Canonicity!

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Abstract. We describe an abstract proof-theoretic framework based on normal-form proofs, defined using well-founded orderings on proof objects. This leads to robust notions of canonical presentation and redundancy. Fairness of deductive mechanisms – in this general framework – leads to completeness or saturation. The method has so far been applied to the equational, Horn-clause, and deduction-modulo cases.

1 Background

In [18], Knuth invented a *completion* procedure that infers new equations by superposing existing equations with one another (using unification) and also uses equations to simplify one another (by rewriting). When completion terminates successfully, the result is a decision procedure for validity in the theory (variety) of the original equations. In [19], Lankford inaugurated a very fruitful research direction in which superposition is incorporated in a general-purpose theorem-prover for first-order logic.

Eventually, it was noticed [12] that the result of completion is unique – modulo the ordering used for simplification (and for orienting derived equations). In this sense, we can think of the rewrite system produced by completion as being a *canonical* presentation of the given theory, one that provides "cheap" rewrite proofs for all identities of the theory. Similarly, Buchberger's algorithm [7] produces a unique *Gröbner basis*, regardless of nondeterministic choices made along the way.

Huet [17] introduced the notion of *fairness* of completion and showed how fair completion may be viewed as an equational theorem prover. Later, in [1], it was shown how to generalize and formalize equational inference using orderings on proofs, and under what conditions the (finite or infinite) outcome is *complete*, in the sense of providing rewrite proofs of all theorems.

Recently, in a series of papers [13,14,15,3], we proposed quite abstract notions of canonicity and of completion, which can be applied to all manners of inference procedures. Promoting the further study of canonical axiomatizations and their derivation by inference is our goal.

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2 Theory of Canonicity

In our abstract view of inference, proofs have little structure, but are endowed with two well-founded orderings: a *proof ordering* (under which only proofs with the same conclusion are comparable); and a *subproof ordering*, which is *compatible* with the proof ordering, in the sense that whenever there is a better subproof, there is also a better proof (using the better subproof). By *better*, we mean smaller in the proof ordering; by *good*, we will mean minimal in the ordering.

As usual, every proof has a formula as its conclusion and a set of formulæ as its premises. Theories, in the sense of deductively-closed sets of formulæ, are presumed to obey the standard properties of Tarskian consequence relations (monotonicity, reflexivity, and transitivity).

An *inference procedure* uses some *strategy* in applying a system of (sound) *inference rules*, usually given in the form

$$\frac{A}{c}\,,$$

where c may be any *theorem* of A (that is, the conclusion of any proof with premises from A). We call such rules *expansions*. Expansion rules add lemmata to the growing set of allowable premises.

Many inference procedures also apply *deletion rules* of the form

$$\frac{A, c}{\overline{\overline{\quad A \quad}}}$$

– provided that every theorem provable from premises $A \cup \{c\}$ is also provable from A alone (or else completeness would be sacrificed). We are using a double inference line here to indicate that formula c is deleted, *replacing* the set of formulæ above the lines by those below.

In *canonical inference*, deletion is restricted to only allow c to be removed if for every proof with c as a premise, there is a *better* proof without c. We call such restricted deletion steps *contractions*. The point is that such formulæ are truly *redundant* (in the sense of [6]). Once redundant, they will stay redundant; thus, they can be safely removed without endangering completeness of any fairly implemented inference engine.

The following are the basic notions of canonical inference:

1. The *theory* of a presentation (set of formulæ) is the set of all conclusions of proofs using premises from the presentation.
2. A proof is *trivial* if its conclusion is its lone premise.
3. A proof is in *normal form* if it is a good proof when considering the whole theory as potential premises.
4. A presentation is *complete* if it affords at least one normal-form proof for each theorem.

5. A presentation is *saturated* if it supports all normal-form proofs for all theorems.
6. A formula is *redundant* in a presentation, if adding it (or removing it) does not affect normal-form proofs.
7. A presentation is *contracted* (or *reduced*) if it contains no redundant formulæ.
8. A presentation is *perfect* if it is both complete and contracted.
9. A presentation is *canonical* if it is both saturated and contracted.
10. A *critical proof* is a good non-normal-form proof, all of whose (proper) subproofs are in normal form.
11. A formula *persists* in a run of an inference procedure if from some point on it is never deleted.
12. The *result* (in the limit) of a run of an inference procedure is its persistent formulæ.
13. An inference procedure is *fair* if all critical proofs with persistent premises have better proofs at some point.
14. An inference procedure is *uniformly fair* if every trivial normal-form proof is eventually generated.

The following consequences follow from these definitions (see [15,3]):

1. A presentation is contracted if it consists only of premises of normal-form proofs.
2. The smallest saturated presentation is canonical.
3. A presentation is canonical if it consists of all non-redundant formulæ of its theory.
4. A presentation is canonical if it consists of the conclusions (or premises, if you will) of all trivial normal-form proofs.
5. The result of a fair inference procedure is complete.
6. The result of a uniformly fair inference procedure is saturated.
7. The result of an inference procedure is contracted if no redundant formula is allowed to persist.

3 Applications of Canonicity

The abstract approach to inference outlined above has to date been applied to the following situations:

- Ground equations (à la [19,16]) in [13].
- Ground resolution in [15].
- Equational theories (à la [18,17,1]) in [8,9].
- Horn theories (à la [2]) in [5,4].
- Natural deduction in [8].
- Deduction modulo rewriting in [10,11].

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