Touchard's Drunkard

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> You're a baby and as stupid as a Frenchman. You persist in thinking that it's the same as it was at Touchard's, and that I'm as stupid as at Touchard's.... But I'm not so silly as I was at Touchard's.... I was drunk yesterday, but not from wine, but because I was excited.
>
> —Fyodor Dostoyevsky, *The Raw Youth* (1875)

Abstract

We give a bijection between N/S/E/W walks and N/S walks that remain on one side of the origin and also a simple derivation of the enumeration of the former, based on Touchard's identity.

1 Introduction: Drunken walks

A drunkard in Nice takes a walk, stepping in cardinal directions, north (N), south (S), east (E), and west (W). In how many possible ways can such walks meander? Let

$$D_n = \begin{cases} \text{the number of walks consisting of } n \text{ steps (N/S/E/W)} \\ \text{beginning at the center of the Promenade des Anglais} \\ \text{(at the southern end of town), ending on the promenade} \\ \text{—all the while remaining on land.} \end{cases}$$

Theorem.

$$D_n = C_{n+1},$$

where C_i is the ith Catalan number, $\frac{1}{i+1}\binom{2i}{i}$ (sequence A000108 in Sloane's *Encyclopedia of Integer Sequences*).

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Guy [1] points out that this fact "is not well known! ... nor can we immediately see any correspondence between [drunken] walks and any of the manifestations [of Catalan objects]."

2 Enumeration: Touchard's identity

Touchard's [4] identity (see, e.g., [2, p. 319]) states:

$$C_{n+1} \;=\; \sum_i C_i 2^{n-2i} \binom{n}{2i}.$$

For a nice proof of this, see [3].

With Touchard's identity, the proof of the above theorem is immediate: C_i counts the patterns consisting of i N-steps and an equal number of S-steps, starting and ending on the promenade, and never venturing further south; 2^{n-2i} counts the patterns of the remaining unconstrained E/W steps; $\binom{n}{2i}$ is the balls-in-bins number of ways of interspersing the two.

3 Bijection: Dyck paths

Walks whose steps are only north or south and stay on land correspond to the well-known Dyck (lattice) paths [2, pp. 151–153]. Whereas the Catalan numbers C_{n+1} count Dyck paths that have a total of $2n+2$ steps of types N/S that meet the requirements that (a) the number of south steps—throughout the walk—does not exceed the number of north ones and (b) that—at the end—they be equal, our enumeration D_n counts the drunken walks with n steps of any type (N/S/E/W) abiding by the identical constraints.

To relate the two kinds of walks, consider a Dyck path of length $2n + 2$. It must start with N and end with S. Forget those two steps. Then start from the beginning and replace as follows: NN $\mapsto$ N, SS $\mapsto$ S, NS $\mapsto$ E, SN $\mapsto$ W. The reverse direction of this bijection is straightforward.

References

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(Concerned with sequence A000108.)