Exact Exploration

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June 10, 2009

In the Beginning

Heaven and Earth

Algorithm

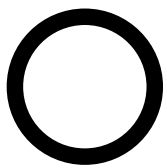
Program

Law and Order

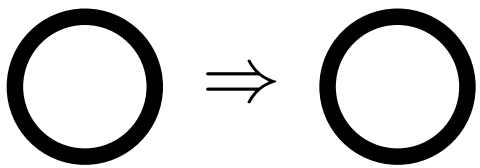
Postulates

ASM

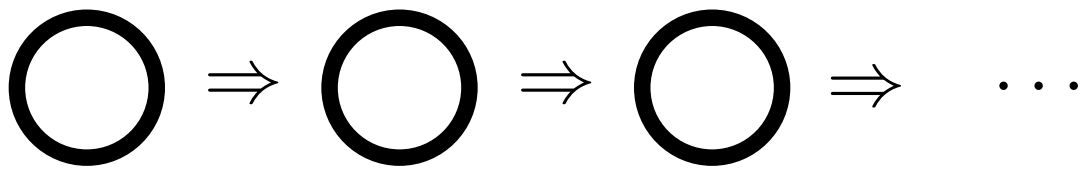
State



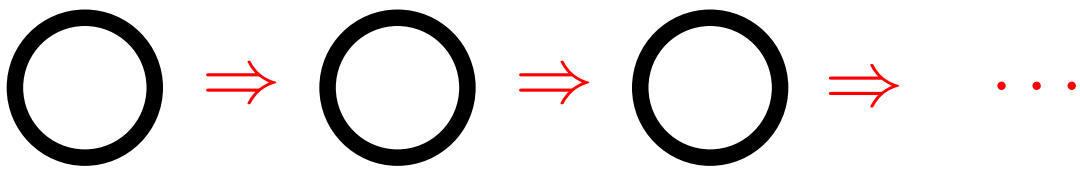
State Transition



Run



Emulation

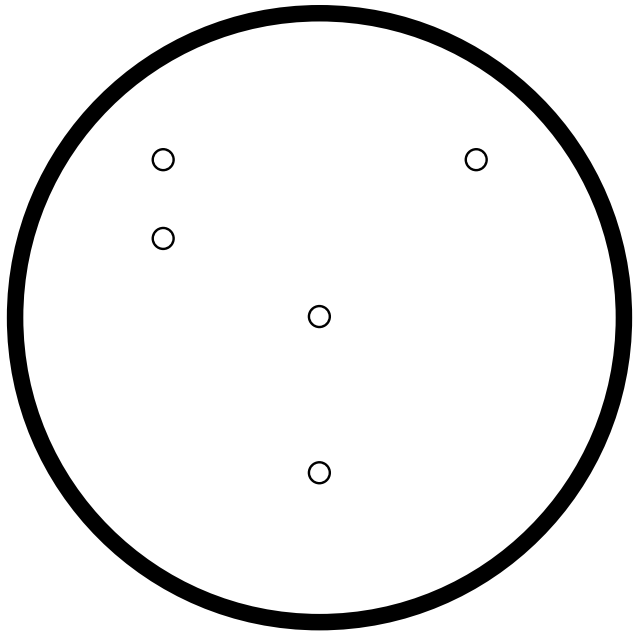


Sequential Time Postulate

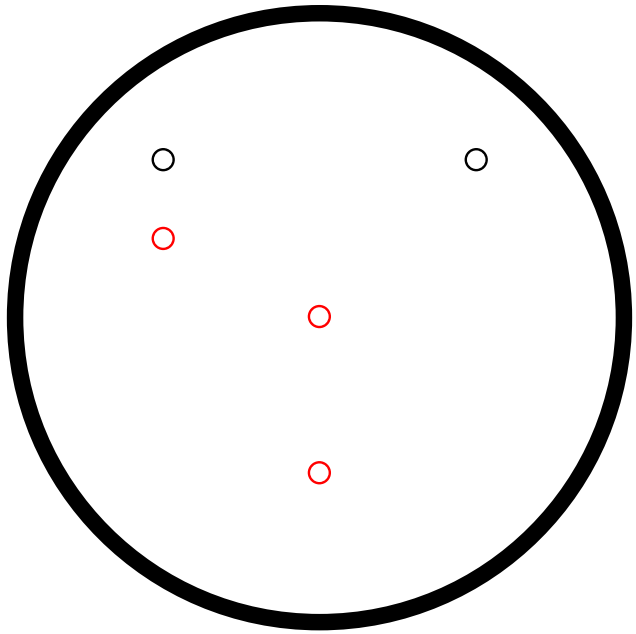
An algorithm determines:

1. A nonempty set $\mathcal{S}$ of *states*.
2. A nonempty subset $\mathcal{I} \subseteq \mathcal{S}$ of *initial states*.
3. A subset $\mathcal{O} \subseteq \mathcal{S}$ of *terminal states*.
4. A *next state* transformation $\tau : \mathcal{S} \setminus \mathcal{O} \to \mathcal{S}$.

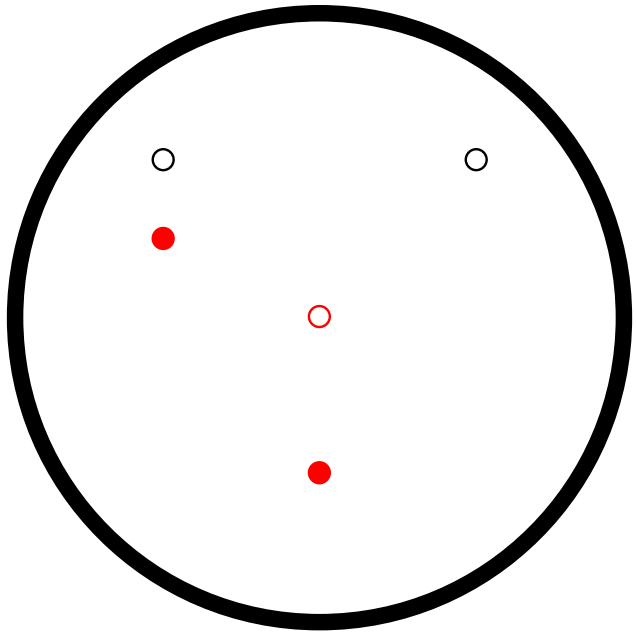
Locations



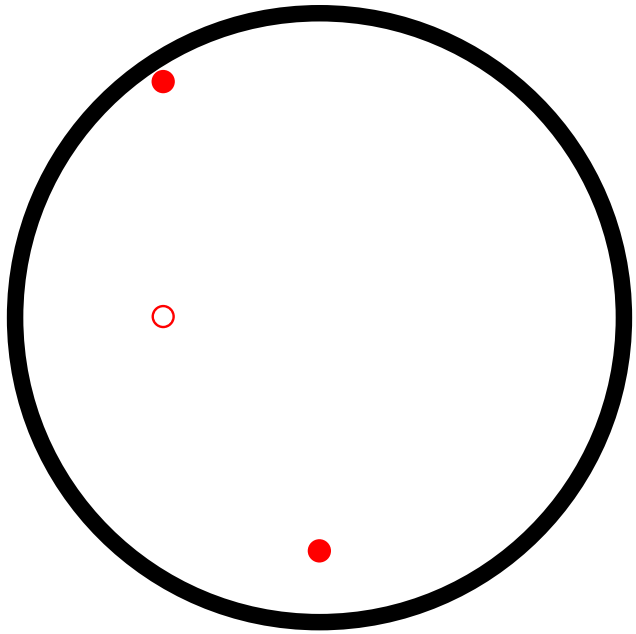
Exploration



Update



State Dependent



ASM Programs

One of the following:

- An *assignment* statement $f(s_1, \ldots, s_n) := t$.
- A *parallel* statement $[P_1 \parallel \cdots \parallel P_n]$, $n \geq 0$.
- A *conditional* statement **if** $C_1 \wedge \cdots \wedge C_n$ **then** P.

Sorting

$$
\begin{array}{l}
[\ \textbf{if}\ j \neq n\ \textbf{then} \\
\quad [\ \textbf{if}\ F(i) > F(j)\ \textbf{then} \\
\qquad [F(i) := F(j) \parallel F(j) := F(i) \parallel i := 0 \parallel j := 1\ ] \\
\quad \parallel \textbf{if}\ F(j) \geq F(i)\ \textbf{then}\ j := j + 1] \\
\parallel \textbf{if}\ j = n \wedge i + 1 \neq n\ \textbf{then}\ [i := i + 1 \parallel j := i + 2\ ]\ ]
\end{array}
$$

Sorting Reals

What happens when the $F(i)$ are real numbers?

It depends!

Sorting States

	State X such that
$\mathcal{I}$	$[\![n]\!] \geq 0,\ [\![i]\!] = 0,\ [\![j]\!] = 1$
$\mathcal{O}$	$[\![j]\!] = [\![n]\!] = [\![i]\!]$
B	$[\![j]\!] \neq [\![n]\!],\ [\![F(i)]\!] > [\![F(j)]\!]$
S	$[\![j]\!] \neq [\![n]\!],\ [\![F(i)]\!] \leq [\![F(j)]\!]$
R	$[\![j]\!] = [\![n]\!] \neq [\![i]\!] + 1$

Update Set

- $\Delta(X) = \tau(X) \setminus X$.
- $\Delta(X) = \perp$ for $X \in \mathcal{O}$.

Sorting Updates

	$\Delta(X)$
$\mathcal{O}$	$\bot$
B	$F([\![i]\!]) \mapsto [\![F(j)]\!],\ F([\![j]\!]) \mapsto [\![F(i)]\!], i \mapsto 0,\ j \mapsto 1$
S	$j \mapsto [\![j]\!] + 1$
R	$i \mapsto [\![i]\!] + 1,\ j \mapsto [\![i]\!] + 2$

$$
\begin{aligned}
\Delta^{+}_{f(s_1,\ldots,s_n):=t}(X) &= \{f([\![s_1]\!]x,\ldots,[\![s_n]\!]x) \mapsto [\![t]\!]x\} \\
\Delta^{+}_{[P_1 \parallel \cdots \parallel P_n]}(X) &= \Delta^{+}_{P_1}(X) \cup \cdots \cup \Delta^{+}_{P_n}(X) \\
\Delta^{+}_{\textbf{if } C \textbf{ then } P}(X) &= \begin{cases} \Delta^{+}_{P}(X) & \text{if } X \models C \\ \varnothing & \text{otherwise} \end{cases}
\end{aligned}
$$

$$\Delta^0(X) = \{f(\bar{a}) \mapsto [\![f(\bar{a})]\!]_X : \bar{a} \in \mathrm{Dom}\ X\}$$

$$\Delta(X) = \begin{cases} \bot & \text{if } \Delta^+(X) = \varnothing \text{ or is inconsistent} \\ \Delta^+(X) \setminus \Delta^0(X) & \text{otherwise} \end{cases}$$

Abstract State Postulate

The states $\mathcal{S}$ are structures over a finite vocabulary $\mathcal{F}$ such that:

1. If $X \in \mathcal{S}$ and $X \cong Y$, then $Y \in \mathcal{S}$.
2. If $X \in \mathcal{I}$ and $X \cong Y$, then $Y \in \mathcal{I}$.
3. If $X \in \mathcal{O}$ and $X \cong Y$, then $Y \in \mathcal{O}$.
4. $\mathrm{Dom}\ \tau(X) = \mathrm{Dom}\ X$ for all $X \in \mathcal{S} \setminus \mathcal{O}$.
5. If $X \cong_\zeta Y$ for $X, Y \in \mathcal{S} \setminus \mathcal{O}$, then $\tau(X) \cong_\zeta \tau(Y)$.

Bounded Exploration Postulate

There is a finite set of terms T, such that, for $X, Y \in \mathcal{S}$

$$\text{if } X =_T Y \text{ then } \Delta(X) = \Delta(Y)$$

Critical Terms

$$j = n$$

$$j \neq n$$

$$i + 1 \neq n$$

$$F(i) > F(j)$$

$$F(j) \geq F(i)$$

$$0,\ 1$$

$$j + 1,\ i + 1,\ i + 2$$

plus subterms

ASM Theorem

$$\textbf{if } (F(i) > F(j)) = \text{true} \wedge j = n \wedge i + 1 \neq n \textbf{ then } j := i + 2$$

ASM Theorem

Every algorithm satisfying these postulates is emulated by an ASM.

The Problem

The ASM may do too much!

Exact Terms

$j = n$

$j \neq n$

$i + 1 \neq n$

$F(i) > F(j)$

$F(j) \geq F(i)$

$0,\ 1$

$j + 1,\ i + 1,\ i + 2$

plus subterms

Gaussian Elimination

- Tests pivot $p \neq 0$ before dividing $a[i, j]/p$.
- Program includes expression $a[i, j]/p$.
- Emulating ASM evaluates $a[i, j]/p$, regardless of p.
- Want ASM that only explores $a[i, j]/p$ when $p \neq 0$.

Matrix Inversion

- Reals; computable reals; complex numbers.
- First compute adjugate.
- Then divide through by determinant.
- No need for equality or disequality.
- Only if $\det A = 0$ is $A^{-1} = \bot$.

Parsimony

- Remove duplicates from a file system.
- First check that sizes are the same.
- ASM always compares *both* size and content!
- Want ASM to only check what algorithm does.

Exact Exploration

There are finite sets of terms $\Gamma(X)$, such that, for $Y \in \mathcal{S}$

$$\text{if } X =_{\Gamma(X)} Y \text{ then } \Delta(X) = \Delta(Y)$$

Exact Exploration

There are finite sets of terms $\Gamma(X)$, such that, for $Y \in \mathcal{S}$

$$\text{if } X =_{\Gamma(X)} Y \text{ then } \Gamma(X) = \Gamma(Y)$$

Explore Sets

	$\Gamma(X)$ (omitting subterms)
$\mathcal{O}$	$j = n,\ j \neq n,\ i+1 \neq n$
B	$j = n,\ j \neq n,\ i+1 \neq n,\ F(i) > F(j),\ F(j) \geq F(i),\ 0,\ 1$
S	$j = n,\ j \neq n,\ i+1 \neq n,\ F(j) > F(i),\ F(j) \geq F(i),\ j+1$
R	$j = n,\ j \neq n,\ i+1 \neq n, i+1,\ i+2$

Exact Terms

$$
\begin{aligned}
\Gamma_{[P_1 \parallel \cdots \parallel P_n]}(X) &= \bigcup_j \Gamma_{P_j}(X) \\
\Gamma_{f(s_1,\ldots,s_n):=t}(X) &= \bigcup_i \Gamma_{s_i}(X) \cup \Gamma_t(X) \\
\Gamma_{f(s_1,\ldots,s_n)}(X) &= \{f(s_1,\ldots,s_n)\} \cup \bigcup_i \Gamma_{s_i}(X) \\
\Gamma_{\textbf{if } C \textbf{ then } P}(X) &= \Gamma_C(X) \cup \begin{cases} \Gamma_P(X) & \text{if } X \models C \\ \varnothing & \text{otherwise} \end{cases} \\
\Gamma_{c_1 \wedge \cdots \wedge c_k}(X) &= \bigcup_j \Gamma_{c_j}(X) \\
\Gamma_{\neg c}(X) &= \Gamma_c(X)
\end{aligned}
$$

Exact Exploration

An algorithm determines a partially-ordered finite explore set $(\Gamma(X), \prec_X)$ of ground terms such that:

- **Determination.** For every other state $Y \in \mathcal{S}$, if $X =_{\Gamma(X)} Y$, then $\Delta(X) = \Delta(Y)$.
- **Discrimination.** For every other state $Y \in \mathcal{S}$, and for every $t \in \Gamma(X) \setminus \Gamma(Y)$, there is a Boolean term $s \in \Gamma(X)$ such that $s \prec_X t$ and $[\![s]\!]_X \neq [\![s]\!]_Y$.
- **Limitation.** The set $\cup_{X \in \mathcal{S}} \Gamma(X)$ of all explore terms is finite.

Exploration Order

	Exploration order $\prec_X$ (omitting subterms)
$\mathcal{O}$	
B	$j \neq n \prec_X F(i) > F(j) \prec_X 0,\ 1;\quad j \neq n \prec_X F(j) \geq F(i)$
S	$j \neq n \prec_X F(j) \geq F(i) \prec_X j+1;\quad j \neq n \prec_X F(j) > F(i)$
R	$i+1 \neq n,\ j = n \prec_X i+1,\ i+2$

$$
\begin{array}{l}
[\ \textbf{if } d \textbf{ then if } b \textbf{ then if } c \textbf{ then } x := 0 \ \| \\
\quad \textbf{if } d \textbf{ then if } \neg b \textbf{ then } x := 1 \ \| \\
\quad \textbf{if } d \textbf{ then if } \neg c \textbf{ then } y := 2 \ ]
\end{array}
$$

Magic Program

$$[\ \textbf{if}\ b\ \textbf{then}\ [b := b \parallel c := c] \quad \parallel \quad \textbf{if}\ \neg b\ \textbf{then}\ a := a\ ]$$

whenever a is true

$$[\ \textbf{if}\ c\ \textbf{then}\ [b := b \parallel c := c] \quad \parallel \quad \textbf{if}\ \neg c\ \textbf{then}\ a := a\ ]$$

whenever a is false

Exact Exploration implies Bounded Exploration

- Let $T = \cup_{X \in \mathcal{S}} \Gamma(X)$ be all the explore terms of the algorithm in question.
- By Limitation, T is finite.
- By Determination, T determines behavior.

Bounded Exploration implies Exact Exploration

- Let $\Gamma(X) = T$.
- Let $\prec_X$ be empty.

Equivalence

Two algorithms $\mathcal{P}$ and $\mathcal{Q}$ are *equivalent* if

- they operate over the same states $\mathcal{S}$,
- have the same initial states $\mathcal{I}$ and terminal states $\mathcal{O}$,
- for all states $X \in \mathcal{S}$, $\Delta_{\mathcal{P}}(X) = \Delta_{\mathcal{Q}}(X)$, and
- for all states $X \in \mathcal{S}$, $\Gamma_{\mathcal{P}}(X) = \Gamma_{\mathcal{Q}}(X)$.

Sets of States

- $\Gamma(V) = \cap_{X \in V} \Gamma(X)$.
- C is *agreeable* if $X =_{\Gamma(V)} Y$ for all $X, Y \in V$.
- Agreeable states should have uniform behavior!

Uniformity

- A set V of states is *uniform* if $\Gamma(X) = \Gamma(Y)$ for all $X, Y \in V$.
- When V is agreeable, then it should also be uniform.

For any discriminating algorithm, agreeability of a set of states implies its uniformity

- Suppose not all states in V have the same explore set.
- Let $t \in \Gamma(X)$ be a minimal explore term of some $X \in V$ that is not also an explore term of all other states.
- Let $Y \in V$ be such that $t \notin \Gamma(Y)$.
- By Discrimination, there is an $s \in \Gamma(X)$ such that $s \prec_X t$ and $[\![s]\!]_X \neq [\![s]\!]_Y$.
- By agreeability, $s \notin \Gamma(V)$.
- Then s must be a smaller choice of an explore term of X than is t, since $s \notin \Gamma(Z)$ for some $Z \in V$.

For any classical algorithm, if every agreeable set of states is uniform, then it's discriminating

$\prec_X := \varnothing$
$V := \mathcal{S}$
while $\Gamma(V) \subsetneq \Gamma(X)$ **do**
  $\prec_X := \prec_X \cup\ \Gamma(V) \times (\Gamma(X) \setminus \Gamma(V))$
  $V := \{Y \in V : Y =_{\Gamma(V)} X\}$

- Consider any $t \in \Gamma(X) \setminus \Gamma(Y)$ for a $Y \in \mathcal{S}$.
- Initially, $t \notin \Gamma(V) = \Gamma(\mathcal{S})$.
- At the end, $t \in \Gamma(V) = \Gamma(X)$.
- So Y is not in the final V.
- When Y is removed from V, there must be an $s \in \Gamma(V)$ that discriminates between X and Y.
- By construction, $s \prec_X t$.

Every ASM program is a discriminating algorithm

- First two postulates hold for ASMs.
- Limitation holds, since programs are finite and all terms in $\Gamma(X)$ appear there.
- If $X =_{\Gamma(X)} Y$, then all tests performed by the algorithm have the same outcome in Y as in X.
- So $\Gamma^+(X) = \Gamma^+(Y)$, and, hence, $\Gamma(X) = \Gamma(Y)$.
- The algorithm for explore sets returns $\Gamma(V)$.
- If V is agreeable, then whenever some state in V does not satisfy C, none do.
- So the computation of $\Gamma(V)$, for agreeable V, proceeds precisely as does the computation of $\Gamma(X)$ for any $X \in V$.
- So agreeability of V implies its uniformity.

Assignments

For every discriminating algorithm, if $f(\bar{a}) \mapsto b$ is an update in $\Delta(X)$ for some state X, then there are terms t and $\bar{s}$ in $\Gamma(X)$ such that $[\![t]\!]_X = b$ and $[\![s_i]\!]_X = a_i$ for each a_i of $\bar{a}$.

Each discriminating algorithm has an equivalent ASM

ASM is $P(\mathcal{S})$, defined as follows:

$$P(V) = \begin{cases} [\,] & \text{if } V = \varnothing \\ [\textbf{if } C_1 \textbf{ then } P(V \restriction C_1) \parallel \cdots \parallel \textbf{if } C_k \textbf{ then } P(V \restriction C_k)] & \\ & \text{if } V \text{ is not agreeable} \\ \textbf{if } C \textbf{ then } R & \text{if } V \text{ is agreeable} \end{cases}$$

C is a conjunction of all the Boolean terms $c \in \Gamma(V)$, or their negations $\neg c$, depending on whether $V \models c$ or $V \models \neg c$.
R is a parallel collection of assignments for all the updates in $\Delta(X)$, for any one state $X \in V$.

Case Statements

$$\textbf{case } q_1, \ldots, q_n \textbf{ of } \begin{cases} \textbf{when } a_{11}, \ldots, a_{1n} \textbf{ then} & S\ ; \\ \textbf{when } a_{21}, \ldots, a_{2n} \textbf{ then} & R\ ; \\ \quad \vdots & \\ \textbf{when } a_{m1}, \ldots, a_{mn} \textbf{ then} & S_m \end{cases}$$

More than Boolean

- **Discrimination.** *For every other state* $Y \in \mathcal{S}$*, and for every* $t \in \Gamma(X) \setminus \Gamma(Y)$*, if any, there is a discrimination term* $s \in \Gamma(X)$ *such that* $s \prec_X t$ *and* $[\![s]\!]_Y \neq [\![s]\!]_X \in \mathbf{K}$.

$$\Delta^{+}_{\textbf{case } \bar{q} \textbf{ of } \overline{W}}(X) = \bigcup_j D^{\bar{q}}_{W_j}(X)$$

$$D^{\bar{q}}_{\textbf{when } \bar{a} \textbf{ then } P}(X) = \begin{cases} \Delta^{+}_{P}(X) & \text{if } X \models \bar{q} = \bar{a} \\ \{\bullet\} & \text{if } \bar{a} \notin K^n \\ \varnothing & \text{otherwise} \end{cases}$$

$$\Gamma_{\textbf{case } \bar{q} \textbf{ of } \overline{W}}(X) = \{q_1, \ldots, q_n\} \cup \bigcup_j G^{\bar{q}}_{W_j}(X)$$

$$G^{\bar{q}}_{\textbf{when } \bar{a} \textbf{ then } P}(X) = \begin{cases} \Gamma_{P}(X) & \text{if } X \models \bar{q} = \bar{a} \\ \varnothing & \text{otherwise} \end{cases}$$

$$\Delta(X) = \begin{cases} \bullet & \text{if } \bullet \in \Delta^{+}(X) \\ \bot & \text{if } \Delta^{+}(X) = \varnothing \text{ or is inconsistent} \\ \Delta^{+}(X) \setminus \Delta^{0}(X) & \text{otherwise} \end{cases}$$

Black Holes

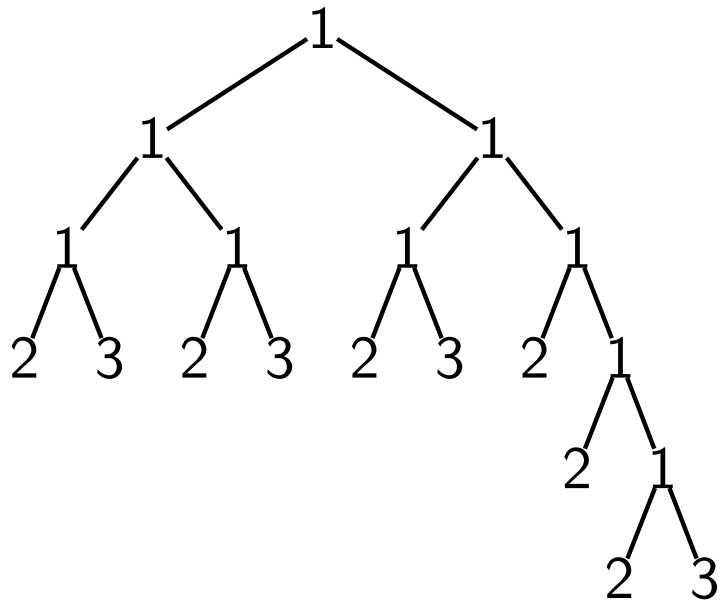
- $\Delta(X) = \tau(X) \setminus X$.
- $\Delta(X) = \perp$ for $X \in \mathcal{O}$.
- $\Delta(X) = \bullet$ for $X \in \mathcal{O}$, but don't know!

Refinement

1. Critical terms are used in the conditional tests of **if** statements and in the queries of the **case** statement.
2. The contents of location indicated by right-hand side of assignement is copied into another location.
3. Critical terms are used indirectly to determine locations needed for tests or updates.

Partition $\Gamma(X)$ into three parts:

1. $\Gamma^D(X)$ for the discriminating terms used in conditional and case statements;
2. $\Gamma^C(X)$ for obtaining the contents of locations indicated by right-hand sides of assignments;
3. $\Gamma^A(X)$ for addressing locations.


1
1
1
1
1
1
1
1
2
3
2
3
2
3
2
1
2
1
2
3

Flat ASMs

$$
\left[\begin{array}{l}
\textbf{if } C_{11} \textbf{ then if } C_{12} \textbf{ then } \cdots \textbf{ then } f_1(s_{11}, \ldots, s_{1n_1}) := t_1 \ \| \\
\textbf{if } C_{21} \textbf{ then if } C_{22} \textbf{ then } \cdots \textbf{ then } f_2(s_{21}, \ldots, s_{2n_2}) := t_2 \ \| \\
\textbf{if } C_{31} \textbf{ then if } C_{32} \textbf{ then } \cdots \textbf{ then } [] \ \| \\
\vdots \\
\textbf{if } C_{k1} \textbf{ then if } C_{k2} \textbf{ then } \cdots \textbf{ then } f_k(s_{k1}, \ldots, s_{kn_k}) := t_k
\end{array}\right]
$$