An Improved Bound on the Number of Unit Area Triangles*

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Abstract

We show that the number of unit-area triangles determined by a set of n points in the plane is $O(n^{9/4+\varepsilon})$, for any $\varepsilon > 0$, improving the recent bound $O(n^{44/19})$ of Dumitrescu et al.

1 Introduction

In 1967, A. Oppenheim (see [6]) asked the following question: Given n points in the plane and $A > 0$, how many triangles spanned by the points can have area A? By applying an affine transformation, one may assume $A = 1$ and count the triangles of *unit* area. Erdős and Purdy [5] showed that a $\sqrt{\log n} \times (n/\sqrt{\log n})$ section of the integer lattice determines $\Omega(n^2 \log\log n)$ triangles of the same area. They also showed that the maximum number of such triangles is at most $O(n^{5/2})$. In 1992, Pach and Sharir [8] improved the bound to $O(n^{7/3})$, using the Szemerédi-Trotter theorem [10] on the number of point-line incidences. Recently, Dumitrescu et al. [4] have further improved the upper bound to $O(n^{44/19}) = O(n^{2.3158})$, by estimating the number of incidences between the given points and a 4-parameter family of quadratic curves.

In this paper we further improve the bound to $O(n^{9/4+\varepsilon})$, for any $\varepsilon > 0$. Our proof borrows some ideas from [4], but works them into a different approach, which reduces the problem to bounding the number of incidences between points and certain kind of surfaces in three dimensions.

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2 Unit-area triangles in the plane

To simplify the notation, we write $O^*(f(n))$ for an upper bound of the form $C_\varepsilon f(n) \cdot n^\varepsilon$, which holds for any $\varepsilon > 0$, where the constant of proportionality C_ε depends on ε.

Our main result is the following result.

Theorem 2.1. *The number of unit-area triangles spanned by n points in the plane is $O^*(n^{9/4})$.*

Proof. We begin by borrowing some notation and preliminary ideas from [4]. Let S be the given set of n points in the plane. Consider a triangle $\Delta = \Delta abc$ spanned by S. We call the three lines containing the three sides of Δabc, *base lines* of Δ, and the three lines parallel to the base lines and incident to the respective third vertices, *top lines* of Δ.

For a parameter k, $1 \le k \le \sqrt{n}$, to be optimized later, call a line ℓ *k-rich* (resp., *k-poor*) if ℓ contains at least k (resp., fewer than k) points of S. Call a triangle Δabc *k-rich* if each of its three top lines is k-rich; otherwise Δ is *k-poor*.

We first observe that the number of k-poor unit-area triangles spanned by S is $O(n^2 k)$. Indeed, assign a k-poor unit-area triangle Δabc whose top line through c is k-poor to the opposite base ab. Then all the triangles assigned to a base ab are such that their third vertex lies on one of the two lines parallel to ab at distance $2/|ab|$, where that line contains fewer than k points of S. Hence, a base ab can be assigned at most $2k$ triangles, and the bound follows.

So far, the analysis follows that of [4]. We now focus the analysis on the set of k-rich unit-area triangles spanned by S, and use a different approach.

Let L denote the set of k-rich lines, and let Q denote the set of all pairs

$$\{(\ell, p) \mid \ell \in L,\, p \in S \cap \ell\}.$$

By the Szemerédi-Trotter theorem [10], we have, for any $k \le \sqrt{n}$, $m := |L| = O(n^2/k^3)$, and $N := |Q| = O(n^2/k^2)$.

A pair (ℓ_1, p_1), (ℓ_2, p_2) of elements of Q is said to *match* if the triangle with vertices p_1, p_2, $\ell_1 \cap \ell_2$ has area 1; see Figure 1.

To upper bound the number of unit-area triangles, all of whose three top lines are k-rich, it suffices to bound the number of matching pairs in Q. Indeed, given such a unit-area triangle $\Delta p_1 p_2 q$, let ℓ_1 (resp., ℓ_2) be the top line of $\Delta p_1 p_2 q$ through p_1 (resp., through p_2). Then (ℓ_1, p_1) and (ℓ_2, p_2) form a matching pair in Q, by definition (again, see Figure 1). Conversely, a matching pair (ℓ_1, p_1), (ℓ_2, p_2) determines at most one unit-area triangle $p_1 p_2 q$, where q is the intersection point of the line through p_1 parallel to ℓ_2 and the line through p_2 parallel to ℓ_1; we get an actual triangle if and only if the point q belongs to S.

In other words, our problem is now reduced to that of bounding the number of matching pairs in Q. (Since we do not enforce the condition that the third point q of the corresponding triangle belong to S, we most likely over-estimate the true bound.)

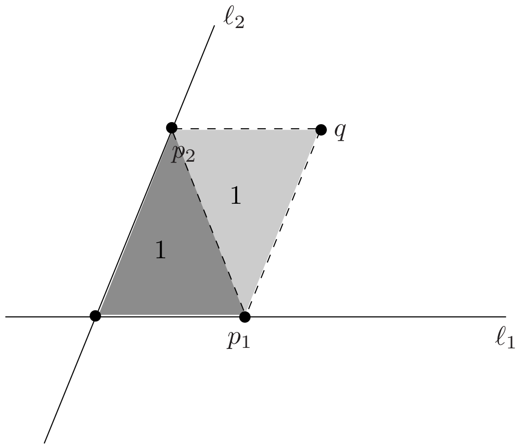

Figure 1: The ordered pair $((\ell_1, p_1), (\ell_2, p_2))$ is a matching pair of elements of Q.

Since elements of Q have three degrees of freedom, we can represent them in an appropriate 3-dimensional parametric space. For example, we can assume that no line in L is vertical, and parametrize an element (ℓ, p) of Q by the triple (a, b, κ), where (a, b) are the coordinates of p, and κ is the slope of ℓ. For simplicity of notation, we refer to this 3-dimensional parametric space as $\mathbb{R}^3$.

So far, the matching relationship is symmetric. To simplify the analysis, and with no loss of generality, we make it assymmetric, by requiring that, in an (ordered) matching pair (ℓ_1, p_1), (ℓ_2, p_2), $\vec{op}_2$ lies counterclockwise to $\vec{op}_1$, where $o = \ell_1 \cap \ell_2$. See Figure 1.

Let us express the matching condition algebraically. Let $(a, b, \kappa) \in \mathbb{R}^3$ be the triple representing a pair (ℓ, p), and $(x, y, w) \in \mathbb{R}^3$ be the triple representing another pair (ℓ', p'). Clearly, $w \neq \kappa$ in a matching pair. The lines ℓ and ℓ' intersect at a point o, for which there exist real parameters t, s which satisfy

$$o = (a + t, b + \kappa t) = (x + s, y + ws),$$

or

$$\begin{aligned} t &= \frac{y - b - w(x - a)}{\kappa - w} \\ s &= \frac{y - b - \kappa(x - a)}{\kappa - w}. \end{aligned}$$

It is now easy to verify that the condition of matching, with $\vec{op'}$ lying counterclockwise to $\vec{op}$, is given by

$$\Big(y - b - \kappa(x - a)\Big)\Big(y - b - w(x - a)\Big) = 2(w - \kappa) \quad \text{and} \quad w \neq \kappa,$$

or, alternatively,

$$w = \frac{\Big(y - b - \kappa(x-a)\Big)(y-b) + 2\kappa}{\Big(y - b - \kappa(x-a)\Big)(x-a) + 2} \quad \text{and} \quad w \neq \kappa. \tag{1}$$

Fix an element (ℓ, p) of Q, and associate with it a surface $\sigma_{\ell,p} \subset \mathbb{R}^3$, which is the locus of all pairs (ℓ', p') that match (ℓ, p) (i.e., $(\ell, p), (\ell', p')$ is an ordered matching pair). By the preceding analysis, $\sigma_{\ell,p}$ satisfies (1), where (a, b, κ) is the parametrization of (ℓ, p), and is thus a 2-dimensional algebraic surface in $\mathbb{R}^3$ of degree 3. We thus obtain a system Σ of N 2-dimensional algebraic surfaces in $\mathbb{R}^3$, and a set Q of N points in $\mathbb{R}^3$, and our goal is to bound the number of incidences between Q and Σ.

The main technical step in the analysis is to rule out the possible existence of *degeneracies* in the incidence structure, where many points are incident to many surfaces; this might happen when many points lie on a common curve which is contained in many surfaces (a situation which might arise, e.g., in the case of planes and points in $\mathbb{R}^3$). However, for the class of surfaces under consideration, namely, the surfaces $\sigma_{\ell,p}$ generated by some line-point incidence pair (ℓ, p), such a degeneracy is impossible, as the following lemma shows.

Lemma 2.2. *Let (ℓ_1, p_1) and (ℓ_2, p_2) be two line-point incidence pairs, let $\gamma = \sigma_{\ell_1,p_1} \cap \sigma_{\ell_2,p_2}$ be the intersection curve of their associated surfaces, and assume that γ is non-empty. Let (ℓ, p) be some incidence pair and assume further that $\sigma_{\ell,p} \supset \gamma$. Then either $(\ell, p) = (\ell_1, p_1)$ or $(\ell, p) = (\ell_2, p_2)$.*

Proof. We establish the equivalent claim that, given a curve γ, which is the intersection of some unknown pair of surfaces σ_{ℓ_1,p_1} and σ_{ℓ_2,p_2}, one can reconstruct (ℓ_1, p_1) and (ℓ_2, p_2) uniquely (up to a swap between the two incidence pairs) from γ. Morever, it is enough to know the projection γ^* of γ onto the xy-plane in order to uniquely reconstruct the incidence pairs (ℓ_1, p_1) and (ℓ_2, p_2) that generated it.

We start by computing the algebraic representation of γ^*. Let (a_1, b_1, κ_1) and (a_2, b_2, κ_2) be the respective parametrizations of (ℓ_1, p_1) and (ℓ_2, p_2). By (1), γ^* satisfies the equation

$$\frac{\Big(y - b_1 - \kappa_1(x-a_1)\Big)(y-b_1) + 2\kappa_1}{\Big(y - b_1 - \kappa_1(x-a_1)\Big)(x-a_1) + 2} = \frac{\Big(y - b_2 - \kappa_2(x-a_2)\Big)(y-b_2) + 2\kappa_2}{\Big(y - b_2 - \kappa_2(x-a_2)\Big)(x-a_2) + 2}. \tag{2}$$

Recall the additional requirement in (1), namely that $w \neq \kappa_1$ and $w \neq \kappa_2$. This requirement is implicit in (1) and in (2), meaning that equation (2) is defined only for values of x and y for which the value of w is not any of κ_1 or κ_2. Consulting (1), this says that (x, y) cannot satisfy $y - b_1 = \kappa_1(x - a_1)$ or $y - b_2 = \kappa_2(x - a_2)$. Put

$$\begin{aligned} L_1 &= y - b_1 - \kappa_1(x - a_1), \quad \text{and} \\ L_2 &= y - b_2 - \kappa_2(x - a_2), \end{aligned}$$

and write (2) as

$$\frac{L_1(y-b_1)+2\kappa_1}{L_1(x-a_1)+2} = \frac{L_2(y-b_2)+2\kappa_2}{L_2(x-a_2)+2},$$

or

$$\big(L_1(y-b_1)+2\kappa_1\big)\big(L_2(x-a_2)+2\big) = \big(L_2(y-b_2)+2\kappa_2\big)\big(L_1(x-a_1)+2\big),$$

which we can rewrite as

$$L_1L_2L_3 + 2L_1L_4 - 2L_2L_5 + 4C = 0,$$

where

$$\begin{aligned} L_3 &= (b_2-b_1)x - (a_2-a_1)y + (a_2b_1 - a_1b_2), \\ L_4 &= y - b_1 - \kappa_2(x-a_1), \\ L_5 &= y - b_2 - \kappa_1(x-a_2), \\ C &= \kappa_1 - \kappa_2. \end{aligned}$$

We can further simplify the equation by noting that $L_6 = L_1L_4 - L_2L_5$ is a linear expression is x, y. That is,

$$L_6 = Dx + Ey + F,$$

where

$$\begin{aligned} D &= 2\kappa_1\kappa_2(a_2-a_1) - (\kappa_1+\kappa_2)(b_2-b_1), \\ E &= 2(b_2-b_1) - (\kappa_1+\kappa_2)(a_2-a_1), \\ F &= \kappa_1\kappa_2(a_1^2-a_2^2) + (\kappa_1+\kappa_2)(a_2b_2-a_1b_1) + (b_1^2-b_2^2). \end{aligned}$$

We can thus write (2) as

$$L_1L_2L_3 + 2L_6 + 4C = 0, \qquad \text{and} \qquad L_1 \neq 0, L_2 \neq 0 \tag{3}$$

All the expressions $L_1, L_2, \ldots, L_6$ are linear in x and y (see Figure 2 for the different lines defined by the equations $L_i = 0$, and their relations with (ℓ_1, p_1) and (ℓ_2, p_2)), so the equation (3) of γ^* is therefore cubic. We have the following two special cases to rule out:

1. If $p_1 = p_2$, that is, $a_1 = a_2$ and $b_1 = b_2$, then $L_3 = 0$, $L_4 = L_2$, and $L_5 = L_1$. But then the equation becomes $4C = 0$, so it has no solutions, meaning that γ is empty and the surfaces do not intersect.

2. If $\ell_1 = \ell_2$ but $p_1 \neq p_2$, that is, $\kappa_1 = \kappa_2 = (b_2-b_1)/(a_2-a_1)$, then $L_1 = L_2 = L_4 = L_5$, $L_3 = (a_1-a_2)L_1$, and $C = 0$, resulting in the equation $(L_1)^3 = 0$, which is not allowed in (3). Hence γ is not defined in this case either.

We can therefore restrict our attention to the general case. Consider the cubic part of the equation $L_1L_2L_3$. In this term, each factor can be thought of as a line defined by the equation $L_i = 0$, for $i = 1, 2, 3$. The lines $L_1 = 0$ and $L_2 = 0$ respectively are simply ℓ_1 and ℓ_2, whereas $L_3 = 0$ defines a third line λ which is the line passing through p_1 and p_2 (see Figure 2). Note

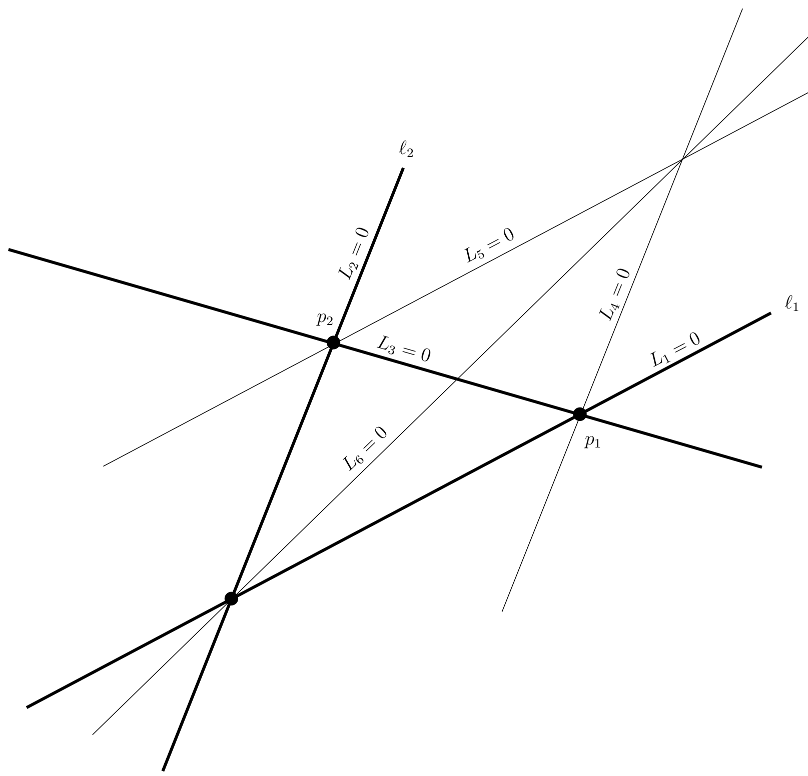

Figure 2: The lines $L_i = 0$, for $i = 1, \ldots, 6$.

that λ may coincide with one of the other two lines. Indeed, if p_1 happens to be incident with ℓ_2, then λ coincides with ℓ_2. Similarly, if $p_2 \in \ell_1$ then λ coincides with ℓ_1 (these are the only possible coincidences, since we have assumed that $\ell_1 \neq \ell_2$). These cases will be handled shortly, and we now consider the general case. In this case, γ^* has three distinct asymptotes given by $L_1 = 0$, $L_2 = 0$, and $L_3 = 0$; for a proof of this fact, see Lemma A.2 in the appendix. Using this fact, one can reconstruct the two line-point pairs that generated γ^*.

Let us explain the details of the reconstruction process. Suppose we are given a curve γ^* generated by some unknown two incidence pairs, and we want to reconstruct the incidence pairs (ℓ_1, p_1) and (ℓ_2, p_2) that generated it. γ^* is given as the zero set of some cubic bivariate polynomial $f(x, y) = 0$, where f is of the form $f(x, y) = c(L_1L_2L_3 + 2L_6 + 4C)$, but the decomposition of f into L_1, L_2, L_3, L_6, C, and c is unknown. First, we find its three asymptotes $\Lambda_1 = 0$, $\Lambda_2 = 0$, and $\Lambda_3 = 0$, where for each $i = 1, 2, 3$, Λ_i is linear in x and y. Since, by Lemma A.2, these asymptotes are $L_1 = 0$, $L_2 = 0$, and $L_3 = 0$, we know that each Λ_i is equal to some L_j multiplied by a constant, but we don't know which is which. To determine the roles of the asymptotes correctly, observe that $\Lambda_1\Lambda_2\Lambda_2 = \mu L_1L_2L_3$ for some constant μ. Thus, there exists some constant ν, such that $f(x, y) - \nu\Lambda_1\Lambda_2\Lambda_2 = \Lambda_4$ is linear in x and y. The line $\Lambda_4 = 0$ is parallel to the line $L_6 = 0$, which happens to be the median of the triangle spanned by the three asymptotes emanating from the vertex $o = \ell_1 \cap \ell_2$, and bisecting the edge p_1p_2; see Figure 2. We thus have enough information to determine which vertex of the triangle is o, and which are p_1 and p_2, and which edges of the triangle are ℓ_1 and ℓ_2. This proves the lemma for the general case where all the points and lines are distinct, and no point coincides with both lines.

Finally, consider the case where $p_2 \in \ell_1$ (a symmetric argument follows when $p_1 \in \ell_2$). In this case, $L_1 = L_5$, and $L_3 = (a_1 - a_2)L_1$, so the equation of the curve γ^* can be rewritten as

$$(a_1 - a_2)L_1^2 L_2 + 2L_1(L_4 - L_2) + 4C = 0.$$

Note that $a_1 \neq a_2$ under the preliminary assumption that there are no vertical lines in the system, since both $p_1 = (a_1, b_2)$ and $p_2 = (a_2, b_2)$ are on ℓ_1. Put $s = L_4 - L_2 = (b_2 - b_1) - \kappa_2(a_2 - a_1)$. Then

$$(a_1 - a_2)L_1^2 L_2 + 2sL_1 + 4C = 0. \tag{4}$$

This equation defines a curve with two asymptotes given by $L_1 = 0$, and $L_2 = 0$, namely, the lines ℓ_1 and ℓ_2; for a proof, see Lemma A.3 in the appendix. In this case, $C = \kappa_1 - \kappa_2 \neq 0$, for otherwise, ℓ_1 and ℓ_2 would have to coincide, which we have ruled out earlier. Hence, γ^* does not intersect $L_1 = 0$, whereas $L_2 = 0$ is intersected at a single point (x, y) for which $L_1 = -2C/s$. Using this point, one can compute the value of s, and hence, reconstruct the line $L_4 = 0$. The point p_1 is then simply the intersection of the lines $L_1 = 0$ and $L_4 = 0$. Thus, one can uniquely reconstruct ℓ_1, ℓ_2, p_1, and p_2 in this case too. This completes the proof of Lemma 2.2. □

Bounding the number of incidences. Recall that we need to bound the number of incidences between the set Σ of surfaces $\sigma_{\ell,p}$, for $(\ell, p) \in Q$, and the set Q of points. This is done by following the standard method of Clarkson et al. [3]. The first step in this method is to derive a simple but weaker bound, usually by extremal graph theory. Then, we strengthen the bound by *cutting* the arrangement of the surfaces into cells, and by summing the number of incidences within each cell, over all the cells.

The first step: A simple bound. Lemma 2.2 implies that the incidence graph between Σ and Q does not contain $K_{3,10}$ as a subgraph, or, in other words, no three distinct surfaces of Σ and ten distinct points of Q can all be incident to one another. Indeed, the intersection points of three surfaces σ_{ℓ_i,p_i}, for $i = 1, 2, 3$, is equal to the intersection points of the two curves $\gamma_{1,2} = \sigma_{\ell_1,p_1} \cap \sigma_{\ell_2,p_2}$, and $\gamma_{1,3} = \sigma_{\ell_1,p_1} \cap \sigma_{\ell_3,p_3}$. These intersection points project to (some of) the intersection points of the projections $\gamma_{1,2}^*$ and $\gamma_{1,3}^*$ of, respectively, $\gamma_{1,2}$, and $\gamma_{1,3}$ onto the xy-plane. By Lemma 2.2, these two curves are distinct (or empty), so (by Bezout's theorem [9]) they intersect in at most nine points. Hence, by the Kővari–Sós–Turán theorem [7], the number of incidences between Σ and Q can be bounded by

$$9^{1/3}|\Sigma||Q|^{2/3} + 2|Q|.$$

By duality, we get that the number of incidences is also bounded by

$$9^{1/3}|Q||\Sigma|^{2/3} + 2|\Sigma|. \tag{5}$$

Cutting. We apply the following fairly standard space decomposition technique. Fix a parameter r, whose specific value will be chosen later, and construct a $(1/r)$-cutting Ξ of $\mathcal{A}(\Sigma)$ [2]. We use the more simple-minded technique in which we choose a random sample R of $O(r\log r)$ surfaces of Σ and construct the vertical decomposition (see e.g. [1]) of the arrangement $\mathcal{A}(R)$. We obtain $O^*(r^3)$ relatively open cells of dimensions 0,1,2, and 3, each of which is crossed by (intersected by, but not contained in) at most $|\Sigma|/r = N/r$ surfaces; this latter property holds with high probability, and we simply assume that our sample R does satisfy it.

Summing over all cells. Fix a cell τ of Ξ, and put $Q_\tau := Q \cap \tau$ and $m_\tau := |Q_\tau|$. Let Σ_τ denote the subset of surfaces of Σ which cross τ, and put $N_\tau := |\Sigma_\tau| \le N/r$.

We now apply the simple bound obtained in the first step (5) to each cell τ of our cutting Ξ, handling, for the time being, only surfaces that *cross* τ. The overall number of incidences is

$$\sum_{\tau\in\Xi} O\left(m_\tau N_\tau^{2/3} + N_\tau\right),$$

which, using the bounds $N_\tau \le N/r$, and $\sum_\tau m_\tau = N$, is

$$O^*\left(N(N/r)^{2/3} + Nr^2\right) = O^*\left(N^{5/3}/r^{2/3} + Nr^2\right).$$

To minimize this expression, we choose $r = N^{1/4}$, making it $O^*(N^{3/2})$.

We also have to take into account incidences between points in a cell τ and surfaces that fully contain τ. This is done separately for cells of dimension 0, 1, and 2 (it is vacuous for cells of dimension 3). Indeed, a 2-dimensional cell τ is contained in exactly one surface, so a point $w \in \tau$ takes part in only one such incidence. Thus, in this case we only need to add N, the number of points, to the above bound.

The same argument applies for points in 1-dimensional cells. Assuming that the vertical decomposition is performed in a generic coordinate frame, it suffices to consider only 1-dimensional cells that are portions of the intersection curves between the surfaces of Σ. By Lemma 2.2, each such cell τ is contained in exactly two surfaces of Σ. Thus, we need to add at most $2N$ to the number of incidences to handle these cells.

Each cell of dimension 0 is a single point w, and, arguing as above, we may assume it to be a vertex of the undecomposed arrangement $\mathcal{A}(R)$. Any surface σ incident to w has to cross or bound an adjacent full-dimensional cell τ^*, so we charge the incidence of σ with w to the pair (τ^*, σ), and note that such a pair can be charged only $O(1)$ times. It follows that the number of incidences with 0-dimensional cells of Ξ is $O^*(r^3 + r^3(N/r)) = O^*(r^2N)$, which is equal to the bound obtained above for the crossing surfaces.

In conclusion, the overall number of incidences between Σ and Q is $O^*(N^{3/2})$.

Recall now that $N = O(n^2/k^2)$, and that we also have the bound $O(n^2k)$ for the number of unit-area triangles with at least one k-poor top line. Thus, the overall bound on the number of unit-area triangles is

$$O^*\left(\frac{n^3}{k^3} + n^2k\right),$$

which, if we choose $k = n^{1/4}$, becomes $O^*(n^{9/4})$, as asserted. $\Box$

Discussion Theorem 2.1 constitutes a major improvement over previous bounds, but it still leaves a substantial gap with the near-quadratic lower bound. One major weakness of our proof is that, in bounding the number of matching pairs, it ignores the constraint that a matching pair is relevant only when the (uniquely defined) third vertex of the resulting triangle belongs to S. It is therefore natural to conjecture that our bound is not tight.

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A Asymptotes of cubic curves

In this appendix, we analyse the class of cubic curves defined by equations (3) and (4) from Section 2, and derive their asymptotes. We start by analysing a normalized version of these equations, in which two of the generating lines (and, as we show hence, the asymptotes) are the x and y axes. We then reduce equation (3) to the normalized case. Finally, we handle equation (4) in a different and simpler way.

Lemma A.1. *Let λ_1 and λ_2 be two lines in $\mathbb{R}^2$, given by the equations $\Lambda_i = 0$, where $\Lambda_i = \alpha_i x + \beta_i y + \gamma_i$, for $i = 1, 2$, such that none of the coefficients is 0. Let Γ be the algebraic curve defined by the equation*

$$xy\Lambda_1 + \Lambda_2 = 0. \tag{6}$$

Then, Γ is asymptotic to the x-axis.

Proof. We shall show that for each arbitrarily small $\delta > 0$, there exists a sufficiently large $M > 0$, such that for any $x > M$, there exists some y, such that $(x, y) \in \Gamma$, and $|y| < \delta$. Rewrite (6) as

$$y + \frac{\Lambda_2}{x\Lambda_1} = 0,$$

and put $f_x(y) = \frac{\Lambda_2}{x\Lambda_1}$, that is, f_x is a function at the variable y with x kept constant. It is easily verified that for $D = \min\{|\alpha_1/2\beta_1|, |\alpha_2/\beta_2|\}$, and for any y in the range $|y| < Dx$, we have

$$|f_x(y)| < \frac{4|\alpha_2 x| + 2|\gamma_2|}{x(||\alpha_1 x| - 2|\gamma_1||)}.$$

By assuming that $x > |4\gamma_1/\alpha_1|$, we have

$$|f_x(y)| < \frac{8|\alpha_2 x| + 4|\gamma_2|}{|\alpha_1|x^2},$$

and by assuming further, that $x > |\gamma_2/\alpha_2|$, we have

$$|f_x(y)| < \frac{12|\alpha_2|}{|\alpha_1|x}.$$

Finally, by assuming that $x > \max\{|12\alpha_2/(\alpha_1\delta)|, \delta/D\}$, we get that for any $|y| \le \delta$, we have $|f_x(y)| < \delta$. But then, $\delta + f_x(\delta) > 0$, and $-\delta + f_x(-\delta) < 0$, so, by the intermediate value theorem, there exists some $y \in (-\delta, \delta)$ for which $y + f_x(y) = 0$, and thus, $(x, y) \in \Gamma$. To wrap things up, we have assumed that x is larger than some positive constant, say M, that depends on $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, and δ, and got that for any such x, there is a point $(x, y) \in \Gamma$ for which $|y| < \delta$. This completes the proof of the lemma. □

We are now ready to prove the more general cases discussed in Section 2.

Lemma A.2. *Let $\ell_1, \ldots, \ell_4$ be four lines in general position in $\mathbb{R}^2$, given by the equations $L_i = 0$, where $L_i = A_i x + B_i y + C_i$, for $i = 1, \ldots, 4$. Let Γ be the algebraic curve defined by the equation*

$$L_1 L_2 L_3 + L_4 = 0.$$

Then, Γ is asymptotic to the lines ℓ_1, ℓ_2, ℓ_3.

Proof. We may assume, by an appropriate change of variables, that one of ℓ_1, ℓ_2, and ℓ_3 is the x-axis and another one is the y-axis. For example, put $u = L_1$, and $v = L_2$. Then we can write $L_3 = \alpha_1 u + \beta_1 v + \gamma_1$, and $L_4 = \alpha_2 u + \beta_2 v + \gamma_2$, for some appropriate coefficients $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$. Note that, by the general position assumption, none of the coefficients is 0. Γ can then be written as

$$uvL_3 + L_4 = 0$$

in the (u, v) coordinate system. Note that the choice of ℓ_1 and ℓ_2 as axes is arbitrary, and we could just as well choose any other pair of lines in any order. Hence, the claim of the lemma is an immediate corollary of Lemma A.1. □

Lemma A.3. *Let ℓ_1 and ℓ_2 be two intersecting lines in $\mathbb{R}^2$, given by the equations $L_i = 0$, where $L_i = A_i x + B_i y + C_i$, for $i = 1, 2$. Let Γ be the algebraic curve defined by the equation*

$$L_1^2 L_2 + L_1 + C = 0,$$

for some constant C. Then, Γ is asymptotic to the lines ℓ_1 and ℓ_2.

Proof. If $C = 0$, then the claim is easy. Indeed, in this case we have $L_1(L_1L_2 + 1) = 0$, so Γ is the union of the line $L_1 = 0$ and the hyperbola $L_1L_2 = -1$, which is asymptotic to the lines $L_1 = 0$, and $L_2 = 0$.

If $C \neq 0$, put $u = L_1$, and $v = L_2$. Then, in the (u, v) coordinate system, Γ is defined by the equation

$$u^2 v + u + C = 0,$$

which can be rewritten as

$$v = -\frac{u + C}{u^2}.$$

Once v has been expressed as a rational function of u, it is easy to see that this function tends to 0 as u tends to ∞, which means it is asymptotic to the u-axis, i.e., to ℓ_2. Furthermore, the function has a pole at $u = 0$, meaning it is asymptotic to the v-axis, i.e., to ℓ_1. □