Problem #91

Originator: Friedrich Otto [OSKM98]
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Summary: Does every automatic group have a presentation through some finite convergent string-rewriting system?

Does every automatic monoid have an automatic structure such that the set of representatives is a prefix-closed cross-section?

For a finite alphabet Σ, we define the *padded extension* $\Sigma_{\#}$ of Σ as

$$\Sigma_{\#} := ((\Sigma \cup \{\#\}) \times (\Sigma \cup \{\#\})) \setminus \{(\#, \#)\},$$

where # is an additional symbol. A mapping $\nu : \Sigma^* \times \Sigma^* \to \Sigma_{\#}^*$ is then used to encode pairs of strings from Σ^* as strings from $\Sigma_{\#}^*$ as follows:
if $u := a_1 a_2 \cdots a_n$ and $v := b_1 b_2 \cdots b_m$, where $a_1, \ldots, a_n, b_1, \ldots, b_m \in \Sigma$, then

$$\nu(u, v) := \begin{cases} (a_1, b_1)(a_2, b_2) \cdots (a_m, b_m)(a_{m+1}, \#) \cdots (a_n, \#), & \text{if } m < n, \\ (a_1, b_1)(a_2, b_2) \cdots (a_m, b_m), & \text{if } m = n, \\ (a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)(\#, b_{n+1}) \cdots (\#, b_m), & \text{if } m > n. \end{cases}$$

Now a subset $L \subseteq \Sigma^* \times \Sigma^*$ is called *synchronously regular*, *s-regular* for short, if $\nu(L) \subseteq \Sigma_{\#}^*$ is accepted by some finite state acceptor (fsa).

An *automatic structure* for a finitely generated monoid-presentation $(\Sigma; R)$ consists of a fsa W over Σ, a fsa $M_=$ over $\Sigma_{\#}$, and fsa's M_a $(a \in \Sigma)$ over $\Sigma_{\#}$ satisfying the following conditions:

1. $L(W) \subseteq \Sigma^*$ is a complete set of (not necessarily unique) representatives for the monoid M_R presented by $(\Sigma; R)$, that is, $L(W) \cap [w]_R \neq \emptyset$ holds for each $w \in \Sigma^*$,
2. $L(M_=) = \{\nu(u, v) \mid u, v \in L(W) \text{ and } u \leftrightarrow_R^* v\}$, and
3. for all $a \in \Sigma$, $L(M_a) = \{\nu(u, v) \mid u, v \in L(W) \text{ and } ua \leftrightarrow_R^* v\}$.

A monoid-presentation is called *automatic* if it admits an automatic structure, and a monoid is called *automatic* if it has an automatic presentation.

Groups with automatic structure have been investigated thoroughly [Eps92], while the automatic monoids have been investigated only recently [CRRT96]. It is known that there exists monoids (in fact, groups) that can be presented through finite convergent string-rewriting systems, but that are not automatic [Ger92a].

QUESTION 1: Does every automatic group have a presentation through some finite convergent string-rewriting system?

For monoids in general the answer is negative as proved by an example given in [OSKM98].

If $(W, M_=, M_a(a \in \Sigma))$ is an automatic structure for a monoid-presentation $(\Sigma; R)$, then the language $L(W)$ contains one or more strings from every congruence class $[w]_R(w \in \Sigma^*)$. Actually, it can be required without loss of generality that $L(W)$ is a *cross-section* for $(\Sigma; R)$, that is, it contains exactly one string from every congruence class [Eps92].

Instead of requiring uniqueness one can also transform the given automatic structure in such a way as to obtain one for which the set of representatives is prefix-closed. However, the following question is still open.

QUESTION 2: Does every automatic monoid have an automatic structure such that the set of representatives is a prefix-closed cross-section?

Gersten stated this question for the special case of groups [Ger92b]. If the language $L(W)$ is a prefix-closed cross-section, then there exists an s-regular convergent prefix-rewriting system P on Σ such that the right-congruence generated by P coincides with the congruence generated by R, and $L(W)$ coincides with the set of irreducible strings mod P. Conversely, if a monoid-presentation admits an s-regular convergent prefix-rewriting system, then it has an automatic structure $(W, M_=, M_a(a \in \Sigma))$ such that the set $L(W)$ is a prefix-closed cross-section. Thus, QUESTION 2 can be reformulated as follows.

QUESTION 2 (restated): Does every finitely presented automatic monoid admit an s-regular convergent prefix-rewriting system?

For additional information on monoid-presentations and convergent string-rewriting systems see e.g. [BO93], and for the notion of prefix-rewriting systems see e.g. [KM89].

Bibliography

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