Confluence by Critical Pair Analysis

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Abstract. Knuth and Bendix showed that confluence of a terminating first-order rewrite system can be reduced to the joinability of its finitely many critical pairs. We show that this is still true of a rewrite system $R_T \cup R_{NT}$ such that R_T is terminating and R_{NT} is a left-linear, rank non-increasing, possibly non-terminating rewrite system. Confluence can then be reduced to the joinability of the critical pairs of R_T and to the existence of decreasing diagrams for the critical pairs of R_T inside R_{NT} as well as for the rigid parallel critical pairs of R_{NT}.

1 Introduction

Rewriting is a non-deterministic rule-based mechanism for describing intentional computations. Confluence is the property expressing that the associated extensional relation is functional. It is well-known that confluence of a set of rewrite rules is undecidable. There are two main methods for showing confluence of a binary relation: the first applies to terminating relations [8] and is the basis of the Knuth-Bendix test, reducing confluence to the *joinability* of its so-called *critical pairs* obtained by unifying left-hand sides of rules at subterms [7]. Based on the Hindley-Rosen Lemma, the second applies to non-terminating relations [9] and is the basis of Tait's confluence proof for the pure λ-calculus. Reduction to critical pairs is also possible under strong linearity assumptions [3], although practice favors orthogonal (left-linear, critical pair free) systems for which there are no pairs. It is our ambition to develop a critical-pair criterion capturing both situations together.

Problem. Van Oostrom succeeded in capturing both confluence methods within a single framework thanks to the notion of *decreasing diagram* of a *labelled abstract relation* [12]. In [5], the method is applied to concrete rewrite relations on terms, opening the way to an analysis of non-terminating rewrite relations in terms of the joinability of their critical pairs. The idea is to split the set of rules into a set R_T of terminating rules and a set R_{NT} of non-terminating ones. While left-linearity is required from R_{NT} as shown by simple examples, it is not from R_T. This problem has however escaped efforts so far.

Contributions. We deliver the first true generalization of the Knuth-Bendix test to rewrite systems made of two subsets, R_T of terminating rules and R_{NT} of possibly non-terminating, rank non-increasing, left-linear rules. Confluence is reduced – via decreasing diagrams – to joinability of the finitely many *critical pairs* of rules in R_T within rules in $R_T \cup R_{NT}$ and the finitely many *rigid parallel critical pairs* of rules in R_{NT} within rules in $R_T \cup R_{NT}$. The result is obtained thanks to a new notion, *sub-rewriting*, which appears as the key to glue together many concepts that appeared before in the study of termination and confluence of union systems, namely: caps and aliens, rank non-increasing rewrites, parallel rewriting, decreasing diagrams, stable terms, and constructor-lifting rules. This culminates with the solution of an old open problem raised by Huet who exhibited a critical pair free, non-terminating, non-confluent system [3]. We show that the computation of critical pairs should then involve unification over infinite rational trees, and then, indeed, Huet's example is no longer critical-pair free.

Organization. Sections 4 and 5 are devoted to the main result, its proof, and extension to Huet's open problem. Relevant literature is analyzed in Sect. 6.

2 Term Algebras

Given a *signature* $\mathcal{F}$ of *function symbols* and a denumerable set $\mathcal{X}$ of *variables*, $\mathcal{T}(\mathcal{F},\mathcal{X})$ denotes the set of *terms* built up from $\mathcal{F}$ and $\mathcal{X}$. Terms are identified with finite labelled trees as usual. *Positions* are strings of positive integers, identifying the empty string Λ with the root position. We use "$\cdot$" for concatenation of positions, or sets thereof. We assume a set of variables $\mathcal{Y}$ disjoint from $\mathcal{X}$ and a bijective mapping ξ from the set of positions to $\mathcal{Y}$. We use $\mathcal{FP}os(t)$ to denote the set of non-variable positions of t, $t(p)$ for the function symbol at position p in t, $t|_p$ for the *subterm* of t at position p, and $t[u]_p$ for the result of replacing $t|_p$ with u at position p in t. We may omit the position p, writing $t[u]$ for simplicity and calling $t[\cdot]$ a *context*. We use $\geq$ for the partial order on positions (further from the root is bigger), $p\#q$ for incomparable positions p, q, called *disjoint*. The order on positions is extended to sets as follows: $P \geq Q$ (resp. $P > Q$) if $(\forall p \in P)(\exists q \in max(Q))\, p \geq q$ (resp. $p > q$), where $max(P)$ is the set of maximal positions in P. We use p for the singleton set $\{p\}$. We write $u[v_1, \ldots, v_n]_Q$ for $u[v_1]_{q_1} \ldots [v_n]_{q_n}$ if $Q = \{q_i\}_1^n$. By $\mathcal{V}ar(t)$ we mean the set of variables occurring in t. We say that t is *linear* if no variable occurs more than once in t.

Substitutions are mappings from variables to terms, called *variable substitutions* when mapping variables onto variables, and *variable renamings* when also bijective. We denote by $\sigma_{|X}$ the restriction of σ to a subset X of variables. We use Greek letters for substitutions and postfix notation for their application. The strict *subsumption order* $\gtrdot$ on terms (resp. substitutions) associated with the quasi-order $s \,\underline{\gtrdot}\, t$ (resp. $\sigma \,\underline{\gtrdot}\, \tau$) iff $s = t\theta$ (resp. $\sigma = \tau\theta$) for some substitution θ, is well-founded. Given terms s, t, computing the substitution σ whenever it exists such that $t = s\sigma$ (resp. $t\sigma = s\sigma$) is called *matching* (resp. unification) and σ is called a *match* (resp. *unifier*). Two unifiable terms s, t have a unique (up to variable renaming) *most general unifier* $mgu(s, t)$, which is the smallest with respect to subsumption. The result remains true when unifying terms $s, t_1, \ldots, t_n$ at a set of disjoint positions $\{p_i\}_1^n$ such that $s|_{p_1}\sigma = t_1\sigma \wedge \ldots \wedge s|_{p_n}\sigma = t_n\sigma$, of which the previous result is a particular case when $n = 1$ and $p_1 = \Lambda$.

Given $F \subseteq \mathcal{F}$, a term t is F-headed if $t(\Lambda) \in F$. The notion extends to substitutions.

3 Rewriting

Our goal is to reduce the Church-Rosser property of the union of a terminating rewrite relation R_T and a non-terminating relation R_{NT} to that of finitely many critical pairs. The particular case where R_{NT} is empty was carried out by Knuth and Bendix and is based on Newman's result stating that a terminating relation is Church-Rosser provided its local peaks are joinable. The other particular case, where R_T is empty, was considered by Huet and is based on Hindley's result stating that a (non-terminating) relation is Church-Rosser provided its local peaks are joinable in at most one step from each side. The general case requires using both, which has been made possible by van Oostrom, who introduced labelled relations and decreasing diagrams to replace joinability.

Definition 1. *A* rewrite rule *is a pair of terms, written $l \rightarrow r$, whose* left-hand side *l is not a variable and whose* right-hand side *r satisfies $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. A rewrite system R is a set of rewrite rules. A rewrite system is* left-linear *(resp.* linear*) if for every rule $l \rightarrow r$, the left-hand side l is a linear term (resp. l and r are linear terms).*

Definition 2. *A term u* rewrites in parallel *to v* at *a set $P = \{p_i\}_1^n$ of pairwise disjoint positions, written $u \Rightarrow^P_{l\rightarrow r} v$, if $(\forall p_i \in P)\, u|_{p_i} = l\sigma_i$ and $v = u[r\sigma_1, \ldots, r\sigma_n]_P$. The term $l\sigma_i$ is a* redex. *We may omit P or replace it by a property that it satisfies.*

We call our notion of parallel rewriting *rigid*. It departs from the literature [3,1] by imposing the use of a *single* rule. Rewriting extends naturally to lists of terms of the same length, hence to substitutions of the same domain. Rewriting *terminates* if there exists no infinite sequence of rewriting issuing from an arbitrary term.

Plain rewriting is obtained as the particular case of parallel rewriting when $n = 1$. We then also write $u \rightarrow^p_{l\rightarrow r} v$. As a consequence, most of the following definitions will be given for parallel rewriting, while also applying to plain rewriting.

Consider two parallel rewrites issuing from the same term u with possibly different rules, say $u \Rightarrow^P_{l\rightarrow r} v$ and $u \Rightarrow^Q_{g\rightarrow d} w$. Following Huet [3], we distinguish three cases,

$P\#Q$, that is, $(\forall p \in P\ \forall q \in Q)\, p\#q$, (disjoint case)
$P = \{p\}, Q > p \cdot \mathcal{FP}os(l)$, (ancestor case)
$P = \{p\}, Q \subseteq p \cdot \mathcal{FP}os(l)$, (critical case)

all other cases being a combination of the above three.

Definition 3 (Rigid parallel critical pairs). *Given a rule $l \rightarrow r$, a set $P = \{p_i \in \mathcal{FP}os(l)\}_1^n$ of disjoint positions and n copies $\{g_i \rightarrow d_i\}_1^n$ of a rule $g \rightarrow d$ sharing no variable among themselves nor with $l \rightarrow r$, such that σ is a most general unifier of the terms $l, g_1, \ldots, g_n$ at P. Then $l\sigma$ is the* overlap *and $\langle r\sigma,\, l\sigma[d_1\sigma, \ldots, d_n\sigma]_P\rangle$ the* rigid (parallel) critical pair *of $\{g_i \rightarrow d_i\}_1^n$ on $l \rightarrow r$ at P (a* critical pair *if $n = 1$).*

Definition 4. *A* labelled rewrite relation *is a pair made of a rewrite relation $\rightarrow$ and a mapping from rewrite steps to a set of labels $\mathcal{L}$ equipped with a partial quasi-order $\trianglerighteq$ whose strict part $\triangleright$ is well-founded. We write $u \Rightarrow^{P,m}_R v$ for a parallel rewrite step from u to v at positions P with label m and rewrite system R. Indexes P, m, R may be omitted. We also write $\alpha \triangleright l$ (resp. $l \triangleright \alpha$) if $m \triangleright l$ (resp. $l \triangleright m$) for all m in the multiset α.*

Given an arbitrary (possibly labelled) rewrite step $\rightarrow^l$, we denote its projection on terms by $\rightarrow$, its inverse by ${}^l\!\leftarrow$, its reflexive closure by $\Rrightarrow^l$, its symmetric closure by $\longleftrightarrow^l$, its reflexive and transitive closure by $\twoheadrightarrow^\alpha$ for some word α on the alphabet of labels, and its reflexive, symmetric, transitive closure, called *conversion*, by $\twoheadleftarrow\!\!\twoheadrightarrow^\alpha$.

We sometimes consider the word α to be a multiset. Given u, $\{v \mid u \twoheadrightarrow v\}$ is the set of reducts of u. We say that a reduct of u is *reachable* from u.

The triple v, u, w is said to be a *local peak* if $v \,{}^{l}\!\leftarrow u \rightarrow^{m} w$, a *peak* if $v \,{}^{\alpha}\!\twoheadleftarrow u \twoheadrightarrow^{\beta} w$, a *joinability diagram* if $v \twoheadrightarrow^{\alpha} u \,{}^{\beta}\!\twoheadleftarrow w$. The local peak $v \,{}^{p,m}_{l\rightarrow r}\!\leftarrow u \rightarrow^{q,n}_{g\rightarrow d} w$ is a *disjoint, critical, ancestor* local peak if $p \# q$, $q \in p \cdot \mathcal{FP}os(l)$, $q > p \cdot \mathcal{FP}os(l)$, respectively. The pair v, w is *convertible* if $v \twoheadleftarrow\!\!\twoheadrightarrow^{\alpha} w$, *divergent* if $v \,{}^{\alpha}\!\twoheadleftarrow u \twoheadrightarrow^{\beta} w$ for some u, and *joinable* if $v \twoheadrightarrow^{\alpha} t \,{}^{\beta}\!\twoheadleftarrow w$ for some t. The relation $\rightarrow$ is *locally confluent* (resp. *confluent*, *Church-Rosser*) if every local peak (resp. divergent pair, convertible pair) is joinable.

Decreasing Diagrams. Given a rewrite relation $\rightarrow$ on terms, we first consider specific conversions made of a local peak and an associated conversion called a *local diagram* and recall the important subclass of van Oostrom's decreasing diagrams and their main property: a relation all whose local diagrams are decreasing enjoys the Church-Rosser property, hence confluence. Decreasing diagrams were introduced in [12], where it is shown that they imply confluence. Van Oostrom's most general form of decreasing diagrams is discussed in [5].

Definition 5 (Local diagrams). *A* local diagram D *is a conversion made of a* local peak $D_{peak} = v \leftarrow u \rightarrow w$ *and a* conversion $D_{conv} = v \twoheadleftarrow\!\!\twoheadrightarrow u$*. We call* diagram rewriting *the rewrite relation $\Rightarrow_{\mathcal{D}}$ on conversions associated with a set $\mathcal{D}$ of local diagrams, in which a local peak is replaced by one of its associated conversions:*

$$P\ D_{peak}\ Q \Rightarrow_{\mathcal{D}} P\ D_{conv}\ Q \text{ for some } D \in \mathcal{D}$$

Definition 6 (Decreasing diagrams [12]). *A local diagram D with peak $v \,{}^{l}\!\leftarrow u \rightarrow^{m} w$ is* decreasing *if $D_{conv} = v \twoheadrightarrow^{\alpha} s \Rrightarrow^{m} s' \twoheadrightarrow^{\delta} \twoheadleftarrow^{\delta'} t' \Lleftarrow^{l} t \twoheadleftarrow^{\beta} w$, with labels in α (resp. β) strictly smaller than l (resp. m), and labels in δ, δ' strictly smaller than l or m. The rewrites $v \twoheadrightarrow^{\alpha} s$ and $t \twoheadleftarrow^{\beta} w$, $s \Rrightarrow^{m} s'$ and $t' \Lleftarrow^{l} t$, $s' \twoheadrightarrow^{\delta} \twoheadleftarrow^{\delta'} t'$ are called the* side steps, facing steps, *and* middle steps *of the diagram, respectively. A decreasing diagram D is* stable *if $C[D\gamma]$ is decreasing for arbitrary context $C[\cdot]$ and substitution γ.*

Theorem 1 ([5]). *The relation $\Rightarrow_{\mathcal{D}}$ terminates for any set $\mathcal{D}$ of decreasing diagrams.*

Corollary 1. *Assume that $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $\mathcal{D}$ is a set of decreasing diagrams in T such that T is closed under $\Rightarrow_{\mathcal{D}}$. Then the restriction of $\rightarrow$ to T is Church-Rosser if every local peak in T has a decreasing diagram in $\mathcal{D}$.*

This simple corollary of Theorem 1 implies van Oostrom decreasing diagram theorem by taking $T = \mathcal{T}(\mathcal{F}, \mathcal{X})$. With a different choice of the set T, it will be the basis of our main Church-Rosser result to come.

Layering. From now on, we assume two signatures F_T and F_{NT} satisfying
(A1) $F_T \cap F_{NT} = \varnothing$.
and proceed by slicing terms into homogeneous subparts, following definitions in [4].

Definition 7. *A term $s \in \mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$ is* homogeneous *if it belongs to $\mathcal{T}(F_T, \mathcal{X})$ or to $\mathcal{T}(F_{NT}, \mathcal{X})$; otherwise it is* heterogeneous*.*

Thanks to assumption (A1), a heterogeneous term can be uniquely decomposed (w.r.t. $\mathcal{Y}$ and ξ introduced in Section 2) into a topmost homogeneous part, its *cap*, and a multiset of remaining subterms, its *aliens*, headed by symbols of the other signature.

Definition 8 (Cap, aliens). *Let $t \in \mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$. An* alien *of t is a maximal non-variable subterm of t whose head does not belong to the signature of t's head. We use $\mathcal{APos}(t)$ for its set of pairwise disjoint* alien positions*, $\mathcal{A}(t)$ for its list of aliens from left to right, and $\mathcal{CPos}(t) = \{p \in \mathcal{Pos}(t) \mid p \not\geq \mathcal{APos}(t)\}$ for its set of* cap positions*. We define the* cap $\bar{t}$ *and* alien substitution $\overline{\gamma}_t$ *of t as follows: (i) $\mathcal{Pos}(\bar{t}) = \mathcal{CPos}(t) \cup \mathcal{APos}(t)$; (ii) $(\forall p \in \mathcal{CPos}(t))$, $\bar{t}(p) = t(p)$; (iii) $(\forall p \in \mathcal{APos}(t))$, $\bar{t}(p) = \xi(p)$ and $\overline{\gamma}_t(\xi(p)) = t|_p$. The* rank *of t, denoted $rk(t)$, is 1 plus the maximal rank of its aliens.*

Fact. Given $t \in \mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$, then $t = \bar{t}\overline{\gamma}_t$.

Example 1. Let $F_T = \{G\}, F_{NT} = \{F, 0, 1\}, t = F(G(0,1,1), G(0,1,x), G(0,1,1))$. Then t has cap $F(y_1, y_2, y_3)$ and aliens $G(0,1,1)$ and $G(0,1,x)$. $G(0,1,1)$ has cap $G(y_1, y_2, y_3)$ and homogeneous aliens 0 and 1, while $G(0,1,x)$ has cap $G(y_1, y_2, x)$ and same set of homogeneous aliens. Hence, the rank of t is 3.

4 From Church-Rosser to Critical Pairs

Definition 9. *A rewrite rule $l \to r$ is* rank non-increasing *iff for all rewrites $u \to_{l \to r} v$, $rk(u) \geq rk(v)$. A rewrite system is rank non-increasing iff all its rules are.*

From now on, we assume we are given two rewrite systems R_T and R_{NT} satisfying:
(A2) R_T is a terminating rewrite system in $\mathcal{T}(F_T, \mathcal{X})$;
(A3) R_{NT} is a set of rank non-increasing, left-linear rules $f(\boldsymbol{s}) \to g(\boldsymbol{t})$ s.t. $f, g \in F_{NT}$, $\boldsymbol{s}, \boldsymbol{t} \in \mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$;
(A4) if $g \to d \in R_T$ overlaps $l \to r \in R_{NT}$ at $p \in \mathcal{FPos}(l)$, then $l|_p \in \mathcal{T}(F_T, \mathcal{X})$.

Our goal is to show that $R_T \cup R_{NT}$ is Church-Rosser provided its critical pairs have appropriate decreasing diagrams.

Strategy. Since R_T and R_{NT} are both rank non-increasing, by assumption for the latter and homogeneity assumption of its rules for the former, we shall prove our result by induction on the rank of terms. To this end, we introduce the set $\mathcal{T}_n(F_T \cup F_{NT}, \mathcal{X})$ of terms of rank at most n. Since rewriting is rank non-increasing, $\mathcal{T}_n(F_T \cup F_{NT}, \mathcal{X})$ is closed under diagram rewriting. This is why we adopted this restricted form of decreasing diagrams rather than the more general form studied in [5].

We say that two terms in $\mathcal{T}_n(F_T \cup F_{NT}, \mathcal{X})$ are *n-($R_T \cup R_{NT}$)-convertible* (in short, *n-convertible*) if their conversion involves terms in $\mathcal{T}_n(F_T \cup F_{NT}, \mathcal{X})$ only. We shall assume that n-($R_T \cup R_{NT}$)-convertible terms are joinable, and show that $(n+1)$-($R_T \cup R_{NT}$)-convertible terms are joinable as well by exhibiting decreasing diagrams for all their local peaks, using Corollary 1.

Since R_{NT} may have non-linear right-hand sides, we classically use parallel rewriting with R_{NT} rules to enable the existence of decreasing diagrams for ancestor peaks in case R_{NT} is below R_{NT}. The main difficulty, however, has to do with ancestor peaks $v \; {}^{q}_{R_{NT}}\!\!\leftarrow u \to^{p}_{R_T} w$ for which R_{NT} is below R_T. Due to non-left-linearity of the rules in R_T, the classical diagram for such peaks, $v \twoheadrightarrow_{R_{NT}} s \to^{p}_{R_T} t \twoheadleftarrow_{R_{NT}} w$, can hardly be made decreasing in case $s \to^{p}_{R_T} t$ must be a facing step and $v \twoheadrightarrow_{R_{NT}} s$ side steps with labels identical to that of the top R_{NT}-step. A way out is to group them together as a single facing step from v to t. To this end, we introduce a specific rewriting relation:

Definition 10 (Sub-rewriting). *A term u* sub-rewrites *to v at $p \in \mathcal{P}os(u)$ with $l \rightarrow r$ in R_T, written $u \rightarrow^{p}_{R_{T_{sub}}} v$ if the following conditions hold: (i) $\mathcal{FP}os(l) \subseteq \mathcal{CP}os(u|_p)$; (ii) $u\ (\rightarrow^{\geq p \cdot \mathcal{AP}os(u|_p)}_{R_T \cup R_{NT}})^*\ w = u[l\sigma]_p$; (iii) $v = u[r\sigma]_p$.*

Condition (ii) allows *arbitrary rewriting* in $\mathcal{A}(u|_p)$ until an R_T-redex is obtained. Thanks to assumptions (A1–3), these aliens remain aliens along the derivation from u to w, implying (i). Condition (i) will however be needed later when relaxing assumptions (A1) and (A3). Note also that the cap of $w|_p$ may collapse in the last step, in which case $v|_p$ becomes F_{NT}-headed.

A Hierarchy of Decompositions. Sub-rewriting needs another notion of cap for F_T-headed terms. Let ζ_n be a bijective mapping from $\mathcal{Y} \cup \mathcal{X}$ to n-($R_T \cup R_{NT}$)-convertibility classes of terms in $\mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$, which is the identity on $\mathcal{X}$. The rank of a term being at least one, 0-($R_T \cup R_{NT}$)-convertibility does not identify any two different terms; hence ζ_0 is a bijection from $\mathcal{Y} \cup \mathcal{X}$ to $\mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$. Similarly we denote by ζ_∞ a bijective mapping from $\mathcal{Y} \cup \mathcal{X}$ to ($R_T \cup R_{NT}$)-convertibility classes, abbreviated as ζ.

Definition 11 (Hat). *The* hat at rank n *of a term $t \in \mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$ is the term $\widehat{t}^n$ defined as: if t is F_{NT}-headed, $\widehat{t}^n = \zeta_n^{-1}(t)$; otherwise, $(\forall p \in \mathcal{CP}os(t))\ \widehat{t}^n(p) = \overline{t}(p)$ and $(\forall p \in \mathcal{AP}os(t))\ \widehat{t}^n(p) = \zeta_n^{-1}(t|_p)$.*

Since n-($R_T \cup R_{NT}$)-convertibility is an infinite hierarchy of equivalences identifying more and more terms, given t, $\widehat{t}^n$ is an infinite sequence of terms, each of them being an instance of the previous one, which is stable from some index n_t. We use $\widehat{t}$ for $\widehat{t}^\infty$.

Lemma 1. *Let $t \in \mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$ and $m \geq n \geq 0$. Then $\widehat{t} \trianglerighteq \widehat{t}^m \trianglerighteq \widehat{t}^n \trianglerighteq \overline{t}$.*

The associated variable substitution from $\widehat{t}^n$ to $\widehat{t}^m$ is $\xi_{n,m}$, omitting m when infinite.

Note that $\xi_{n,m}$ does not actually depend on the term t, but only on the m- and n-convertibility classes. Also, $\widehat{t}^0$ corresponds to the case where identical terms only are identified by ζ_0^{-1}, while $\widehat{t}$ corresponds to the case where any two ($R_T \cup R_{NT}$)-convertible terms are identified by ζ^{-1}. In the literature, $\widehat{t}^0$ is usually called a hat (or a cap!).

Example 2. Let $F_{NT} = \{F\}$, $F_T = \{G, 0, 1\}$ and $R_T = \{1 \rightarrow 0\}$. Then, $G(F(1, 0, x), F(1, 0, x), 1) \rightarrow^{2 \cdot 1}_{1 \rightarrow 0} G(F(1, 0, x), F(0, 0, x), 1)$. 0-hats of these terms are $G(y, y, 1)$ and $G(y, y', 1)$, respectively. Their 1-hats are the same as their 0-hats, since their aliens have rank 2, hence cannot be 1-convertible. On the other hand, their $(i \geq 2)$-hats are $G(y, y, 1)$ and $G(y, y, 1)$, since $F(1, 0, x)$ and $F(0, 0, x)$ are 2-convertible.

The following lemmas are standard, with $\zeta_t = \zeta_{0|\mathcal{V}ar(\widehat{t}^0)}$.

Lemma 2. *Let $t \in \mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$. Then $t = \widehat{t}^0 \zeta_t$.*

Lemma 3. *Let $u \rightarrow^{p}_{R_T} v$, $p \in \mathcal{CP}os(u)$. Then $\widehat{u}^0 \rightarrow^{p}_{R_T} \widehat{v}^0$ and $(\forall y \in \mathcal{V}ar(\widehat{v}^0))\ \zeta_u(y) = \zeta_v(y)$.*

Lemma 4. *Let $u(\Lambda) \in F_T$ and $u \rightarrow^{p}_{R_T \cup R_{NT}} v$ at $p \geq \mathcal{AP}os(u)$. Then $\mathcal{CP}os(u) = \mathcal{CP}os(v)$, $(\forall q \in \mathcal{CP}os(u))\ u(q) = v(q)$, $\mathcal{AP}os(u) = \mathcal{AP}os(v)$, $(\forall q \in \mathcal{AP}os(u))\ u|_q \Rrightarrow_{R_T \cup R_{NT}} v|_q$.*

Key properties of sub-rewriting are the following:

Lemma 5. *Let u be an F_T-headed term of rank $n+1$ s.t. $u \rightarrow^{\geq \mathcal{AP}os(u)}_{R_T \cup R_{NT}} v$. Then, $(\forall i \geq n) \widehat{u}^i = \widehat{v}^i$.*

Proof. Rules in R_{NT} being F_{NT}-headed, $\mathcal{AP}os(u) = \mathcal{AP}os(v)$, and rewriting in aliens does not change $\mathcal{CP}os(u)$. It does not change $(i \geq n)$-convertibility either, hence the statement. □

Lemma 6. *Let u of rank $n{+}1$, $p \in \mathcal{CP}os(u)$, and $u \rightarrow^p_{R_{T_{sub}}} v$. Then, $(\forall i \geq n)\, \widehat{u}^i \rightarrow^p_{R_T} \widehat{v}^i$.*

Proof. By definition of sub-rewriting, we get $u(\rightarrow^{\geq \mathcal{AP}os(u)}_{R_T \cup R_{NT}})^* w \rightarrow^p_{l \rightarrow r \in R_T} v$, therefore $w|_p = l\sigma$ for some substitution σ and $v = w[r\sigma]_p$. Let $i \geq n$.

By Lemma 3, $\widehat{w}^0 \rightarrow^p_{l \rightarrow r} \widehat{v}^0$. By repeated applications of Lemma 4, $\mathcal{CP}os(u) = \mathcal{CP}os(w)$, $(\forall q \in \mathcal{CP}os(u))\, u(q) = w(q)$, and $\mathcal{A}(u)$ rewrites to $\mathcal{A}(w)$; hence aliens in $\mathcal{A}(u)$ are n-convertible iff the corresponding aliens in $\mathcal{A}(w)$ are n-convertible. By definition 11, we get $\widehat{u}^n = \widehat{w}^n$.

Putting things together, $\widehat{u}^i = \widehat{u}^n \xi_{n,i} = \widehat{w}^n \xi_{n,i} = \widehat{w}^0 \xi_{0,n} \xi_{n,i} \rightarrow \widehat{v}^0 \xi_{0,n} \xi_{n,i} = \widehat{v}^i$. □

Definition 12 (Rewrite root). *The* root *of a rewrite $u \rightarrow^p_{R_{T_{sub}}} v$ is the minimal position, written $\widehat{p}$, such that $(\forall q : p \geq q \geq \widehat{p})\, u(q) \in F_T$.*

Note that $u|_p$ is a subterm of $u|_{\widehat{p}}$. By monotony of rewriting:

Corollary 2. *Let $u \rightarrow^p_{R_{T_{sub}}} v$. Then $\widehat{u|_{\widehat{p}}} \rightarrow_{R_T} \widehat{v|_{\widehat{p}}}$.*

Main Result. We assume from here on that rules are indexed, those in R_T by 0, and those in R_{NT} by (non-zero) natural numbers, making R_{NT} into a disjoint union $\{R_i\}_{i \in I}$ where $I \subseteq i > 0$. Having a strictly smaller index for R_T rules is no harm nor necessity.

Our relations, parallel rewriting with R_{NT} and sub-rewriting with R_T, are labelled by triples made of the rank of the rewritten term first, the index of the rule used, and – approximately – the hat of the considered redex, ordered by the well-founded order $\rhd := (>, >, \rightarrow^+_{R_T})_{lex}$. More precisely,

$u \Rightarrow^P_{R_{i>0}} v$ is given label $\langle k, i, \_ \rangle$, where $k = max\{rk(u|_{p_i})\}_{p_i \in P}$;

$u \rightarrow^q_{R_{T_{sub}}} v$ is given label $\langle k, 0, \widehat{u|_{q'}} \rangle$, where $k = rk(u|_q)$ and q' is the root $\widehat{q}$ of q.

The third component of an R_{NT}-rewrite is never used. Decreasing diagrams for critical pairs need be stable and satisfy a *variable condition* introduced by Toyama, see also [1]:

Definition 13. *The R_{NT} rigid critical peak $v \leftarrow^{\Lambda} u \Rightarrow^{Q} w$ (resp. rigid critical pair (v, w)) is* naturally decreasing *if it has a stable decreasing diagram in which:*
(i) step $s \Rightarrow^{Q'} s'$ facing $u \Rightarrow w$ uses the same rule and satisfies $\mathcal{V}ar(s'|_{Q'}) \subseteq \mathcal{V}ar(u|_Q)$;
(ii) step $t \Rightarrow t'$ facing $u \rightarrow v$ uses the same rule.

Note the variable condition is automatically satisfied for an overlapping at the root.

Definition 14. *The R_{NT}-R_T critical peak $v \leftarrow^{\Lambda}_{R_{NT}} u \rightarrow^q_{R_T} w$ (resp. critical pair (v, w)) is* naturally decreasing *if it has a stable decreasing diagram whose step $t \Rrightarrow^P t'$ facing $u \rightarrow v$ uses the same rule.*

Theorem 2 **(Church-Rosser unions).** *A rewrite union $R_T \cup R_{NT}$ satisfying: (A1–4), R_{NT}-R_T critical pairs are naturally decreasing, R_{NT} rigid critical pairs are naturally decreasing, is Church-Rosser iff its R_T critical pairs are joinable in R_T.*

Proof. While the "only if" direction is trivial, we are going to prove the "if" direction.

Since $\rightarrow_{R_T \cup R_{NT}} \subseteq \rightarrow_{R_{T_{sub}}} \cup \Rightarrow_{R_{NT}}$ and $(\rightarrow_{R_{T_{sub}}} \cup \Rightarrow_{R_{NT}})^* = (\rightarrow_{R_T \cup R_{NT}})^*$, $R_T \cup R_{NT}$ is Church-Rosser iff $\rightarrow_{R_{T_{sub}}} \cup \Rightarrow_{R_{NT}}$ is. By induction on the rank, we therefore show that every local peak $v \; ({}_{R_{T_{sub}}}\!\!\leftarrow \cup \Leftarrow_{R_{NT}}) \; u \; (\rightarrow_{R_{T_{sub}}} \cup \Rightarrow_{R_{NT}}) \; w$, where $rk(u) = n+1$, enjoys a decreasing diagram, implying confluence on terms of rank $n+1$ by Corollary 1.

The proof is divided into three parts according to the considered local peak. Each key case is described by a picture to ease the reading, in which $\rightarrow$, $\rightarrow$ and $\rightarrow$ are used for plain steps with R_T, $R_{T_{sub}}$ and $R_T \cup R_{NT}$, respectively, while $\rightarrow$ is used for parallel (sometimes plain) steps with R_{NT}. Every omitted case is symmetric to some considered case, or is easily solved by induction in case all rewrites take place in the aliens of u.

1) Consider a local peak $v \Leftarrow^{P,\langle k,i,\_\rangle}_{R_{NT}} u \Rightarrow^{Q,\langle m,j,\_\rangle}_{R_{NT}} w$. Following [1], we carry out first the particular case of a root peak, for which a rule $l \rightarrow r \in R_i$ applies at the root of u

(a) Root case. Although our labelling technique is different from [1], with ranks playing a prominent role here, the proof can be adapted without difficulty, as described in Fig. 1. Let $Q_1 := \{q \in Q \mid q \in \mathcal{FP}os(l)\}$. We first split the parallel rewrite from u to w into two successive parallel steps, at positions in Q_1 first, then at positions in $Q_2 = Q \setminus Q_1$. Note that the peak is specialized into ancestor peak when $Q_1 = \varnothing$. The inner part of the figure uses the fact that l unifies at Q_1 with some R_{NT} rule, yielding a rigid critical peak (v', u', w') of which the peak $(v, u, w'\sigma)$ is a σ-instance. By assumption, (v', w') has a stable diagram which is instantiated by σ in the figure. Since $Q_1 \cup Q_2$ are pairwise disjoint positions and $Q_2 > \mathcal{FP}os(w')$, by left-linearity of R_{NT}, $w'\sigma \Rightarrow^{Q_2}_{R_j} w'\sigma' = w$. Now, we can push that parallel rewrite from $w'\sigma$ to $s'\sigma$ as indicated, using stability and monotony of rewriting, thereby making ancestor redexes commute.

Finally, Toyama's variable condition ensures that Q'_1 and Q'_2 are disjoint sets of positions; hence $s\sigma$ rewrites to $s'\sigma'$ in one parallel step with the same j-rule as $u \Rightarrow w$. The obtained diagram is decreasing as a consequence of stability of the rigid critical pair diagram and rank non-increasingness of rewrites.

(b) For the general case, we proceed again as in [1]. For every position $p \in min(P \cup Q)$, the peak $v \Leftarrow^{P,\langle k,i,\_\rangle}_{R_{NT}} u \Rightarrow^{Q,\langle m,j,\_\rangle}_{R_{NT}} w$ induces a root-peak $v|_p \Leftarrow^{P',\langle k',i,\_\rangle}_{R_{NT}} u|_p \Rightarrow^{Q',\langle m',j,\_\rangle}_{R_{NT}} w|_p$. As just shown, root-peaks have decreasing diagrams; hence, for each p, we have a decreasing diagram between $v|_p$ and $w|_p$. Notice that in the decreasing diagram we have shown, each facing step – if it exists – uses the same rule as that one it faces. Since positions in $min(P \cup Q)$ are pairwise disjoint, these decreasing diagrams combine into a single decreasing diagram: in particular, the facing steps $\Rightarrow^{\langle m',j,\_\rangle}_{R_{NT}}$ (resp. $\Leftarrow^{\langle k',i,\_\rangle}_{R_{NT}}$) yield the facing step $\Rightarrow^{\langle m,j,\_\rangle}_{R_{NT}}$ (resp. $\Leftarrow^{\langle k,i,\_\rangle}_{R_{NT}}$).

2) Consider a local peak $v \; {}^{p,\langle k,0,\widehat{u|_{\hat{p}}}\rangle}_{R_{T_{sub}}}\!\!\leftarrow u \rightarrow^{q,\langle m,0,\widehat{u|_{\hat{q}}}\rangle}_{R_{T_{sub}}} w$. We denote by $l \rightarrow r$ and $g \rightarrow d$ the R_T-rules applied from u to v at p and u to w at q, respectively. We discuss cases depending on $p, \hat{p}, q, \hat{q}$, instead of only p, q as usual.

(a) Disjoint case: $p \# q$. The usual commutation lemma yields $v \rightarrow^{q,\langle m,0,\widehat{v|_{\hat{q}}}\rangle}_{R_{T_{sub}}} t \; {}^{p,\langle k,0,\widehat{w|_{\hat{p}}}\rangle}_{R_{T_{sub}}}\!\!\leftarrow w$ for some t. It is decreasing easily by Corollary 2 or Lemma 5, decided by $\widehat{p}, \widehat{q}$.

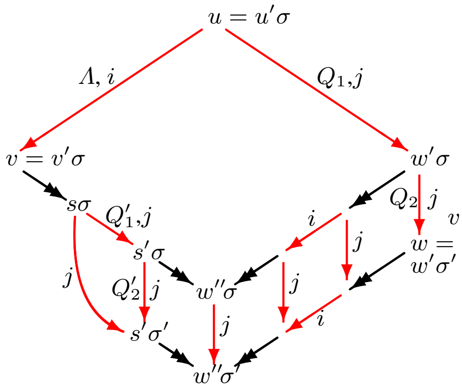


Fig. 1. R_{NT} root peak

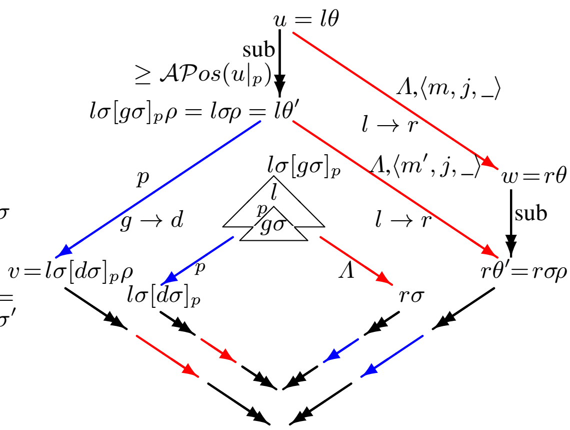


Fig. 2. R_{NT} above R_T critical peak

(b) Root ancestor case: $\hat{q} > p$. By Definition 12, $m < k$; hence $q \geq \mathcal{AP}os(u|_{\hat{p}})$. This case is thus similar to the R_T above R_{NT} ancestor case considered later, pictured at Fig. 4.

(c) Ancestor case: $\hat{q} = \hat{p}$; hence $k = m$, with $q > p \cdot \mathcal{FP}os(l)$. This is the usual ancestor case, within a given layer. The proof is depicted in Fig. 3, simplified by taking $p = \Lambda$.

Using Definition 8 and Lemma 2, then, by Definition 10, the rewrite from $u = \overline{u}\,\overline{\gamma}_u$, to $v = \widehat{v}^0\zeta_v$ (resp. $w = \widehat{w}^0\zeta_w$) factors out through $v' = \widehat{v'}^0\zeta_{v'}$ (resp., $w' = \widehat{w'}^0\zeta_{w'}$). By Lemma 3, ζ_v and $\zeta_{v'}$ coincide on $\mathcal{V}ar(\widehat{v'}^0)$, and so do ζ_w and $\zeta_{w'}$ on $\mathcal{V}ar(\widehat{w'}^0)$. By Lemma 4, $\mathcal{A}(u)$ rewrites to both $\mathcal{A}(v')$ and $\mathcal{A}(w')$, hence each alien in $\mathcal{A}(v)$ and $\mathcal{A}(w)$ originates from some in $\mathcal{A}(u)$. It follows that the aliens in $\mathcal{A}(v)$ and $\mathcal{A}(w)$ originating from the same one in $\mathcal{A}(u)$ are n-convertible. For each $y \in \mathcal{V}ar(\widehat{v}^n) \cup \mathcal{V}ar(\widehat{w}^n)$, we choose all aliens of v and w which belong to the n-convertibility class $\zeta^n(y)$, and apply induction hypothesis to get a common reduct t_y of them, mapping y to t_y to construct the substitution $\zeta_v{\downarrow}^n_w$. Letting v_n be the term $\widehat{v}^n\zeta_v{\downarrow}^n_w$, v rewrites to v_n. Similarly, w rewrites to w_n. This technique, which we call **equalization**, of equalizing all n-convertible aliens to construct $\zeta_v{\downarrow}^n_w$ is somewhat crucial in our proof. The last three steps follow from the inner ancestor diagram between hats of u, v, w, which upper part follows from Lemma 6 and bottom part from the fact that $q > p \cdot \mathcal{FP}os(l)$, resulting in an ancestor peak between homogeneous terms. Such an ancestor peak has an easy stable decreasing diagram, which bottom part can be therefore lifted to the outside diagram. Checking that the obtained diagram is decreasing is routine.

(d) Critical case: $\hat{q} = \hat{p}$; hence $k = m$, with $q \in \mathcal{FP}os(l)$. This is the usual critical case, happening necessarily within same layer. The proof works as in Case (2c), except that the inner diagram is now of a critical peak. Since the R_T critical peak has a joinability diagram by assumption, thanks to stability of rewriting, it can be lifted to the outer diagram, yielding a decreasing diagram for the starting peak.

3) Consider a local peak $v \underset{R_{T_{sub}}}{\overset{p,\langle k,0,\widehat{u|_{\hat{p}}}\rangle}{\longleftarrow}} u \underset{R_{NT}}{\overset{Q,\langle m,j,\_\rangle}{\Rightarrow}} w$. There are three cases.

(a) Disjoint case: $p \# Q$. We get the usual commuting diagram with two facing steps.

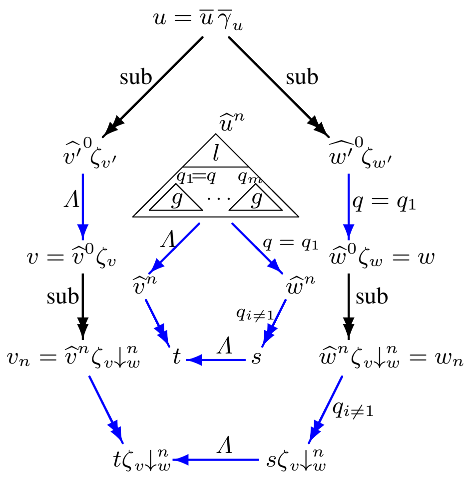


Fig. 3. R_T ancestor peak

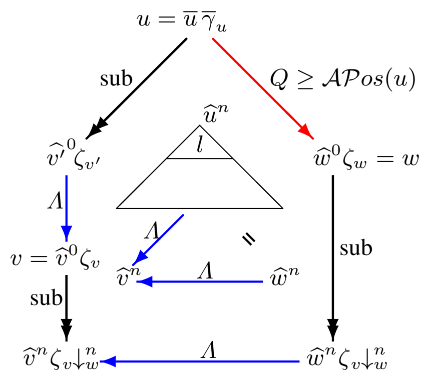


Fig. 4. R_T above R_{NT} ancestor peak

(b) Ancestor case. There are two sub-cases: (α) $p > Q$; hence $m > k$. Since R_{NT} is left-linear, then $v \Rightarrow^{\langle m',j,\_\rangle}_{R_{NT}} t \twoheadleftarrow^{\langle k,0,?\rangle}_{R_{T_{sub}}} w$ for some t and $m' \leq m$, being a clearly decreasing diagram. (β) $p < Q$. This case is a little bit more delicate, since the R_T-rule $l \to r$ used at position p may be non-left-linear. We use equalization as for Case (2c), depicted in Fig. 4 in the particular case where $p = \Lambda$ for simplicity. The main difference with Case (2c) is that the R_{NT}-step must occur in an alien; hence $\widehat{w}^n = \widehat{u}^n$, which somewhat simplifies the figure.

(c) Critical case. By assumption (A1-3), $Q = \{q_i\}_i$ and $p \in q_i \cdot \mathcal{FPos}(l)$ for some q_i. The proof is depicted at Fig. 2 with $Q = \{\Lambda\}$ for simplicity, implying a unique redex for that parallel rewrite at the top. Note that the R_T- and R_{NT}-redexes must have different ranks, hence $m > k$.

By assumption, $u = l\theta \Rightarrow^{\Lambda}_{l\to r} r\theta = w$ and $u(\to^{\geq \mathcal{APos}(u|_p)}_{R_T \cup R_{NT}})^* u[g\theta]_p \to^p_{g\to d} v$ for some substitution θ (assuming l and g are renamed apart). The key of the proof is the fact that $u[g\theta]_p = l\theta'$ for some substitution θ' such that $\theta \twoheadrightarrow \theta'$. By assumption (A4), if o is a variable position in g and $p \cdot o \in \mathcal{FPos}(l)$, then $l|_{p\cdot o} \in \mathcal{T}(F_T, \mathcal{X})$. This indeed ensures that the sub-rewrites from u to v cannot occur at positions in $\mathcal{FPos}(l)$, therefore ensuring the fact $u[g\theta]_p = l\theta'$ since l is linear. It follows that $l\theta'$ rewrites to $r\theta'$ at the root, and to v at $p \in \mathcal{FPos}(l)$, which proves the existence of a critical pair of R_T inside R_{NT}. The rest of the proof is routine, the lifting part being ensured by stability.

To conclude, we simply remark that any two $(R_T \cup R_{NT})$-convertible terms are n-$(R_T \cup R_{NT})$-convertible for some n possibly strictly larger than their respective ranks. $\square$

5 Relaxing Assumptions

One must understand that there is no room for relaxing the conditions on R_T and little for R_{NT}. Left-linearity is mandatory, rank non-increasingness as well, and the fact that left-hand sides are headed by symbols which do not belong to F_T serves avoiding critical pairs of R_{NT} inside R_T. This does not forbid left-hand sides to stretch over possibly several layers, making our result very different from known modularity results. Therefore, the only potential relaxations apply to the right-hand sides of R_{NT}-rules, which need not be headed by F_{NT}-symbols, as we assumed to make the proof more comfortable. We will allow them to be headed by some symbols from F_T.

From now on, we replace our assumption (A1) by the following: Let $F_C = F_T \cap F_{NT}$ be the set of *constructor symbols* s.t. no rule in $R_T \cup R_{NT}$ can have an F_C-headed left-hand side. We use $F_{T\setminus C}$ and $F_{NT\setminus C}$ as shorthand for $F_T \setminus F_C$ and $F_{NT} \setminus F_C$, respectively.

Terms in $\mathcal{T}(F_C, \mathcal{X})$ are *constructor terms*, trivial ones if in $\mathcal{X}$. The definitions of rank, cap and alien for terms headed by $F_{T\setminus C}$- or $F_{NT\setminus C}$-symbols are as before with respect to F_T and F_{NT}, respectively. An F_C-headed term has its cap and aliens defined with respect to F_C, and its rank is the maximal rank of its aliens, which are headed in $F_{T\setminus C}$ or $F_{NT\setminus C}$. The rank of a homogeneous constructor term is therefore 0, which explains why we started with rank 1 before.

Definition 15. *We introduce names for three important categories of terms:*
– *type 1: $F_{NT\setminus C}$-headed terms have a variable as cap and themselves as alien;*
– *type 2: terms u whose cap $\overline{u} \in \mathcal{T}(F_C, \mathcal{Y})$ and aliens are all $F_{NT\setminus C}$-headed;*
– *type 3: $F_{T\setminus C}$-headed terms whose cap $u \in \mathcal{T}(F_T, \mathcal{X} \cup \mathcal{Y})$, and aliens are $F_{NT\setminus C}$-headed.*

We also modify our assumption (A3), which becomes:
(A3) R_{NT} is a left-linear, rank non-increasing rewrite system whose rules have the form $f(\boldsymbol{l}) \to r$, $f \in F_{NT\setminus C}$, $\boldsymbol{l} \in \mathcal{T}(F_T \cup F_{NT}, \mathcal{X})$, r is a term of type 2.
Previous assumption (A3) is a particular case of the new one when r has type 1 $\subseteq$ type 2.

The proof structure of Theorem 2 depends on layering and labelling. Allowing constructor lifting rules in R_{NT} invalidates Lemmas 5, 6 used to control the label's third component of R_T-sub-rewriting steps, since R_{NT}-rewrites in aliens may now modify the cap of an F_T-headed term. Our strategy is to modify the notion of hat and get analogs of Lemmas 5, 6, making the whole proof work by changing the third component of the label of an R_T-sub-rewriting step. Following [4], the idea is to *estimate* the constructors which can pop up at the head of a given $F_{NT\setminus C}$-headed term, by rewriting it until stabilization.

From here on, we assume the Church-Rosser property for n-convertible terms of rank up to n. Being fixed throughout this section, the rank n will often be left implicit.

Finite Constructor Lifting.

Definition 16. *A derivation $s \twoheadrightarrow u$, where $s : type\, 1$ and $u : type\, 2 \setminus type\, 1$, is said to be* constructor lifting. *$R_T \cup R_{NT}$ is a* finite constructor lifting *rewrite system if* $(\forall s : type\, 1)\, \exists n_s \geq 0$ *s.t. for all constructor lifting derivation $s \twoheadrightarrow u$, $|\overline{u}| \leq n_s$.*

Definition 17 (Stable terms). *A term whose multiset M of aliens only contains $F_{NT\setminus C}$-headed terms of rank at most n, is* stable *if M is stable. A multiset M of $F_{NT\setminus C}$-headed terms of rank at most n is* stable *if (i) reducts of terms in M are $F_{NT\setminus C}$-headed; (ii) any two convertible terms in M are equal.*

Example 3. Let $R_T = \{G(x,x,y) \to y, G(x,y,x) \to y, G(y,x,x) \to y, 1 \to 0\}$, $R_{NT} = \{F(0,1,x) \to F(x,x,x), F(1,0,x) \to F(x,x,x), F(0,0,x) \to F(x,x,x)\}$. Then, $u = G(F(0,1,G(0,0,0)), F(0,0,0), F(1,0,0))$ is not stable since its aliens are all convertible but different. But u rewrites to stable $G(F(0,0,0), F(0,0,0), F(0,0,0))$.

From rank non-increasingness and the Church-Rosser assumption, we get:

Lemma 7. *Let u a stable term of type 1 s.t. $u \twoheadrightarrow v$. Then v is a stable term of type 1.*

Lemma 8. *Let u a stable term whose aliens are of rank up to n. Then, $(\forall i \leq n)\, \widehat{u}^i = \widehat{u}^0$.*

Lemma 9 (Stabilization). *A term s of type 1, 2, 3 whose aliens have rank up to n has a stable term t such that $\widehat{t}^n = \widehat{s}^n\, \theta$ for some constructor substitution θ which depends only on the aliens of s.*

Proof. Let M be a multiset of type 1 terms, and $u \in M$. By assumption (A3), the set of constructor positions on top can only increase along a derivation from u. Being bounded, it has a maximum. Let v be such a reduct. If v is of type 1, then it is stable. Otherwise, we still needto equalize its convertible aliens, using the Church-Rosser property of terms of rank up to n, and we are done. Applying this procedure to all terms in M, we are left equalizing as above the convertible stable terms which are stable by Lemma 7. Taking now a type 2/3 term, we apply the procedure to its multiset of aliens, all of which have type 1. The relationship between the hats of s and t is clear: θ is generated by constructor lifting, which is the same for equivalent aliens, hence for equal aliens. □

Lemma 10 (Structure). *Let s be a term of type 1,2,3 whose aliens have rank up to n, and u, v be two stable terms obtained from s by stabilization. Then, $(\forall i \leq n)\, \widehat{u}^i = \widehat{v}^i$.*

Proof. Let $p \in \mathcal{AP}os(s)$. By stabilization $u|_p$ and $v|_p$ are convertible stable terms of type 2. By Church-Rosser assumption $u|_p \twoheadrightarrow t \twoheadleftarrow v|_p$. Since constructors cannot be rewritten, $u|_p$ and $v|_p$ must have the same constructor cap, thus u, v have the same cap. Since they are stable, two convertible aliens of u (resp., v) must be equal, hence u, v have the same 0-hat. We conclude by Lemma 8. □

Definition 18 (Estimated hat). *Let u be a term of type 1,2,3 whose aliens have rank up to n and v a stable term obtained from u by stabilization. The* estimated hat $\overset{\triangle}{u}{}^n_v$ *of u w.r.t. v is the term $\widehat{v}^n$.*

By Lemma 10, the choice of v has no impact on $\overset{\triangle}{u}{}^n_v$, hence the short notation $\overset{\triangle}{u}$.

Lemma 11 (Alien rewriting). *Let u, v be terms of type 3 whose aliens are of rank up to n, such that $u \rightarrow^{\geq \mathcal{AP}os(u)}_{R_T \cup R_{NT}} v$. Then $\overset{\triangle}{u} = \overset{\triangle}{v}$.*

Proof. Follows from Lemmas 9 and 10: any stable term for v is a stable term for u. □

Lemma 12. *Let u be a term of type 3 whose aliens have rank up to n, s.t. $u \rightarrow^{p}_{R_{T_{sub}}} v$ with $p \in \mathcal{CP}os(u)$. Then $\overset{\triangle}{u} \rightarrow_{R_T} \overset{\triangle}{v}$.*

Proof. By definition of sub-rewriting $u \twoheadrightarrow^{\geq \mathcal{AP}os(u)} w \rightarrow^{p}_{R_T} v$. By Lemma 11, $\overset{\triangle}{u} = \overset{\triangle}{w}$. By Lemma 6, $\widehat{w}^n \rightarrow^{p}_{R_T} \widehat{v}^n$, and aliens of v are aliens of w. Let now w', v' be stable terms obtained from w, v by stabilization, hence $\widehat{w'}^n = \widehat{w}^n \theta_w$ and $\widehat{v'}^n = \widehat{v}^n \theta_v$ by Lemma 9, where θ_v, θ_w depend only on the aliens of v, w, respectively; hence θ_v and θ_w coincide on $\mathcal{V}ar(\widehat{v}^n) \subseteq \mathcal{V}ar(\widehat{w}^n)$ and $\widehat{v'}^n = \widehat{v}^n \theta_w$. We conclude by stability of rewriting and definition of estimated hats. □

Theorem 3. *Theorem 2 holds with finite constructor lifting.*

Proof. Same as for Theorem 2, with the exception of the crucial sub-rewriting cases, which are marginally modified by using stabilization instead of equalization of terms. □

Infinite Constructor Lifting. It is easy to see that the only difficult case in the main proof is the elimination of sub-rewriting critical peaks. Consider the critical peak $v \; {}_{l\to r}\!\overset{\Lambda}{\leftarrow} v' \twoheadleftarrow^{\geq \mathcal{A}\mathcal{P}os(u)}_{R_T \cup R_{NT}} u \twoheadrightarrow^{\geq \mathcal{A}\mathcal{P}os(u)}_{R_T \cup R_{NT}} w' \rightarrow^{p}_{g \to d} w$, $p \in \mathcal{FP}os(l)$ and $l \to r, g \to d \in R_T$. To obtain a term instance of l whose subterm at position p is an instance of g, v' and w' must be equalized into a term s whose hat rewrites at Λ with $l \to r$ and at p with $g \to d$ to the hats of the corresponding equalizations of v and w. The heart of the problem lies therefore in equalization which constructs here a solution in the signature of F_T to F_T-unification problems associated with critical pairs by rewriting in $R_T \cup R_{NT}$. Hence,

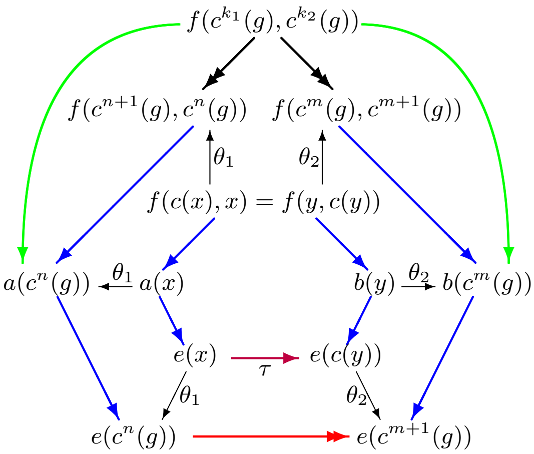
Theorem 4. *With new assumption (A3), Theorem 2 holds if R_T critical pairs modulo $R_T \cup R_{NT}$ are joinable in R_T.*

Because sub-rewriting can only equalize aliens, $R_T \cup R_{NT}$-unification sole purpose is to solve *occurs-check* failures that occur in the plain unification problem $l|_p = g$.

Definition 19. *Let $l \to r$ and $g \to d$ be two rules in R_T s.t. g Prolog unifies with l at position $p \in \mathcal{FP}os(l)$. Let $\bigwedge_i x_i = s_i \wedge \bigwedge_j y_j = t_j$ be a dag solved form returned by Prolog unification, where $\bigwedge_i x_i = s_i$ is the finite substitution part, and $\bigwedge_j y_j = t_j$ the occurs-check part. Let now σ be the substitution $\{x_i \mapsto s_i\}_i$ and $\tau = \{y_j \mapsto t_j\}_j$. Then $\langle r\sigma, l\sigma[d\sigma]_p \rangle$ is a Prolog critical pair of R_T, constrained by the occurs checks $y_j = t_j$.*

If the critical pairs obtained by Prolog unification are joinable in R_T constrained by the occurs-check equations, then the Church-Rosser property is satisfied:

Conjecture 1. With new assumption (A3), Theorem 2 holds if R_T critical pairs are joinable in R_T and Prolog critical pairs of R_T are joinable in R_T modulo their occurs checks.



Example 4 (Variation of Huet's example [3]). Let

$$R_T = \{\; f(c(x), x) \to a(x),\; f(y, c(y)) \to b(y),\; a(x) \to e(x),\; b(y) \to e(c(y)) \;\},$$
$$R_{NT} = \{\; g \to c(g) \;\}.$$

Then the unification problem $f(c(x), x) = f(y, c(y))$ results in an empty substitution and the occurs-check equations $\tau = \{x = c(y), y = c(x)\}$. The critical pair $\langle a(x), b(y) \rangle$ is then joinable by $a(x) \to e(x) = e(c(y)) \leftarrow b(y)$, as exemplified in the figure, where $\theta_1 = \{x \mapsto c^n(g)\}, \theta_2 = \{y \mapsto c^m(g)\}$.

Fig. 5. Variation of Huet's example

The idea is shown in Fig. 5. Note that the red bottom steps operate on aliens, hence have a small rank, making the whole joinability diagram decreasing. We have no clear formulation of the converse yet. Confluence is indeed satisfied if the occurs check is unsolvable, that is, when there exists no $F_{NT\setminus C}$-headed substitution θ of the y_j's such that $y_j\theta \twoheadleftarrow\!\!\twoheadrightarrow_{R_T \cup R_{NT}} t_j\theta$. We suspect this condition can be reinforced as $y_j\theta \twoheadrightarrow_{R_T \cup R_{NT}} t_j\theta$, possibly leading to interesting sufficient conditions for unsolvability of occurs checks.

6 Related Work

In [5], it is shown that confluence can be characterized by the existence of decreasing diagrams for the critical pairs in $R_T \cup R_{NT}$ provided all rules are linear (an assumption that was forgotten [but used] for R_T, as pointed out to the third author by Aart Middeldorp). This is a particular case of a recent result of Felgenhauer [1] showing that R_{NT} is confluent if rules are left-linear and parallel critical pairs have decreasing diagrams with respect to rule indexes used as labels. When F_T is empty, all terms have rank 1, hence our labels for non-linear rules reduce to his. A difference is that we assume R_{NT}-rules to be non-collapsing. One could argue that R_{NT} collapsing rules can be moved to R_T, but this answer is not satisfactory for two different reasons: the resulting change of labels may affect the search for decreasing diagrams, and it can also impact condition (A1). A second difference is that we use rigid parallel rewriting, which yields exponentially fewer parallel critical pairs than when allowing parallel steps with different rules of a given index (which we could have done too). The price to pay – having less flexibility for finding decreasing diagrams – should not make a difference in practice.

A very recent result of Klein and Hirokawa, generalizing [2], extends Knuth and Bendix's critical pair test to relatively terminating systems [6]. It is an extension in the sense that it boils down to it when $R_{NT} = \varnothing$. Otherwise, it requires computing critical pairs of R_T *modulo* a confluent R_{NT}, hence modifies the critical pair test for the subset of terminating rules. Further, it requires proving relative termination (termination of $\twoheadrightarrow_{R_{NT}} \rightarrow_{R_T} \twoheadrightarrow_{R_{NT}}$), complete unification modulo R_{NT}, and absence of critical pairs between R_T and R_{NT}, all tests implemented in CSI[http://dx.doi.org/10.1007/978-3-642-22438-6_38] – to our surprise! This is used to detect that Huet's example is non-confluent.

Theorem 2 can be seen as a modularity theorem to some extent, since rewriting a term in $\mathcal{T}(F_T, \mathcal{X})$ can only involve R_T rules. But left-hand sides of R_{NT} rules may have F_T-symbols. That is why we need to compute critical pairs of R_T inside R_{NT}. Our proof uses many concepts and techniques inherited from previous work on modularity, such as the decomposition of terms (caps and aliens, hats and estimated caps [10]). We have not tried using van Oostrom's notion of cap, in which aliens must have maximal rank [13], nor the method developed by Klein and Hirokawa for studying the Church-Rosser property of *disjoint* rewrite relations on terms [6], which we could do by considering cap rewriting with R_T-rules and alien rewriting with all rules. This remains to be done.

7 Conclusion

Decreasing diagrams opened the way for generalizing Knuth and Bendix's critical-pair test for confluence to non-terminating systems, re-igniting these questions. Our results answer important open questions, in particular by allowing both non-left-linear and non-terminating rules. While combining many existing as well as new techniques, our proof has proved quite robust. Two technical questions have been left open: having collapsing rules in R_{NT}, following [1], and eliminating assumption (A4).

A major theoretical question is whether layering requires assumption (A1). Our proof is based on two key properties, layering and the absence of overlaps of R_{NT} inside R_T. Currently, (A1) serves both purposes. The question is however open whether the latter property is sufficient to define some form of layering, as we suspect.

We end up with our long term goal, applying this technique in practice. The need for showing the Church-Rosser property of mixed terminating and non-terminating rewrite computations arises in at least two areas, first-order and higher-order. The development of sophisticated type theories with complex elimination rules requires proving Church-Rosser *before* strong-normalization and type preservation, directly on untyped terms. Unfortunately, besides being collapsing, β-reduction is also rank-increasing in the presence of another signature. We therefore need to develop another notion of rank that would apply to pure λ-calculus, a question related to the previous one.

Transformation valuation is a static analysis that tries to verify that an optimizer is semantics preserving by constructing a *value graph* for both programs and showing their equivalence by rewriting techniques [11]. Here, the user has a good feeling of which subset of rules is a candidate for R_{NT}. Where this is not the case, work is of course needed to find good splits automatically. Implementers are invited to lead the way.

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