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Critical Pair Criteria for Completion †

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We formulate the Knuth-Bendix completion method at an abstract level, as an equational inference system, and formalize the notion of critical pair criterion using orderings on equational proofs. We prove the correctness of standard completion and verify all known criteria for completion, including those for which correctness had not been established previously. What distinguishes our approach from others is that our results apply to a large class of completion procedures, not just to a particular version. Proof ordering techniques therefore provide a basis for the design and verification of specific completion procedures (with or without criteria).

1. Introduction

Rewrite techniques, such as standard completion (Knuth and Bendix, 1970), have been applied to a variety of problems including word problems in universal algebra, proofs of inductive properties of data types (Musser, 1980; Huet and Hullot, 1982), equational programming (Dershowitz, 1985; Fribourg, 1985), and theorem proving in first order logic (Hsiang, 1985). Rewrite systems are sets of directed equations (rewrite rules) that may be used for computation by simplification. Completion tackles the problem of constructing a canonical (i.e., terminating and Church-Rosser) rewrite system for a given set of

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equational axioms. The validity problem is decidable in equational theories that can be represented as canonical systems: two terms are equivalent if and only if they simplify to an identical form. A large number of canonical systems have been derived using completion (e.g., Hullot, 1980; Le Chenadec, 1986).

In constructing a canonical system, completion generates rules by orienting equations (with respect to a given well-founded ordering on terms) and derives new equations, called critical pairs, by unifying left-hand sides of existing rules. The procedure may fail if an equation is generated that can not be oriented in the given term ordering. Mechanisms that permit control over the number of rules and critical pairs that have to be computed are indispensable for efficiency. Mutual simplification of rules, as suggested by Knuth and Bendix (1970), may considerably reduce the number of rules and, consequently, also the number of critical pairs. Schemes for sifting out superfluous critical pairs, called critical pair criteria, have been described by Buchberger (1979), Winkler (1984), Winkler and Buchberger (1983), Küchlin (1985), and Kapur, Musser, and Narendran (1985).

A simplification or deletion scheme for completion is correct if its use does not preclude the construction of a canonical system. Simplification of rules was first proved correct by Huet (1981) and, in a more general framework, by Bachmair, Dershowitz, and Hsiang (1986). A major difficulty in verifying critical pair criteria consists in showing their compatibility with simplification schemes. Criteria based on connectedness (a smaller proof of a critical pair exists) have been verified for specific cases (Küchlin, 1986a; Winkler, 1985). We generalize these results and also establish the correctness of criteria based on compositeness (a third rewrite applies to the unified left-hand sides), proving that composite criteria can be combined with any correct strategy for simplification of rules. Furthermore, we show that the two types of criteria, connectedness and compositeness, can be combined.

After introducing basic definitions in Section 2, we present, in Section 3, an equational inference system for standard completion and introduce the notion of proof ordering. In Section 4, we formalize the notion of critical pair criterion in the proof ordering framework. Connected criteria are described in Section 5; composite criteria, in Section 6.

2. Definitions

We consider *terms* over some (finite) set of operator symbols F and some set of variables V. The symbols s, t, u, $\cdots$ denote terms; f, g, $\cdots$ denote operator symbols; and x, y, z, $\cdots$ denote variables. A subterm of a term t is called *proper* if it is distinct from t. The expression t/p denotes the subterm of t at position p (positions may, for instance, be represented as sequences of indices). We write $s[t]$ to indicate that a term s contains t as a

subterm and (ambiguously) denote by $s[u]$ the result of replacing a particular occurrence of t by u.

A binary relation $\rightarrow$ on terms is *monotonic* (with respect to the term structure) if $s \rightarrow t$ implies $u[s] \rightarrow u[t]$, for all terms s, t, and u. It is *stable* (under substitution) if $s \rightarrow t$ implies $s\sigma \rightarrow t\sigma$, for any substitution σ. The symbols $\rightarrow^+$, $\rightarrow^*$ and $\leftrightarrow$ denote the transitive, transitive-reflexive, and symmetric closure of $\rightarrow$, respectively. The inverse of $\rightarrow$ is denoted by $\leftarrow$. We call $\rightarrow$ an (strict partial) *ordering* if it is irreflexive and transitive. An ordering $\rightarrow$ is *well-founded* if there is no infinite sequence $t_1 \rightarrow t_2 \rightarrow t_3 \cdots$. A *reduction ordering* is a well-founded ordering that is stable and monotonic.

An *equation* is a pair (s, t) of terms, written $s = t$. For any set of equations E, $\leftrightarrow_E$ denotes the smallest symmetric relation that contains E and is stable and monotonic. That is, $s \leftrightarrow_E t$ if and only if, for some term w and some substitution σ, s is $w[u\sigma]$ and t is $w[v\sigma]$, where $u \doteq v$ is in E ($u \doteq v$ denotes, ambiguously, $u = v$ or $v = u$). The relation $\leftrightarrow_E^*$ is the smallest stable congruence that contains E; a congruence is, by definition, monotonic.

Directed equations are also called *rewrite rules* and are written $s \rightarrow t$. A *rewrite system* is any set R of rewrite rules. The *rewrite relation* $\rightarrow_R$ is the smallest stable and monotonic relation that contains R. That is, $s \rightarrow_R t$ (s *rewrites to* t) if and only if s is $w[u\sigma]$ and t is $w[v\sigma]$, for some rewrite rule $u \rightarrow v$ in R, term w, and substitution σ. A term t is in *normal form* with respect to R if there is no term u, such that $t \rightarrow_R u$.

A rewrite system R is *Church-Rosser* if, for all terms s and t with $s \leftrightarrow_R^* t$, there exists a term u, such that $s \rightarrow_R^* u \leftarrow_R^* t$. A rewrite system R *terminates* if $\rightarrow_R^+$ is well-founded. Thus, a rewrite system terminates if and only if it is contained in some reduction ordering. A terminating Church-Rosser system is called *canonical*. A canonical system defines a unique normal form for each term.

Let E be a set of equations and R be a rewrite system. A *proof* of $s = t$ in $E \cup R$ (or a proof $s \leftrightarrow_{E \cup R}^* t$) is a sequence $(s_0, \ldots, s_n)$, such that s_0 is s, s_n is t and, for $1 \leq i \leq n$, one of $s_{i-1} \leftrightarrow_E s_i$, $s_{i-1} \rightarrow_R s_i$, or $s_{i-1} \leftarrow_R s_i$ holds. Every single proof step (s_{i-1}, s_i) has to be justified by an equation $u_i = v_i$, a substitution σ_i, and a position p_i, such that s_{i-1}/p_i is $u_i\sigma_i$, s_i is $s_{i-1}[v_i\sigma_i]$ (where the replacement takes place at position p_i), and $u_i \doteq v_i$ is in $E \cup R$. The *justification* of a proof is the sequence of all tuples $(s_{i-1}, s_i, u_i, v_i, \sigma_i, p_i)$, $1 \leq i \leq n$. It may be (partially) indicated by writing the proof as, for instance, $s_0 \leftrightarrow_E s_1 \rightarrow_R \cdots \leftarrow_R s_n$, etc.

A proof step $s \leftrightarrow_E t$ is called an *equality step*; a step $s \rightarrow_R t$, a *rewrite step*; a proof $s \leftarrow_R u \rightarrow_R t$, a *peak*. We usually abbreviate a proof of the form $s_0 \rightarrow_R \cdots \rightarrow_R s_n$ by $s_0 \rightarrow_R^* s_n$. A proof $s_0 \rightarrow_R^* s_k \leftarrow_R^* s_n$ is called a *rewrite proof*. A *subproof* of $(s_0, \ldots, s_n)$ is any proof $(s_i, \ldots, s_j)$, where $0 \leq i \leq j \leq n$. The notation $P[P']$ indicates that P contains P' as a subproof.

A *proof pattern* in $E \cup R$ is a schema for a class of proofs; it describes proofs that share a common structure. For example, the pattern $s \rightarrow_R t$, where the metavariables s and t denote arbitrary terms and R denotes an arbitrary rewrite system, characterizes all single step rewrite proofs in R; $s \rightarrow_R^* u \leftarrow_R^* t$ describes all rewrite proofs in R; $s \leftarrow_R u \rightarrow_R t$, all peaks. An *instance* of a pattern is any specific proof of the given structure.

3. Standard Completion

We first describe the *Knuth-Bendix completion method* for constructing a canonical rewrite system R for a given set of equations E. If R is finite and canonical, and the congruence relations $\leftrightarrow_E^*$ and $\leftrightarrow_R^*$ are the same, then R may be used as a *decision procedure* for the *validity problem* in E: two terms s and t are equivalent in E if and only if they reduce to identical normal forms with respect to R. In particular, canonical systems may be used for solving word problems in equational theories. The unsolvability of the word problem for certain (even finitely-based) equational theories implies that the construction of a canonical system R is not always possible. Thus, a completion procedure may terminate either with success or failure, or it may not terminate and instead compute successive approximations R_n of an infinite canonical system R.

We will formulate completion as an equational inference system. Since we distinguish between equations and rewrite rules, the objects of this inference system are pairs (E, R), where E is a set of equations and R is a set of rules. Let $>$ be a reduction ordering on terms. *Standard completion* is the proof system **C** consisting of the following *inference rules*, where R is any rewrite system contained in $>$:

1) Orienting an equation.

$$\frac{(E \cup \{s \doteq t\}, R)}{(E, R \cup \{s \rightarrow t\})} \qquad \text{if } s > t$$

2) Adding an equational consequence.

$$\frac{(E, R)}{(E \cup \{s = t\}, R)} \qquad \text{if } s \leftarrow_R u \rightarrow_R t$$

3) Simplifying an equation.

$$\frac{(E \cup \{s \doteq t\}, R)}{(E \cup \{u \doteq t\}, R)} \qquad \text{if } s \rightarrow_R u$$

4) Deleting a trivial equation.

$$\frac{(E \cup \{s = s\}, R)}{(E, R)}$$

The following simplification rules are also part of standard completion and are indispensable for efficiency:

5) Simplifying the right-hand side of a rewrite rule

$$\frac{(E,R\cup\{s\to t\})}{(E,R\cup\{s\to u\})} \qquad \text{if } t\to_R u$$

6) Simplifying the left-hand side of a rewrite rule

$$\frac{(E,R\cup\{s\to t\})}{(E\cup\{u=t\},R)} \qquad \text{if } s\to_R u \text{ at a position not at the top,}$$

$$\frac{(E,R\cup\{s\to t\})}{(E\cup\{u=t\},R)} \qquad \text{if } s\to_R u \text{ by } l\to r \text{ and } s\rhd l.$$

The symbol $\rhd$ denotes the proper subsumption ordering: $s\rhd l$ *if and only if* s *is a proper instance of* l. For example, $f(z,g(z))$ and $f(z,z)$ are proper instances of $f(x,y)$, but $f(x,z)$ is not.

We write $(E,R)\vdash(E',R')$ if (E',R') can be obtained from (E,R) by an application of an inference rule of C. A *derivation* is a (possibly infinite) sequence $(E_0,R_0)\vdash(E_1,R_1)\vdash\cdots$. The *limit* of a derivation is the pair (E^∞,R^∞), where E^∞ is the set $\bigcup_{i\geq 0}\bigcap_{j\geq i}E_j$ of all *persisting equations*, and R^∞ is the set $\bigcup_{i\geq 0}\bigcap_{j\geq i}R_j$ of all *persisting rules*.

Standard completion is *sound*:

LEMMA 1. *If* $(E,R)\vdash(E',R')$, *then the congruence relations* $\leftrightarrow^*_{E\cup R}$ *and* $\leftrightarrow^*_{E'\cup R'}$ *are the same.*

We are interested in derivations for which the limit R^∞ is canonical. If an equational theory $E\cup R$ can be represented by a canonical system, then any equation valid in $E\cup R$ can be proved by simple rewriting. A rewrite proof in $E\cup R$ can be characterized as a proof that contains no equality step $s\leftrightarrow_E t$ and no peak $s\leftarrow_R u\to_R t$. The application of a completion inference rule allows us to eliminate (or simplify) such "undesirable" subproofs. In other words, the inference rules of C are reflected on the proof level by a simplification or reduction relation on proofs. Proof orderings (Bachmair, Dershowitz, and Hsiang, 1986) are the key to formalizing this aspect of completion.

A binary relation $\Rightarrow$ on proofs is called *monotonic* if $P\Rightarrow P'$ implies $Q[P]\Rightarrow Q[P']$, for all proofs P, P', and Q. It is *stable* if

$$(s,\ldots,u_i,\ldots,t)\Rightarrow(s,\ldots,v_j,\ldots,t)$$

implies

$$(w[s\sigma],\ldots,w[u_i\sigma],\ldots,w[t\sigma])\Rightarrow(w[s\sigma],\ldots,w[v_j\sigma],\ldots,w[t\sigma]),$$

for all proofs, terms w, and substitutions σ. A *proof ordering* is a stable, monotonic, and well-founded ordering on proofs.

An *elimination pattern* is a pair of proof patterns. If S is a set of elimination patterns, then $\Rightarrow_S$ denotes the smallest stable and monotonic relation on proofs that contains each instance of an elimination pattern of S. In other words, $\Rightarrow_S$ is the rewrite relation on proofs induced by S.

For completion we have *equality patterns*

$$\begin{array}{lcll} s \leftrightarrow_E t & \Rightarrow & s \rightarrow_{R'} t & \text{if } s > t \\ s \leftrightarrow_E t & \Rightarrow & s \leftarrow_{R'} t & \text{if } t > s \\ s \leftrightarrow_E t & \Rightarrow & s \rightarrow_{R'} u \leftrightarrow_{E'} t & \\ s \leftrightarrow_E t & \Rightarrow & s \leftrightarrow_{E'} u \leftarrow_{R'} t & \\ s \leftrightarrow_E s & \Rightarrow & s & \end{array}$$

overlap patterns

$$\begin{array}{lcl} s \leftarrow_R u \rightarrow_R t & \Rightarrow & s \rightarrow_R^{*\prime} v \leftarrow_R^{*\prime} t \\ s \leftarrow_R u \rightarrow_R t & \Rightarrow & s \leftrightarrow_{E'} t \end{array}$$

and *simplification patterns*

$$\begin{array}{lcl} s \rightarrow_R t & \Rightarrow & s \rightarrow_{R'} u \leftarrow_{R'} t \\ t \leftarrow_R s & \Rightarrow & t \rightarrow_{R'} u \leftarrow_{R'} s \\ s \rightarrow_R t & \Rightarrow & s \rightarrow_{R'} v \leftrightarrow_{E'} t \\ t \leftarrow_R s & \Rightarrow & t \leftrightarrow_{E'} v \leftarrow_{R'} s \end{array}$$

(see Figures 1, 2, and 3). In all patterns above R and R' have to be contained in the given reduction ordering $>$. In the simplification patterns, $s \rightarrow_R t$ is by application of a rule $l \rightarrow r$ at position p; $s \rightarrow_{R'} u$ is by application of $l \rightarrow r'$ at position p; and $s \rightarrow_{R'} v$ is either strictly below p, or at position p by application of a rule $l' \rightarrow r'$ with $l \rhd l'$. By $\Rightarrow_C$ we denote the rewrite relation on proofs induced by the above patterns with these restrictions.

LEMMA 2. *Whenever* $(E,R) \vdash (E',R')$ *and* P *is a proof in* $E \cup R$, *then there exists a proof* P' *in* $E' \cup R'$, *such that* $P \Rightarrow_C^* P'$.

We next prove that the ordering $\Rightarrow_C^+$ is well-founded. In this context the concept of *multiset orderings* is of importance. A *multiset* is a finite unordered collection of elements in which elements may appear more than once. If $>$ is a partial ordering on a set S, then the corresponding multiset ordering $\gg$ on the set of all multisets of elements in S is the smallest transitive relation such that

$M \cup \{x\} \gg M \cup \{y_1, \ldots, y_n\}$, whenever $n \geq 0$ and $x > y_i$, for $1 \leq i \leq n$.

According to this ordering an element of a multiset can be replaced by any finite number of elements that are smaller in $>$. Dershowitz and Manna (1979) have proved that *the multiset ordering* $\gg$ *is well-founded if and only if* $>$ *is well-founded.*

LEMMA 3 (Bachmair, Dershowitz, and Hsiang, 1986). *The ordering* $\Rightarrow_C^+$ *is a proof ordering.*

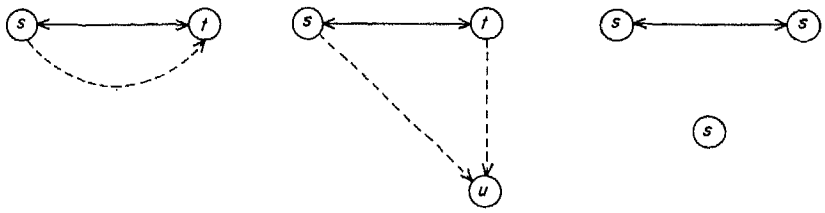

Figure 1. *Equality patterns*

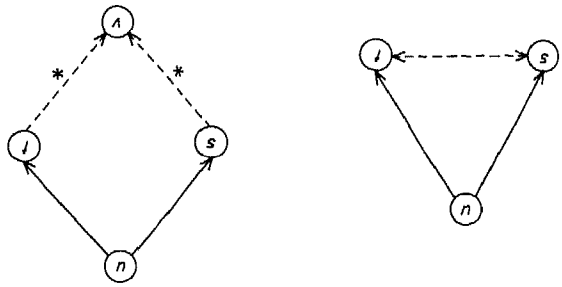

Figure 2. *Overlap patterns*

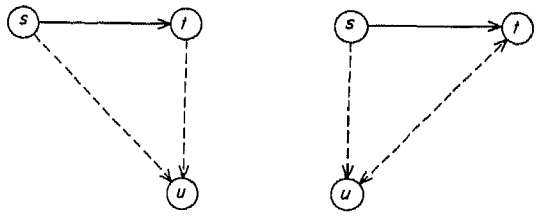

Figure 3. *Simplification patterns*

Proof. We construct a well-founded ordering $>_C$ and prove that $P \Rightarrow_C P'$ implies $P >_C P'$, for all proofs P and P'.

First we define the *complexity* $c(s,t)$ of a single proof step (s,t) by:

if $s \rightarrow_R t$ by $l \rightarrow r$ at position p, then $c(s,t)$ is $(\{s\},s/p,l,t)$;
if $s \leftarrow_R t$ by $l \rightarrow r$ at position p, then $c(s,t)$ is $(\{t\},t/p,l,s)$;
if $s \leftrightarrow_E t$, then $c(s,t)$ is $(\{s,t\},-,-,-)$.

Only the first component is relevant in the last case. The ordering $>^c$ is the lexicographic combination of the multiset extension $\gg$ of the reduction ordering $>$, the proper subterm ordering, the proper subsumption ordering $\rhd$, and the reduction ordering $>$. We define: $(s_0, \ldots, s_m) >_C (t_0, \ldots, t_n)$ *if and only if* $\{c(s_0,s_1), \ldots, c(s_{m-1},s_m)\} \gg^c \{c(t_0,t_1), \ldots, c(t_{n-1},t_n)\}$. Since

the reduction ordering $>$ and the proper subterm ordering are well-founded, monotonic, and stable (under substitution) and the subsumption ordering $\rhd$ is well-founded, we can readily infer that $>_C$ is a proof ordering. Therefore it suffices to show that $>_C$ contains any instance of an elimination pattern for completion. For equality patterns:

a) $(s \leftrightarrow_E t) >_C (s \rightarrow_R t)$, since $\{s,t\} \gg \{s\}$;

b) $(s \leftrightarrow_E t) >_C (s \rightarrow_R u \leftrightarrow_E t)$,
since $\{s,t\} \gg \{s\}$ and $\{s,t\} \gg \{u,t\}$;

c) $(s \leftrightarrow_E s) >_C (s)$, since $\{\{s,s\}\} \gg \emptyset$.

For overlap patterns:

d) $(s \leftarrow_R u \rightarrow_R t) >_C (s \rightarrow_R^* v \leftarrow_R^* t)$,
since all terms on the right-hand side are smaller than u;

e) $(s \leftarrow_R u \rightarrow_R t) >_C (s \leftrightarrow_E t)$, since $\{u\} \gg \{s,t\}$.

For simplification patterns:

f) $\{(\{s\},s/p,l,t)\} \gg^c \{(\{s\},s/p,l,u),(\{t\},t/q,l',u)\}$,
since $t > u$ and $s > t$;

g) $\{(\{s\},s/p,l,t)\} \gg^c \{(\{t,u\},-,-,-),(\{s\},s/q,l',v)\}$,
since $s > t$, $s > u$, and either q is strictly below p, or $l \rhd l'$.

□

This lemma shows that the inference system C can be used to *simplify* proofs containing equality steps $s \leftrightarrow_E t$ or peaks $s \leftarrow_R u \rightarrow_R t$. Equality steps can be eliminated by orienting, simplifying, or deleting equations. To eliminate peaks it suffices to generate certain equational consequences called critical pairs.

Let $s \rightarrow t$ and $l \rightarrow r$ be rules in R with no variables in common (the variables of one rule are renamed if necessary) and suppose that, for some position p, s/p is not a variable and is unifiable with l, σ being the most general unifier (thus $s\sigma/p$ and $l\sigma$ are identical). Then the *superposition* of $l \rightarrow r$ on $s \rightarrow t$ at position p determines a *critical pair* $t\sigma = s\sigma[r\sigma]$ (where the replacement in $s\sigma$ takes place at position p). The proof $t\sigma \leftarrow_R s\sigma \rightarrow_R s\sigma[r\sigma]$ is called a *critical overlap*; the term $s\sigma$, the *overlapped term*; the position p, the *critical pair position*. By $CP(R)$ we denote the set of all critical pairs between rules of R.

CRITICAL PAIR LEMMA (Knuth and Bendix, 1970; Huet, 1980). *For each peak* $s \leftarrow_R u \rightarrow_R t$ *there exists a term* v, *such that either* $s \rightarrow_R^* v \leftarrow_R^* t$, *or else* s *is* $v[s'\sigma]$ *and* t *is* $v[t'\sigma]$, *for some critical pair* $s' = t'$ *in* $CP(R)$.

The considerations above lead to

DEFINITION 1. A derivation $(E_0,R_0) \vdash (E_1,R_1) \vdash \cdots$ is *fair* if (a) $E^\infty = \emptyset$ and (b) $CP(R^\infty)$ is a subset of $\cup_k E_k$.

A *completion procedure* is any procedure that accepts as input a set of equations E, a rewrite system R, and a reduction ordering $>$ containing R,

and generates a derivation $(E,R)\vdash(E_1,R_1)\vdash \cdots$, using applications of the inference rules of C as the only elementary computation steps. Since a fair derivation may not be possible from an arbitrary pair (E_i,R_i), or may require backtracking (Dershowitz, Marcus, and Tarlecki, 1987), we have to allow for the possibility of *failure* for certain inputs E, R, and $>$. A completion procedure may terminate with output "failed," even when it need not. We ignore derivations for which a procedure explicitly fails, and call the procedure *fair* if all its non-failing derivations are fair.

THEOREM 1 (Huet, 1981; Bachmair, Dershowitz, and Hsiang, 1986). *If a completion procedure is fair, and does not fail for inputs E, R, and $>$, then R^∞ is canonical.*

Proof. Let $(E_0,R_0)\vdash(E_1,R_1)\vdash \cdots$ be a fair derivation. We show that R^∞ is canonical. Since R^∞ is contained in $>$, it is terminating. For the Church-Rosser property we prove, by induction on $\Rightarrow_C^+$, that for any arbitrary proof P in $E_i \cup R_i$, $i \geq 0$, there exists a rewrite proof P' in R^∞ with $P \Rightarrow_C^* P'$. We assume that each proof Q with $P \Rightarrow_C^+ Q$ can be transformed into a persisting rewrite proof.

If P contains an equality step $s \leftrightarrow_{E_i} t$ in which an equation $u = v$ is used, then, by fairness, the equation $u = v$ will eventually be formed into a rewrite rule, simplified, or deleted. By Lemma 2, there is a proof Q in $E_j \cup R_j$, for some $j \geq i$, such that $P \Rightarrow_C^+ Q$. Likewise, if P contains a non-persisting rewrite step $s \rightarrow_{R_i} t$, then simplification will eventually result in a proof Q, such that $P \Rightarrow_C^+ Q$.

If P is a proof in R^∞, but not a rewrite proof, then it must contain a peak $s \leftarrow_{R_i} u \rightarrow_{R_i} t$. By the Critical Pair Lemma, if this peak is not a critical overlap, then there is a rewrite proof $s \rightarrow_{R_i}^* v \leftarrow_{R_i}^* t$; hence $P \Rightarrow_C Q$, for some proof Q in $E_i \cup R_i$. If the peak is a critical overlap, then $s = t$ can be written as $v[s'\tau] = v[t'\tau]$, where $s' = t'$ is in $CP(R^\infty)$. By fairness, $s' = t'$ is contained in E_k, for some k. Using Lemma 2, we may conclude that there is a proof Q in $E_j \cup R_j$, for some $j \geq i$, such that $P \Rightarrow_C^+ Q$.

In summary, there exists a proof Q in $E_j \cup R_j$, for some $j \geq i$, such that $P \Rightarrow_C^+ Q$. By the induction hypothesis, there is a rewrite proof Q' in R^∞ with $Q \Rightarrow_C^* Q'$. Therefore we have $P \Rightarrow_C^+ Q'$, which concludes the proof. □

The notion of completion as formalized above covers a wide variety of specific completion procedures, including those given in Knuth and Bendix (1970) and Huet (1981). Any particular completion procedure has to specify in which order the inference rules of C are to be applied to given sets of equations and rules. We call such a selection strategy *fair* if it gives rise only to fair or failing derivations. By Theorem 1, any implementation using a fair selection strategy is guaranteed to construct a (possibly infinite) canonical system, provided it does not fail. Such an implementation is therefore called

correct. The correctness—in this sense—of a specific completion procedure was first proved by Huet (1981). Huet's proof requires intricate arguments by induction on certain orderings on *terms.* One of the main differences with our approach is that we use orderings on *proofs.* The use of multisets of terms, as in Jouannaud and Kirchner (1986), may be regarded as a simple instance of a proof ordering that makes no use of the (additional) information contained in the proof steps. The full potential of proof orderings is only realized when this information is utilized.

4. Critical Pair Criteria

The efficiency of the completion process depends on the number of rewrite rules and critical pairs generated. Simplification can be a very effective mechanism for eliminating superfluous equations and rules. For instance, whenever a critical pair $s=t$ has been computed, both s and t can be reduced to normal forms s' and t'. If the normal forms are identical, then the equation $s'=t'$ can be deleted, indicating that the original equation $s=t$ was not needed in the first place. Normalization is done systematically in most completion procedures, but can be costly. The redundancy of $s=t$ can often be determined more efficiently by looking at the structure of its associated critical overlap $s \leftarrow_R u \rightarrow_R t$. Characterizations of redundant critical pairs, called critical pair criteria, can be conveniently described by proof orderings.

We say that a set CPC of elimination patterns of the form

$$s \leftarrow_R u \rightarrow_R t \;\Rightarrow\; s \leftrightarrow^*_{E \cup R} t\,,$$

where R is contained in the given reduction ordering $>$, specifies a *critical pair criterion.* By $\Rightarrow_{CPC}$ we denote the corresponding rewrite relation on proofs. We use this proof relation to sift out redundant critical pairs. By $CPC(E,R)$ we denote the set of all critical pairs $s=t$ in $CP(R)$ for which the critical overlap $s \leftarrow_R u \rightarrow_R t$ can be reduced via $\Rightarrow^+_{CPC}$; that is, for which there exists a proof P in $E \cup R$ with $(s \leftarrow_R u \rightarrow_R t) \Rightarrow^+_{CPC} P$. Critical pairs in $CPC(E,R)$ are meant to be treated as superfluous.

DEFINITION 2. A derivation $(E_0,R_0) \vdash (E_1,R_1) \vdash \cdots$ is *fair relative to a critical pair criterion CPC* if (a) $E^\infty = \emptyset$, and (b) $CP(R^\infty) - \cup_i CPC(E_i,R_i)$ is a subset of $\cup_k E_k$.

($A-B$ denotes the set of all elements of A that are not in B.) Fairness relative to the trivial criterion CPC, for which $CPC(E,R)$ is always empty, corresponds to fairness in the usual sense. Thus, Definition 1 is a special case of Definition 2.

A criterion CPC is *correct* if, for all derivations that are fair relative to CPC, the limit R^∞ is canonical. If CPC is correct, then critical pairs in $CPC(E,R)$ may be ignored by completion.

THEOREM 2. *Let CPC be a correct critical pair criterion and C be a completion procedure that is fair relative to CPC. If C does not fail for inputs E, R, and $>$, then R^∞ is canonical.*

Proof. Immediate from the definition of correctness of a criterion. □

The following lemma is useful for establishing the correctness of a criterion.

LEMMA 4. *A critical pair criterion CPC is correct if the ordering induced by $\Rightarrow_C \cup \Rightarrow_{CPC}$ is well-founded.*

Proof. Let *CPC* be a critical pair criterion for which the ordering induced by $\Rightarrow_C \cup \Rightarrow_{CPC}$ is well-founded. Let $(E_0,R_0)\vdash(E_1,R_1)\vdash \cdots$ be a derivation that is fair relative to CPC. We have to show that R^∞ is canonical. Since R^∞ is contained in the reduction ordering $>$, it is terminating. For the Church-Rosser property, it suffices to show that any arbitrary proof P in $E_i \cup R_i$ can be transformed, via $(\Rightarrow_C \cup \Rightarrow_{CPC})^+$, into a rewrite proof in R^∞. We assume that this assertion holds for every proof Q with $P\,(\Rightarrow_C \cup \Rightarrow_{CPC})^+ Q$.

Let P be a proof in $E_i \cup R_i$. Using fairness and Lemma 2, we may conclude that whenever P contains a non-persisting proof step or a non-critical overlap, then there is a proof Q in $E_j \cup R_j$, for some $j \geq i$, such that $P \Rightarrow_C^+ Q$. Suppose P contains a persisting critical overlap $s \leftarrow_{R_i} u \rightarrow_{R_i} t$. Thus, $s = t$ must involve some critical pair $s' = t'$ in $CP(R^\infty)$. If this critical pair is not contained in $\cup_j CPC(E_j,R_j)$, then, by fairness, it is in E_k, for some k. By Lemma 2, there is a proof Q in $E_j \cup R_j$, for some $j \geq i$, such that $P \Rightarrow_C^+ Q$. If $s' = t'$ is contained in some set $CPC(E_k,R_k)$, then there is, by definition, a proof Q' in $E_k \cup R_k$, such that $P \Rightarrow_{CPC}^+ Q'$. Using Lemma 2, we may conclude that there is a proof Q in $E_j \cup R_j$, for some $j \geq i$, such that $Q' \Rightarrow_C^+ Q$.

In summary, we have shown that there exists a proof Q in $E_j \cup R_j$, for some $j \geq i$, such that $P\,(\Rightarrow_C \cup \Rightarrow_{CPC})^+ Q$. By the induction hypothesis, Q (and therefore P) can be transformed into a rewrite proof in R^∞. □

A criterion can considerably decrease the total number of critical pairs generated by completion. This advantage may be offset, however, by the additional cost of checking whether the criterion applies to a given critical pair.

Critical pair criteria have also been applied to testing the Church-Rosser property.

DEFINITION 3. A criterion *CPC* is *sound* if, for each rewrite system R contained in $>$, the following property holds: *R is Church-Rosser if and only if there exists a rewrite proof in R, for each critical pair in $CP(R)$-$CPC(\emptyset,R)$.*

A sound criterion, whose applicability can be effectively tested, can be used for testing the Church-Rosser property of terminating systems. While soundness

of a criterion can usually be established without difficulty, correctness can be considerably more difficult to verify.

PROPOSITION 1. *Any correct criterion is sound.*

Proof. Let R be a rewrite system contained in the given reduction ordering $>$, and let CPC be a correct criterion. Furthermore, suppose that, for each critical pair $s = t$ in $CP(R) - CPC(\emptyset, R)$, there exists a term v, such that $s \rightarrow_R^* v \leftarrow_R^* t$. We have to show that R is Church-Rosser.

Under the above assumptions, there exists a finite derivation $(CP(R) - CPC(\emptyset, R), R) \vdash \cdots \vdash (\emptyset, R)$. (Each equation in the initial set can be reduced to a trivial one, and then deleted.) Since $CP(R) - \bigcup_i CPC(E_i, R)$ is a subset of the initial set of equations $CP(R) - CPC(\emptyset, R)$, the above derivation is fair relative to criterion CPC. By correctness, R is canonical. □

Formalizing critical pair criteria in terms of proof orderings greatly facilitates the task of proving correctness. We will present correctness proofs for all known criteria, including those for which correctness had not been established previously.

A first example of a critical pair criterion is *blocking*, a concept introduced by Slagle (1974) and applied to rewriting by Lankford and Ballantyne (1979).

DEFINITION 4. Let R be a rewrite system and $t\sigma \leftarrow_R s\sigma[l\sigma] \rightarrow_R s\sigma[r\sigma]$ be a critical overlap between rules $s \rightarrow t$ and $l \rightarrow r$. The critical pair $t\sigma = s\sigma[r\sigma]$ is called *blocked* if $x\sigma$ is irreducible, for all variables x in s or l. Otherwise, it is called *unblocked.*

If $t\sigma = s\sigma[r\sigma]$ is a unblocked critical pair, then $x\sigma \rightarrow_R w$, for some variable x in s or l, and some term w. Let σ' be the substitution for which $x\sigma'$ is w, and $y\sigma'$ is $y\sigma$, for all variables y distinct from x. Then there is a proof $t\sigma \rightarrow_R^* t\sigma' \leftarrow_R s\sigma' \rightarrow_R s\sigma'[r\sigma'] \leftarrow_R^* s\sigma[r\sigma]$, and we also have $s\sigma \rightarrow_R^+ s\sigma'$. We define BCP as the set of all elimination patterns

$$t\sigma \leftarrow_R s\sigma[l\sigma] \rightarrow_R s\sigma[r\sigma] \Rightarrow t\sigma \rightarrow_R^* t\sigma' \leftarrow_R s\sigma' \rightarrow_R s\sigma'[r\sigma'] \leftarrow_R^* s\sigma[r\sigma],$$

where $s \rightarrow t$, $l \rightarrow r$, σ, and σ' are as described above. The set $BCP(E, R)$ contains all unblocked critical pairs in $CP(R)$.

PROPOSITION 2. *The unblocked criterion BCP is correct.*

Proof. It suffices to show that $\Rightarrow_{BCP}$ is contained in $>_c$. This is trivial, since each left-hand side of an elimination pattern of BCP contains a term $s\sigma$ that is bigger than all terms on the corresponding right-hand side. □

The proposition shows that unblocked critical pairs may be ignored by completion. The unblocked criterion BCP is a special case of both the connected criterion and the composite criterion discussed below.

5. Connectedness

Several critical pair criteria have been proposed that are based on the concept of *connectedness.*

DEFINITION 5. Let R be a rewrite system and $>$ be a reduction ordering. Two terms s and t are *connected* in $E \cup R$ below u (relative to $>$) if $s \leftrightarrow_{E \cup R} u_1 \leftrightarrow_{E \cup R} \cdots \leftrightarrow_{E \cup R} u_n \leftrightarrow_{E \cup R} t$, for some terms $u_1, \ldots, u_n$, $n \geq 0$, with $u > u_i$, for $1 \leq i \leq n$.

This concept was introduced in a more restricted form by Buchberger (1984) and can be readily utilized for a critical pair criterion. Completion can be viewed as a process of establishing, for every critical overlap $s \leftarrow_R u \rightarrow_R t$, the connectedness of s and t below the overlapped term u. For instance, adding the critical pair $s = t$ as an equation is one possible way of establishing connectedness. Conversely, if s and t are already connected, then the critical pair $s = t$ is superfluous. Thus, we define the set CCP as consisting of all elimination patterns, for $n \geq 0$, of the form

$$s \leftarrow_R u \rightarrow_R t \quad \Rightarrow \quad s \leftrightarrow_{E \cup R} u_1 \leftrightarrow_{E \cup R} \cdots u_n \leftrightarrow_{E \cup R} t,$$

where $>$ contains R and $u > u_i$, for $1 \leq i \leq n$. The set $CCP(E, R)$ contains all critical pairs in $CP(R)$ that are connected below their associated overlapped term.

PROPOSITION 3. *The connected criterion CCP is correct.*

Proof. It suffices to prove that $\Rightarrow_{CCP}$ is contained in the proof ordering $>_c$. Suppose that $P \Rightarrow_{CPC} P'$, where in P some peak $s \leftarrow_R u \rightarrow_R t$ is replaced by $u_0 \leftrightarrow_{E \cup R} \cdots \leftrightarrow_{E \cup R} u_{n+1}$, with u_0 being s, u_{n+1} being t, and $u > u_i$, for $0 \leq i \leq n+1$. The first component of the quadruple $c(s, u)$ is $\{u\}$, and the first component of $c(u_{i-1}, u_i)$ is $\{u_{i-1}, u_i\}$, $\{u_{i-1}\}$, or $\{u_i\}$. Since $u > u_i$, we have $c(s, u) >^c c(u_i, u_{i+1})$, for all i, $0 \leq i \leq n+1$. This implies $P >_c P'$. □

A criterion based on connectedness was first formulated by Buchberger (1979) for a completion-like algorithm for constructing canonical bases for polynomial ideals. This criterion has been adapted to completion by Winkler and Buchberger (1983), Winkler (1984, 1985), and Küchlin (1985, 1986a). Each criterion checks whether a critical pair is connected relative to the ordering $\rightarrow_R^+$ induced by R. The criteria differ in the respective tests used to ensure connectedness. We sketch the basic idea.

Suppose that $s \leftarrow_R u \rightarrow_R t$ is a critical overlap and that u reduces to a term v. Thus, we can decompose the original overlap into two peaks $s \leftarrow_R u \rightarrow_R v$ and $v \leftarrow_R u \rightarrow_R t$. If $s \leftarrow_R u \rightarrow_R v$ is no overlap or is a variable overlap, then s and v are connected below u. Otherwise, $s = v$ must involve an instance of a critical pair $s' = v'$. If this critical pair has already been computed, then s and v are also connected below u. Similar arguments

apply to v and t. Thus, connectedness can often be verified by checking whether certain critical pairs have already been computed. Various bookkeeping mechanisms to that end have been proposed by Küchlin (1985, 1986a) and Winkler (1985). The test described by Winkler restricts the position at which the rewrite step $u \to_R v$ may apply. No such restriction is imposed by Küchlin.

The emphasis in the papers cited above is on soundness and practicality. Winkler (1985) and Küchlin (1986a) also show the correctness of specific versions of completion that incorporate tests for connectedness. Winkler's proof is similar to the proof of correctness of standard completion in Huet (1981); Küchlin's proof is based on multiset induction. Both are quite complicated. Our correctness proof is not only considerably simpler, but also applies to a large class of completion procedures. The flexibility of our approach should be particularly helpful in establishing the correctness of other implementations of completion procedures and criteria.

6. Compositeness

A different type of criterion was suggested by Kapur, Musser, and Narendran (1985).

DEFINITION 6. Let R be a rewrite system and $t\sigma \leftarrow_R s\sigma[l\sigma] \to_R s\sigma[r\sigma]$ be a critical overlap between rules $s \to t$ and $l \to r$ in R. The critical pair $t\sigma = s\sigma[r\sigma]$ is called *composite* if some proper subterm of $l\sigma$ is reducible in R.

For example, suppose R contains rewrite rules $-(-x + y) \to -y + -(-x)$, $x + -x \to 0$, and $-(-x) \to x$. The first two rules define a critical overlap

$$-(-(-x)) + -(-x) \leftarrow_R -(-x + -(-x)) \to_R -0.$$

This overlap is composite, since the subterm $-(-x)$ of $-x + -(-x)$ is reducible.

Let PCP be the set of elimination patterns

$$s \leftarrow_R u \to_R t \;\Rightarrow\; s \leftarrow_R u \to_R v \leftarrow_R u \to_R t$$

where the rewrite step $u \to_R v$ applies strictly below $u \to_R s$, and $u \to_R s$ applies below $u \to_R t$. The rewrite relation $\Rightarrow_{PCP}$ induced by PCP can be used to eliminate composite overlaps. The set $PCP(E, R)$ consists of all composite critical pairs in $CP(R)$.

LEMMA 5 (Kapur, Musser, and Narendran, 1985). *The composite criterion PCP is sound.*

This result is also implied by:

PROPOSITION 4. *The composite criterion PCP is correct.*

Proof. Let $(E_0,R_0)\vdash(E_1,R_1)\vdash\cdots$ be a derivation fair relative to PCP. We show that whenever P is a proof in $E_i\cup R_i$, for some $i\geq 0$, then there is a rewrite proof P' in R^∞, such that $P\Rightarrow_C^+ P'$. This obviously implies that R^∞ has the Church-Rosser property.

Let P be a proof in $E_i\cup R_i$. We assume that the assertion is true for all proofs Q with $P\Rightarrow_C^+ Q$. Using fairness and Lemma 2, we may conclude that whenever P contains a non-persisting proof step or a non-critical overlap, then there is a proof Q in $E_j\cup R_j$, for some $j\geq i$, such that $P\Rightarrow_C^+ Q$. Suppose P contains a persisting critical overlap $s\leftarrow_{R_i} u\rightarrow_{R_i} t$. Thus, $s=t$ must involve some critical pair $s'=t'$ in $CP(R^\infty)$.

Suppose $s'=t'$ is not contained in $\cup_j PCP(E_j,R_j)$. By fairness, it is in some set E_k, which, by Lemma 2, implies $P\Rightarrow_C^+ Q$, for some proof Q in $E_j\cup R_j$, $j\geq i$. On the other hand, if $s'=t'$ is contained in some set $PCP(E_k,R_k)$, then the overlap $s\leftarrow_{R_k} u\rightarrow_{R_k} t$ can be decomposed into two peaks, $s\leftarrow_{R_k} u\rightarrow_{R_k} v$ and $v\leftarrow_{R_k} u\rightarrow_{R_k} t$. Since both peaks are smaller than P, they can, by the induction hypothesis, be transformed via $\Rightarrow_C^+$ into rewrite proofs $s\rightarrow_{R^\infty}^* w\leftarrow_{R^\infty}^* v$ and $v\rightarrow_{R^\infty}^* w'\leftarrow_{R^\infty}^* t$, respectively. The concatenation $s\rightarrow_{R^\infty}^* w\leftarrow_{R^\infty}^* v\rightarrow_{R^\infty}^* w'\leftarrow_{R^\infty}^* t$ of these two proofs is also smaller than P, hence can be transformed into a persisting rewrite proof. The overlap $s\leftarrow_{R_i} u\rightarrow_{R_i} t$ can therefore be replaced by a rewrite proof in R^∞. By Lemma 2, there is a proof Q in $E_j\cup R_j$, for some $j\geq i$, such that $P\Rightarrow_C^+ Q$.

Using the induction hypothesis, we may conclude that Q (and therefore P) can be transformed via $\Rightarrow_C^+$ into a rewrite proof P' in R^∞. □

The correctness of criterion PCP can also be proved by constructing a well-founded ordering $>_{PCP}$ that contains both $\Rightarrow_{PCP}$ and $\Rightarrow_C$.

Let P be an overlap $s\leftarrow_R u\rightarrow_R t$, and P' be $s\leftarrow_R u\rightarrow_R v\leftarrow_R u\rightarrow_R t$, where $u\rightarrow_R v$ by $l''\rightarrow r''$ at a position q strictly below p (see Figure 4). Since both P and P' contain the proof steps $u\rightarrow_R s$ and $u\rightarrow_R t$, we have $P'>_C P$! However, including a measure of the overlap between successive proof steps in the complexity of a proof allows us to distinguish between occurrences of these single proof steps in P and P', respectively, so that we can design a proof ordering $>_{PCP}$ wherein $P>_{PCP} P'$.

Let P be a proof $(s_0,\ldots,s_n)$ and p_i be the position of the i-th proof step (s_{i-1},s_i). By $M(P)$ we denote the multiset $\{d(s_0,s_1,P),\ldots,d(s_{n-1},s_n,P)\}$, where $d(s_{i-1},s_i,P)$ is

$(\{s_{i-1}\},s_{i-1}/p_i,l,s_i,s_{i-1}/p_{i-1})$, if $s_{i-1}\rightarrow_R s_i$ by $l\rightarrow r$
(where s_{i-1}/p_{i-1} is $-$, if i is 1);
$(\{s_i\},s_i/p_i,l,s_{i-1},s_i/p_{i+1})$, if $s_{i-1}\leftarrow_R s_i$ by $l\rightarrow r$
(where s_i/p_{i+1} is $-$, if i is n);
$(\{s_i,s_{i-1}\},-,-,-,-)$, if $s_{i-1}\leftrightarrow_E s_i$.

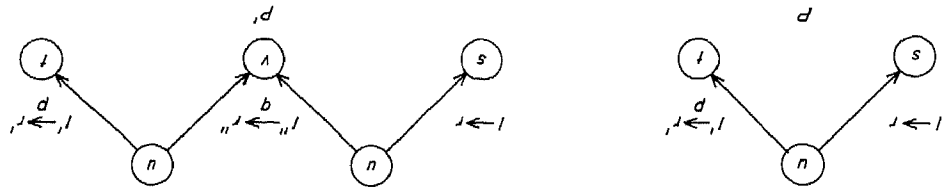

Figure 4. *Composite overlap*

The first four components of d are the same as for the complexity measure c. The additional fifth component reflects the amount of the overlap of a rewrite step with its neighboring step. The ordering $>^d$ is the lexicographic combination of the multiset extension $\gg$ of the reduction ordering $>$, the proper subterm ordering, the proper subsumption ordering $\rhd$, the reduction ordering $>$, and the proper subterm ordering. This ordering is well-founded and stable, but not monotonic. The ordering $>_{PCP}$, defined by: $P >_{PCP} P'$ *if and only if* $M(P) \gg^d M(P')$, contains both proof relations, $\Rightarrow_C$ and $\Rightarrow_{PCP}$. The proof of this fact is not difficult, but rather technical. Details can be found in Bachmair (1987).

The unblocked criterion BCP can also be regarded as a special case of compositeness, since any unblocked critical pair is composite. Furthermore, composite and connected criteria can be combined.

PROPOSITION 5. *The combined criterion* $CCP \cup PCP$ *is correct.*

Proof. The proof of Proposition 4 can be easily adapted to the criterion $CCP \cup PCP$. The correctness also follows from the fact that the rewrite relations $\Rightarrow_C$, $\Rightarrow_{CCP}$, and $\Rightarrow_{PCP}$ are all contained in $>_{PCP}$; hence (the transitive closure of) their union is well-founded. □

Experimental results that give some indication of the utility of critical pair criteria have been reported by Kapur, Musser and Narendran (1985)—for compositeness—and by Küchlin (1985)—for connectedness.

7. Summary

We have presented a general formalism for describing critical pair criteria for completion and have demonstrated that proof orderings provide a powerful tool for reasoning about completion with criteria. Proof ordering techniques facilitate relatively simple and intuitive correctness proofs and are useful for both designing and verifying critical pair criteria.

The approach described here can also be used in the more general context of rewriting modulo a congruence (Bachmair and Dershowitz, 1987a). For

instance, we have shown recently (Bachmair and Dershowitz, 1987b) that a restricted version of blocking can be used with the associative-commutative completion procedure of Peterson and Stickel (1981). Other critical pair criteria for rewriting modulo a congruence have been studied by Winkler (1984) and Küchlin (1986b).

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References

Bachmair, L. (1987). Proof methods for equational theories. Dissertation, Univ. of Illinois at Urbana-Champaign.

Bachmair, L., Dershowitz, N., and Hsiang, J. (1986). Orderings for equational proofs. *Proc. IEEE Symp. Logic in Computer Science,* Cambridge, Massachusetts, pp. 346-357.

Bachmair, L., and Dershowitz, N. (1987a). Completion for rewriting modulo a congruence. In: (Lescanne, P., ed.) Proc. Second Int. Conf. on Rewriting Techniques and Applications. *Springer Lec. Notes Comp. Sci.* **256**, 192-203.

Bachmair, L., and Dershowitz, N. (1987b). Critical pair criteria for rewriting modulo a congruence. To appear in *Proc. Eurocal '87.* Leipzig, German Democratic Republic.

Buchberger, B. (1979). A criterion for detecting unnecessary reductions in the construction of Gröbner bases. In: (Ng, W., ed.) Proc. Eurosam '79. *Springer Lec. Notes Comp. Sci.* **72**, 3-21.

Buchberger, B. (1984). A critical-pair/completion algorithm for finitely generated ideals in rings. In: (Boerger, E., et al., eds.) Proc. Symp. Rekursive Kombinatorik. *Springer Lec. Notes Comp. Sci.* **171**, 137-161.

Dershowitz, N. (1985). Computing with rewrite systems. *Inf. Control* **64**, 122-157.

Dershowitz, N., and Manna, Z. (1979). Proving termination with multiset orderings. *Comm. ACM* **22**, 465-476.

Dershowitz, N., Marcus, L., and Tarlecki, A. (1987). Existence, uniqueness, and construction of rewrite systems. To appear in *SIAM J. Comput.*

Fribourg, L. (1985). SLOG: A logic programming language interpreter based on clausal superposition and rewriting. *Proc. 1985 Symp. on Logic Programming,* Boston, Massachusetts, pp. 172-184.

Hsiang, J. (1985). Refutational theorem proving using term-rewriting systems. *Artif. Intell.* **25**, 255-300.

Huet, G. (1980). Confluent reductions: abstract properties and applications to term rewriting systems. *J. ACM* **27**, 797-821.

Huet, G. (1981). A complete proof of correctness of the Knuth and Bendix completion algorithm. *J. Comp. Syst. Sci.* **23**, 11-21.

Huet, G. and Hullot, J.M. (1982). Proofs by induction in equational theories with constructors. *J. Comp. Syst. Sci.* **25**, 239-266.

Hullot, J.M. (1980). A catalogue of canonical term rewriting systems. Tech. Rep. CSL-113, SRI International, Menlo Park, California.

Jouannaud, J.-P., and Kirchner, H. (1986). Completion of a set of rules modulo a set of equations. *SIAM J. Comput.* **15**, 1155-1194.

Kapur, D., Musser, D.R., and Narendran, P. (1985). Only prime superpositions need be considered in the Knuth-Bendix procedure. Unpublished manuscript, Computer Science Branch, Corporate Research and Development, General Electric, Schenectady, New York.

Knuth, D., and Bendix, P. (1970). Simple word problems in universal algebras. In: (Leech, J., ed.) *Computational Problems in Abstract Algebra,* pp. 263-297. Oxford: Pergamon Press.

Küchlin, W. (1985). A confluence criterion based on the generalised Newman lemma. In:

(Caviness, B., ed.) Proc. Eurocal '85. *Springer Lec. Notes Comp. Sci.* **204**, 390-399.

Küchlin, W. (1986a). A generalized Knuth-Bendix algorithm. Rep. 86-01, Dept. of Mathematics, ETH Zürich, Switzerland.

Küchlin, W. (1986b). Equational Completion by Proof Simplification. Report 86-02, Dept. of Mathematics, ETH Zürich, Switzerland.

Lankford, D., and Ballantyne, A. (1979). The refutation completeness of blocked permutative narrowing and resolution. In: (Joyner, W. H., Jr., ed.) *Proc. Fourth Workshop on Automated Deduction,* Austin, Texas, pp. 168-174.

Le Chenadec, P. (1986). *Canonical forms in finitely presented algebras.* London: Pitman.

Musser, D.L. (1980). On proving inductive properties of abstract data types. *Proc. 7th ACM Symp. on Principles of Programming Languages,* Las Vegas, Nevada, pp. 154-162.

Peterson, G. E., and Stickel, M. E. (1981). Complete sets of reductions for some equational theories. *J. ACM* **28**, 233-264.

Slagle, J. R. (1974). Automated theorem proving for theories with simplifiers, commutativity, and associativity. *J. ACM* **21**, 622-642.

Winkler, F. (1984). The Church-Rosser property in computer algebra and special theorem proving: An investigation of critical-pair/completion algorithms. Dissertation, Johannes Kepler Universität Linz, Austria.

Winkler, F. (1985). Reducing the complexity of the Knuth-Bendix completion algorithm: a 'unification' of different approaches. In: (Caviness, B., ed.) Proc. Eurocal '85. *Springer Lec. Notes Comp. Sci.* **204**, 378-389.

Winkler, F., and Buchberger, B. (1983). A criterion for eliminating unnecessary reductions in the Knuth-Bendix algorithm. *Proc. Coll. on Algebra, Combinatorics and Logic in Computer Science,* Györ, Hungary.