subterm v that is an instance of ("matches") the pattern $\ell_i\lceil\overline{\alpha}\rceil$, i.e. $v=\ell_i\lceil\overline{s}\rceil$ with (possibly open) terms $\overline{s}$ of T substituted for the variables $\overline{\alpha}$. Multiple occurrences of the same variable α_j in ℓ_i must be matched by occurrences of the same subterm s_j in v. The rule is applied to the term t by replacing its

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subterm v with the term $r_i\lceil \overline{s} \rceil$. For example, applying the rule $(\alpha+\beta)+\gamma \rightarrow \alpha+(\beta+\gamma)$ to the term (a+b)+((c+d)+e) at the subterm (c+d)+e yields the new term (a+b)+(c+(d+e)). (The square-bracket notation $t=u\lceil s_1,\cdots,s_n \rceil$ means that the term t can be parsed into a tree of operators u connecting distinct occurrences of the subterms s_1, ..., s_n. We assume that the set of terms T is closed under the "subterm" operation, i.e. any subterm of a term in T is also a term in T.)

The choice of which rule to apply is made nondeterministically from amongst the applicable rules; similarly, the choice of which subterm to apply a rule to is nondeterministic. We write $t \Rightarrow_S t'$ to indicate that the term $t' \varepsilon T$ may be derived from the term $t \varepsilon T$ by a single application of some rule in S. (The subscript S is sometimes left implicit.)

For example, the system $\{(\alpha+\beta)+\gamma \rightarrow \alpha+(\beta+\gamma)\}$ reparenthesizes a summation by associating to the right. Applying that system to the term t=(a+b)+((c+d)+e), yields $t \Rightarrow a+(b+((c+d)+e)) \Rightarrow a+(b+(c+(d+e)))$ or $t \Rightarrow (a+b)+(c+(d+e)) \Rightarrow a+(b+(c+(d+e)))$. In either case, no further applications of the rule are possible. Similarly, the system could be applied to an open term: $((\alpha+\beta)+\gamma)+\delta \Rightarrow (\alpha+(\beta+\gamma))+\delta \Rightarrow \alpha+((\beta+\gamma)+\delta) \Rightarrow \alpha+(\beta+(\gamma+\delta))$. Obviously, any substitution of terms for variables in a derivation of open terms would give an equally valid derivation.

The property of term-rewriting systems that we deal with in this paper is termination. A term-rewriting system S <u>terminates</u> for a set of terms T if there exist no infinite sequences of terms $t_i \varepsilon T$ such that $t_1 \Rightarrow t_2 \Rightarrow t_3 \Rightarrow \cdots$; conversely, a system is <u>nonterminating</u> if there exists any such infinite derivation. For example, the above reparenthesizing system terminates for all inputs. We shall assume that no left-hand side is just a variable; otherwise the system clearly would be nonterminating. For the same reason, the variables appearing in each right-hand side must be a subset of those in the corresponding left-hand side.

We consider several limitations on the form of the rules in the term-rewriting systems. These restrictions make it possible to restrict the class of possible derivations that must be considered when proving the termination or nontermination of the system.

A term t is said to <u>overlap</u> a term t′ if t′ can be unified with some (not necessarily proper) subterm s of t, i.e. if by substituting terms for the variables in t′ and the variables in s the two can be made equal. (Whenever we speak of unifying two open terms, we consider their variables to be disjoint and insist that neither of the terms be just a variable.) We say that there is no overlap between two terms t and t′ if neither t overlaps t′ nor t′ overlaps t. A term-rewriting system S is said to be <u>non-overlapping</u> (or "have no critical pairs", in the terminology of Knuth and Bendix [1969]) if there is no overlap among the left-hand sides of S, i.e. no left-hand side ℓ_i overlaps a different left-hand side ℓ_j and no proper subterm of a left-hand side ℓ_i overlaps all of ℓ_i.

A system is <u>left-linear</u> if no variable occurs more than once on the left-hand side of a rule; it is <u>right-linear</u> if no variable has more than one occurrence on the right-hand side. We say that a system is <u>linear</u> if it is both left- and right-linear.

For example, the linear system $\{(\alpha+\beta)+\gamma\to\alpha+(\beta+\gamma)\}$ is overlapping since $(\alpha+\beta)+\gamma$ is unifiable with $\alpha+\beta$. The system $\{\alpha\times(\beta+\gamma)\to(\alpha\times\beta)+(\alpha\times\gamma)\}$ is left-linear but not right-linear; the system $\{(\alpha\times\beta)+(\alpha\times\gamma)\to\alpha\times(\beta+\gamma)\}$ is right-linear but not left-linear. Both are non-overlapping.

In the next section we present sufficient methods of proving the termination of term-rewriting systems that are restricted in combinations of the above ways. It is important to note, though, that nontermination is not semi-decidable even if all three conditions are present. This follows from Huet and Lankford's [1978] simulation of a deterministic Turing machine by a non-overlapping and monadic (hence linear) term-rewriting system. In the last section we give sufficient criteria for the union of two terminating term-rewriting systems to be terminating.

Other investigations of some of these classes of systems may be found in Rosen [1973], O'Donnell [1977], Huet [1979], and Huet and Levy [1979]. For a survey of term-rewriting systems and their use, see Huet and Oppen [1980].

II. TERMINATION

In this section we show how derivations of open terms may be used in termination proofs for right-linear or non-overlapping left-linear systems. We first need the following definitions:

<u>Definition</u>: The <u>active</u> area of a term t_i in a derivation $t_1\Rightarrow t_2\Rightarrow\cdots\Rightarrow t_i\Rightarrow\cdots$ is that part of t_i that has been created by the nonvariable portions of the right-hand sides of the rules that have been applied. Only the outermost operator of the initial term t_1 is considered active. More precisely, if a (possibly non-left-linear) rule $\ell\lceil\bar{\alpha}\rceil\to r\lceil\bar{\alpha}\rceil$ is applied to a term $t\lceil\ell\lceil\bar{s}\rceil\rceil$ to obtain $t\lceil r\lceil\bar{s}\rceil\rceil$, then in the latter term all the operators in r are active if the outermost operator of $\ell\lceil\bar{s}\rceil$ was active, those operators in t that were already active remain active, and the operators that were active in any of the previous occurrences of some s_i in $\ell\lceil\bar{s}\rceil$ are active in each new occurrence of s_i in $r\lceil\bar{s}\rceil$. We say that a term is active if its outermost operator is and we identify active operators or terms by underlining them.

Example: In the following derivations for the system

$$\{\ (\alpha+\beta)+\gamma\rightarrow\alpha+(\beta+\gamma),\ (\alpha+\alpha)\rightarrow\alpha\ \}$$

applied to the term

$$t=(((a+b)+c)\underline{+}((a+b)+c)),$$

the active operators have been underlined:

$$t \Rightarrow ((a+b)+c) \Rightarrow (a+(b+c));$$

$$t \Rightarrow ((a+b)\underline{+}(c\underline{+}((a+b)+c))) \Rightarrow (a\underline{+}(b\underline{+}(c\underline{+}((a+b)+c)))) \Rightarrow (a\underline{+}(b\underline{+}(c\underline{+}(a+(b+c)))));$$

$$t \Rightarrow ((a+(b+c))\underline{+}((a+b)+c)) \Rightarrow ((a+(b+c))\underline{+}(a+(b+c))) \Rightarrow (a+(b+c));$$

$$t \Rightarrow (((a+b)+c)\underline{+}(a+(b+c))) \Rightarrow ((a+(b+c))\underline{+}(a+(b+c)))$$
$$\Rightarrow (a\underline{+}((b+c)\underline{+}(a+(b+c)))) \Rightarrow (a\underline{+}(b\underline{+}(c\underline{+}(a+(b+c))))).$$

Definition: A chain $c_1\Rightarrow c_2\Rightarrow\cdots\Rightarrow c_i\Rightarrow\cdots$ of open terms is a derivation in which rules are applied only in the active region of terms. Furthermore, the derivation must be minimal in the sense that it is not an instance of any other chain. (Two chains are considered equal if they can be obtained one from the other by variable renaming.)

Example: Consider the system

$$\{\ -(\alpha+\beta)\rightarrow(-\alpha+-\beta),\ --\alpha\rightarrow\alpha\ \}.$$

The derivation

$$\underline{-}((\alpha+-\beta)+-\gamma) \Rightarrow (\underline{-}(\alpha+-\beta)\underline{+-}\gamma) \Rightarrow ((\underline{-}\alpha\underline{+-}\beta)\underline{+-}\gamma) \Rightarrow ((\underline{-}\alpha\underline{+}\beta)\underline{+-}\gamma) \Rightarrow ((\underline{-}\alpha\underline{+}\beta)\underline{+}\gamma)$$

is a chain for that system; the derivations $\underline{-}(\alpha+--\beta) \Rightarrow \underline{-}(\alpha+\beta) \Rightarrow (\underline{-}\alpha\underline{+-}\beta)$ and $\underline{-}(\alpha+--\beta) \Rightarrow (\underline{-}\alpha\underline{+-}--\beta) \Rightarrow (\underline{-}\alpha\underline{+-}\beta)$ are not.

The set of all chains for a given term-rewriting system S is denoted Ch(S). To inductively generate Ch(S), begin with all possible derivations issuing from the left-hand sides of the rules. If a left-hand side $\ell_i[\bar{\alpha}]$ is unifiable with a (nonvariable) subterm $v[\bar{\beta}]$ of the last term c_n in any chain $c_1[\bar{\beta}]\Rightarrow c_2[\bar{\beta}]\Rightarrow\cdots\Rightarrow c_n[\bar{\beta}]$ already included in Ch(S), then all derivations beginning $c_1[\bar{s}]\Rightarrow c_2[\bar{s}]\Rightarrow\cdots\Rightarrow c_n[\bar{s}]\Rightarrow u[r_i[\bar{\alpha}']]\Rightarrow\cdots$, are also included in Ch(S), where $c_n[\bar{\beta}]=u[v[\bar{\beta}]]$ and $\bar{s}$ are the most general (i.e. least defined) open terms such that $v[\bar{s}]$ is an instance of $\ell_i[\bar{\alpha}]$. This procedure is related to "narrowing", as defined in Slagle [1974].

Example: The term-rewriting system

$$\{\ f(\alpha,g\beta)\rightarrow gf(g\beta,\alpha)\ \}$$

has two chains:

$$f(\alpha,g\beta) \Rightarrow gf(g\beta,\alpha)$$

and

$$f(g\beta,g\beta) \Rightarrow gf(g\beta,g\beta) \Rightarrow ggf(g\beta,g\beta) \Rightarrow \cdots.$$

Example:

$$Ch(\{ff\alpha\rightarrow f\alpha\}) = \{f^{i+1}\alpha\Rightarrow f^i\alpha\Rightarrow\cdots\Rightarrow f\alpha \mid i\geq 1\}.$$

Observation: In general a term-rewriting system need not terminate even if all its chains do. For example, the non-right-linear and overlapping system

$$\{ f(a,b,\alpha)\rightarrow f(\alpha,\alpha,b),\ b\rightarrow a \}$$

has two finite chains:

$$f(a,b,\alpha) \Rightarrow f(\alpha,\alpha,b) \Rightarrow f(\alpha,\alpha,a)$$

and

$$b \Rightarrow a.$$

Nevertheless, the system does not terminate. To wit,

$$f(a,b,b) \Rightarrow f(b,b,b) \Rightarrow f(a,b,b).$$

Similarly, the system

$$\{ f(a,b,\alpha,\alpha)\rightarrow f(\alpha,\alpha,b,b),\ b\rightarrow a \}$$

does not terminate though its chains do.

Theorem: A right-linear term-rewriting system terminates if and only if it has no infinite chains.

Example: The term-rewriting system

$$\{ fh\alpha \rightarrow fgh\alpha \},$$

over terms constructed from integers and the unary operators f, g, and h, is right-linear and has only one chain:

$$fh\alpha \Rightarrow fgh\alpha\ .$$

Since this chain is finite, the system must terminate. The termination of this system could not be proved using "simplification orderings" nor with any "monotonic" well-ordering (see Dershowitz [1979]).

Example: $Ch(\{fg\alpha\rightarrow ggf\alpha\}) = \{ fgg^{i}\alpha\Rightarrow ggfg^{i}\alpha\Rightarrow\cdots\Rightarrow g^{2i}f\alpha \mid i>0 \}$. Since the system is right-linear and all its chains are finite, by the theorem, it must terminate for all inputs.

Example: $Ch(\{fg\alpha\rightarrow ggff\alpha\})$ contains $fg\alpha \Rightarrow ggff\alpha$ and the infinite chains $fggg^{i}\alpha \Rightarrow ggffgg^{i}\alpha \Rightarrow ggfggffg^{i}\alpha \Rightarrow \cdots$ for all $i>0$. Thus, the system does not terminate.

Proof: If the system has an infinite chain, then clearly it has an infinite derivation. For the converse, we show that if a system has an infinite derivation, then there is a similar derivation that contains an instance of some infinite chain.

Consider an infinite derivation $t_1\Rightarrow t_2\Rightarrow\cdots$. There must be some (not necessarily proper) subterm s_1 of t_1 that initiates an infinite derivation $s_1\Rightarrow s_2\Rightarrow\cdots\Rightarrow s_i\Rightarrow s_{i+1}\Rightarrow\cdots$ itself, while no proper subterm of s_1 initiates such an infinite derivation. It follows that a rule is applied to the outermost operator of some s_i in that derivation.

On account of right-linearity, applying a rule at an active operator cannot create any new inactive operators. Thus, in the infinite derivation $s_1\Rightarrow s_2\Rightarrow\cdots$ there are only a finite number of rule applications at inactive operators (each proper subterm of s_1 can initiate only a finite derivation). Those inactive applications can be pushed back to the beginning of the derivation, since if $\underline{s}\lceil r,\cdots,r\rceil \Rightarrow_a \underline{s'}\lceil r\rceil \Rightarrow_i \underline{s'}\lceil r'\rceil$, then that part of the derivation could be replaced with $\underline{s}\lceil r,\cdots,r\rceil \Rightarrow_i \cdots \Rightarrow_i \underline{s}\lceil r',\cdots,r'\rceil \Rightarrow_a \underline{s'}\lceil r'\rceil$, where $\Rightarrow_a$ and $\Rightarrow_i$ denote applications at active and inactive operators, respectively. That is the case because the r in $\underline{s'}\lceil r\rceil$ is inactive, which means that it is not part of s' and could only have influenced the derivation $\underline{s}\lceil r,\cdots,r\rceil \Rightarrow_a \underline{s'}\lceil r\rceil$ by being matched with other subterms in a non-left-linear rule that rewrites $s\lceil \alpha,\cdots,\alpha\rceil$ to $s'\lceil\alpha\rceil$. But if all the corresponding occurrences of r in the other subterms matching α are also rewritten, then $\underline{s}\lceil r',\cdots,r'\rceil \Rightarrow_a \underline{s'}\lceil r'\rceil$ as well. (An active application cannot be below an inactive operator, while if the two applications are not nested, they clearly can be reordered.) After a finite number of such reorderings, from some point on in the reordered infinite derivation there are only active applications. By definition, a derivation with only active applications must be an instance of a chain. □

Example: Consider the system

$$\{\ g\alpha\rightarrow h\alpha,\ f(\alpha,\alpha)\rightarrow f(a,\alpha),\ b\rightarrow a,\ a\rightarrow b\ \}$$

and the infinite derivation

$$gf(b,a) \Rightarrow hf(b,a) \Rightarrow hf(b,b) \Rightarrow hf(a,b) \Rightarrow hf(a,a) \Rightarrow hf(a,a) \Rightarrow \cdots.$$

The subterm f(b,a) of gf(b,a) initiates the infinite derivation

$$\underline{f}(b,a) \Rightarrow_i \underline{f}(b,b) \Rightarrow_a \underline{f}(\underline{a},b) \Rightarrow_i \underline{f}(\underline{a},a) \Rightarrow_a \underline{f}(\underline{a},a) \Rightarrow \cdots.$$

Reordering yields

$$\underline{f}(b,a) \Rightarrow_i \underline{f}(b,b) \Rightarrow_i \underline{f}(b,a) \Rightarrow_i \underline{f}(a,a) \Rightarrow_a \underline{f}(\underline{a},a) \Rightarrow_a \underline{f}(\underline{a},a) \Rightarrow \cdots$$

which contains an instance of the chain

$$\underline{f}(a,a)\Rightarrow\underline{f}(\underline{a},a)\Rightarrow\cdots.$$

Theorem: A non-overlapping left-linear term-rewriting system terminates if and only if it has no infinite chains.

Example: None of the chains of the non-overlapping left-linear system

$$\{\ Dx\rightarrow 1,\ Dy\rightarrow 0,\ D(\alpha+\beta)\rightarrow(D\alpha+D\beta),\ D(\alpha\times\beta)\rightarrow((\beta\times D\alpha)+(\alpha\times D\beta)),\ \alpha+\alpha\rightarrow 2\times\alpha\ \}$$

for differentiation with respect to x have nested D operators. (This can be shown by induction.) Thus, the finiteness of those chains — and consequently the termination of the system -- can be easily proved by considering the multiset of the sizes of the arguments of the D′s. Any rule application reduces that value under the multiset ordering defined in Dershowitz and Manna [1979].

Proof: The proof is similar to the previous one. Let $s_1\Rightarrow s_2\Rightarrow\cdots$ be an infinite derivation such that no proper subterm of s_1 initiates an infinite derivation itself. Applications at inactive operators can be pushed back in the following manner: if $\underline{s}\lceil r\rceil \Rightarrow_a \underline{s'}\lceil r,\cdots,r\rceil \Rightarrow_i \underline{s'}\lceil r',\cdots,r\rceil \Rightarrow\cdots$ is infinite, then $\underline{s}\lceil r\rceil \Rightarrow_i \underline{s}\lceil r'\rceil \Rightarrow_a \underline{s'}\lceil r',\cdots,r'\rceil \Rightarrow\cdots$ is also infinite. This is the case since no subsequent rule application above r can depend on r on account of the left-linear and non-overlapping conditions. This pushing back can only be done a finite number of times, since there is no infinite derivation $r \Rightarrow r' \Rightarrow \cdots$. □

Conjecture: A non-overlapping term-rewriting system terminates if and only if it has no infinite chains. In this case, the reordered derivation may require additional applications of $r \Rightarrow r'$ to ensure that subterms match for a subsequent application of a non-left-linear rule.

The two theorems give necessary and sufficient conditions for a right-linear or non-overlapping left-linear system to terminate. One of the advantages in using chains is that nontermination is more easily detectable, as the next theorem will demonstrate.

Definition: A chain $c_1 \Rightarrow c_2 \Rightarrow \cdots \Rightarrow c_i \Rightarrow \cdots$ cycles if for some $i>1$ c_i has a subterm that is an instance of c_1.

Example: The chain

$$-((\alpha|\beta)|\gamma) \Rightarrow ((--(\alpha|\beta)|--\gamma)|(--(\alpha|\beta)|--\gamma))$$
$$\Rightarrow ((-((--\alpha|--\beta)|(--\alpha|--\beta))|--\gamma)|(--(\alpha|\beta)|--\gamma)) \Rightarrow \ldots$$

cycles, since $-((--\alpha|--\beta)|(--\alpha|--\beta))$ is an instance of $-((\alpha|\beta)|\gamma)$.

Theorem: A right-linear or non-overlapping left-linear term-rewriting system is nonterminating if and only if it has infinitely many noncycling infinite chains or it has a cycling chain.

Example: The system (for the Sheffer stroke) consisting of the non-overlapping left-linear rule

$$-(\alpha|\beta) \rightarrow ((--\alpha|--\beta)|(--\alpha|--\beta))$$

has the cycling chain illustrated above.

Example: The system

$$\{\ fg\alpha \rightarrow fh\alpha,\ hg\alpha \rightarrow gh\alpha,\ ha \rightarrow gga\ \}$$

is linear and non-overlapping. Its chains are all of one of the following forms:

$$fgg^i\alpha \Rightarrow fhg^i\alpha \Rightarrow \cdots \Rightarrow fg^ih\alpha$$
$$fgg^ia \Rightarrow fhg^ia \Rightarrow \cdots \Rightarrow fg^iha \Rightarrow fg^igga \Rightarrow \cdots$$
$$hg^i\alpha \Rightarrow \cdots \Rightarrow g^ih\alpha$$
$$hg^ia \Rightarrow \cdots \Rightarrow g^iha \Rightarrow g^igga.$$

Though none of these chains cycle, those of the second type are infinite.

Proof: The "if" direction is trivial. For the converse, let $c_1 \Rightarrow c_2 \Rightarrow \cdots$ be an infinite chain and consider the infinite derivation $c_2 \Rightarrow c_3 \Rightarrow \cdots$. By the previous theorems, there is some subterm of a c_i, $i>2$, that is an instance of the first term in some chain. (In the previous proofs, the subterm $s[r,\cdots,r]$ of a derivation and the term $s[r',\cdots,r']$ of the reordered derivation are both instances of a term $s[\alpha,\cdots,\alpha]$ in a chain.) Thus, if there are only finitely many infinite chains, one of them must cycle. □

Corollary: The termination of a right-linear or non-overlapping left-linear term-rewriting system is decidable if the number of chains issuing from different initial terms is finite.

Proof: To decide termination, generate all chains until either one cycles or all terminate. □

Example: The non-overlapping left-linear system

$$\{\ f(a,\alpha)\rightarrow f(\alpha,g(\alpha)),\ g(a)\rightarrow a\ \}$$

has three chains:

$$g(a) \Rightarrow a,$$
$$f(a,\alpha) \Rightarrow f(\alpha,g(\alpha)),$$
$$f(a,a) \Rightarrow f(a,g(a)) \Rightarrow f(a,a) \Rightarrow \cdots.$$

Since its third chain cycles, it does not terminate. On the other hand, the system

$$\{\ f(a,\alpha)\rightarrow f(\alpha,g(\alpha)),\ g(a)\rightarrow b\ \}$$

has the chains:

$$g(a) \Rightarrow b,$$
$$f(a,\alpha) \Rightarrow f(\alpha,g(\alpha)),$$
$$f(a,a) \Rightarrow f(a,g(a)) \Rightarrow f(a,b) \Rightarrow f(b,g(b)).$$

Since none of its three chains cycles, it does terminate.

Note: Even if the set of chains is infinite the above theorems can be used to constrain the form of terms that must be considered in a termination proof. For example, the right-linear system $\{-(\alpha+\beta)\rightarrow(-\alpha+-\beta),--\alpha\rightarrow\alpha\}$ has a chain $--\alpha\Rightarrow\alpha$ and all chains of the form $-\Sigma{\sim}\alpha_i\Rightarrow\cdots\Rightarrow\Sigma{-\!\sim}\alpha_i$, where Σs_i denotes any summation of two or more summands s_i, $\sim\alpha$ denotes either α or $-\alpha$, and $-\!\sim\alpha$ is $-\alpha$ or α, respectively. Since chains cannot begin with a term having a minus other than for the outermost or innermost operator, the termination of all chains can be easily proved using a multiset ordering on the sizes of the arguments to the minuses.

Corollary: The termination of a term-rewriting system containing no variables on the right-hand sides is decidable.

Proof: Since there are no variables on the right-hand sides, the system is obviously right-linear. Furthermore, without variables all chains must issue from the rules themselves; thus there can only be as many chains with different initial terms as there are rules. □

Example: $Ch(\{f(\alpha,\alpha)\rightarrow f(a,b),b\rightarrow c\}) = \{b\Rightarrow c, f(\alpha,\alpha)\Rightarrow f(a,b)\Rightarrow f(a,c)\}$. Since the chains do not cycle, the system terminates.

Example: $Ch(\{f(\alpha,\alpha)\rightarrow f(a,b),b\rightarrow a,b\rightarrow c\})$ consists of the two finite chains $b\Rightarrow a$ and $b\Rightarrow c$, the infinite cycling chain $f(\alpha,\alpha)\Rightarrow f(a,b)\Rightarrow f(a,a)\Rightarrow\cdots$, and an infinite number of finite chains $f(\alpha,\alpha)\Rightarrow f(a,b)\Rightarrow f(a,a)\Rightarrow\cdots\Rightarrow f(a,b)\Rightarrow f(a,c)$ with the same initial term.

Corollary (Huet and Lankford [1978]): The termination of a term-rewriting system containing no variables (a ground system) is decidable.

III. COMBINED SYSTEMS

In this section we consider the termination of combinations of term-rewriting systems. If R and S are two terminating systems, we wish to know under what conditions the system R+S, containing all the rules of both R and S, also terminates.

Theorem: Let R and S be two terminating term-rewriting systems over some set of terms T. If R is left-linear, S is right-linear, and there is no overlap between left-hand sides of R and right-hand sides of S, then the combined system R+S also terminates.

Example: $R=\{\alpha\times(\beta+\gamma)\rightarrow(\alpha\times\beta)+(\alpha\times\gamma)\}$ and $S=\{(\alpha\times\alpha)\rightarrow\alpha,(\alpha+\alpha)\rightarrow\alpha\}$ each terminate; therefore R+S also does.

Proof: Assume that R+S is nonterminating and consider an infinite derivation $t_1 \Rightarrow_{R+S} t_2 \Rightarrow_{R+S} \cdots$. Were there only a finite number of applications of R in that derivation, then from some point on there would be an infinite derivation for S alone, contradicting the given fact that S terminates. So we may assume that the derivation contains an infinite number of applications of R. We show that those applications of R can be pushed back to the beginning of the derivation, i.e. that if $t \Rightarrow_S t' \Rightarrow_R t''$, then $t \Rightarrow_R t''' \Rightarrow_{R+S} \cdots \Rightarrow_{R+S} t''$. Thus, were there a derivation for R+S with an infinite number of applications of R, then there would also be an infinite derivation for R alone.

We must consider two cases. In the first case the application of R is below the application of S. On account of the right-linearity of S and the fact that no right-hand side of S overlaps a left-hand side of R, the two applications must be of the form $s[r,\cdots,r] \Rightarrow_S s'[r] \Rightarrow_R s'[r']$. But then $s[r,\cdots,r] \Rightarrow_R s[r',\cdots,r] \Rightarrow_R \cdots \Rightarrow_R s[r',\cdots,r'] \Rightarrow_S s'[r']$ would also be a possible derivation, since the derivation $s[r,\cdots,r] \Rightarrow_S s'[r]$ could not have depended on the form of r, or else s' would overlap r in $s[r,\cdots,r] \Rightarrow s'[r]$.

In the second case the application of R is above that of S. On account of the left-linearity of R and the fact that no left-hand side of R overlaps a right-hand side of S, the two applications must be of the form $r[s] \Rightarrow_S r[s'] \Rightarrow_R r'[s',\cdots,s']$. But then $r[s] \Rightarrow_R r'[s,\cdots,s] \Rightarrow_S \cdots \Rightarrow_S r'[s',\cdots,s']$ would also be a possible derivation, since r does not overlap s'. □

Observation: Each of the three requirements of the above theorem is necessary, as evidenced by the following nonterminating systems R+S:

$$\{f(\alpha,\alpha)\rightarrow f(a,b)\} + \{b\rightarrow a\}$$

has the infinite derivation $f(a,a)\Rightarrow f(a,b)\Rightarrow f(a,a)\Rightarrow\cdots$, though both R and S terminate, R is right-linear, S is linear, and there is no overlap (but R is not left-linear).

$$\{b\rightarrow a\} + \{f(a,b,\alpha)\rightarrow f(\alpha,\alpha,\alpha)\}$$

has the infinite derivation $f(a,b,b)\Rightarrow f(b,b,b)\Rightarrow f(a,b,b)\Rightarrow\cdots$, though both R and S terminate, R is linear, S is left-linear, and there is no overlap (but S is not right-linear).

$$\{b\rightarrow ga\} + \{a\rightarrow gb\}$$

has the infinite derivation $b\Rightarrow ga\Rightarrow ggb\Rightarrow\cdots$, though both R and S terminate and both are linear (but there is overlap).

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