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## Distinct distances on two lines<sup>☆</sup>


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### ABSTRACT

Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two finite sets of points in the plane, so that  $\mathcal{P}_1$  is contained in a line  $\ell_1$ ,  $\mathcal{P}_2$  is contained in a line  $\ell_2$ , and  $\ell_1$  and  $\ell_2$  are neither parallel nor orthogonal. Then the number of distinct distances determined by the pairs of  $\mathcal{P}_1 \times \mathcal{P}_2$  is

$$\Omega(\min\{|\mathcal{P}_1|^{2/3}|\mathcal{P}_2|^{2/3}, |\mathcal{P}_1|^2, |\mathcal{P}_2|^2\}).$$

In particular, if  $|\mathcal{P}_1| = |\mathcal{P}_2| = m$ , then the number of these distinct distances is  $\Omega(m^{4/3})$ , improving upon the previous bound  $\Omega(m^{5/4})$  of Elekes (1999) [3].

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## 1. Introduction

Given a set  $\mathcal{P}$  of  $m$  points in  $\mathbb{R}^2$ , let  $D(\mathcal{P})$  denote the number of distinct distances that are determined by pairs of points from  $\mathcal{P}$ . Let  $D(m) = \min_{|\mathcal{P}|=m} D(\mathcal{P})$ ; that is,  $D(m)$  is the minimum number of distinct distances that any set of  $m$  points in  $\mathbb{R}^2$  must always determine. In his celebrated 1946 paper [6], Erdős derived the bound  $D(m) = O(m/\sqrt{\log m})$ . For the celebrations of his 80<sup>th</sup> birthday, Erdős compiled a survey of his favorite contributions to mathematics [7], in which he wrote “My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances. This can be found in many of my papers on combinatorial and geometric problems”. Recently, after 65 years and a series of increasingly larger lower bounds (comprehensively described in the book [8]), Guth and Katz [9] provided an almost matching lower bound  $D(m) = \Omega(m/\log m)$ . For this, Guth and Katz developed several novel techniques, relying on tools from algebraic geometry.

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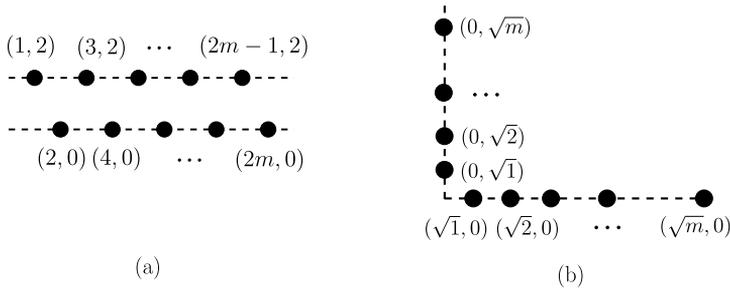


Fig. 1. (a) Two parallel lines with  $D(\mathcal{P}_1, \mathcal{P}_2) = \Theta(m)$ . (b) Two orthogonal lines with  $D(\mathcal{P}_1, \mathcal{P}_2) = \Theta(m)$ .

While the problem of obtaining the asymptotic value of  $D(m)$  is almost settled, many other variants of the distinct distances problem are still widely open. For example, see [2,10] regarding the conjecture that any  $m$  points in convex position in the plane determine at least  $\lfloor m/2 \rfloor$  distinct distances, and [13] for a study of the minimum number of distinct distances in higher dimensions.

In this paper we consider the following variant of the distinct distances problem in the plane. Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two finite sets of points, such that all the points of  $\mathcal{P}_1$  lie on a line  $\ell_1$ , and all the points of  $\mathcal{P}_2$  lie on a line  $\ell_2$ . Let  $D(\mathcal{P}_1, \mathcal{P}_2)$  denote the number of distinct distances between the points of  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , i.e.,

$$D(\mathcal{P}_1, \mathcal{P}_2) = |\{\text{dist}(p, q) \mid p \in \mathcal{P}_1, q \in \mathcal{P}_2\}|.$$

Consider first the “balanced” case, where  $|\mathcal{P}_1| = |\mathcal{P}_2| = m$ . When the two lines are parallel or orthogonal, the points can be arranged such that  $D(\mathcal{P}_1, \mathcal{P}_2) = \Theta(m)$ ; for example, see Fig. 1. Purdy conjectured that if the lines are neither parallel nor orthogonal then  $D(\mathcal{P}_1, \mathcal{P}_2) = \omega(m)$  (e.g., see [1, Section 5.5]). Elekes and Rónyai [4] proved that the number of distinct distances in such a scenario is indeed superlinear. They did not give an explicit bound, but a brief analysis of their proof shows that  $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(m^{1+\delta})$  for some  $\delta > 0$ . Elekes [3] derived the improved bound  $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(m^{5/4})$  (when the lines are neither parallel nor orthogonal) and gave a construction, reminiscent of the one by Erdős [6], with  $D(\mathcal{P}_1, \mathcal{P}_2) = O(m^2/\sqrt{\log m})$ , in which the angle between the two lines is  $\pi/3$ . Previously, these were the best known bounds for  $D(\mathcal{P}_1, \mathcal{P}_2)$  for the balanced case. The unbalanced case, where  $|\mathcal{P}_1| \neq |\mathcal{P}_2|$ , has recently been studied by Schwartz, Solymosi, and de Zeeuw [12], who have shown, among several other related results, that the number of distinct distances remains superlinear when  $|\mathcal{P}_1| = m^{1/2+\epsilon}$  and  $|\mathcal{P}_2| = m$ , for any  $\epsilon > 0$ .

In this paper we derive the following result, for point sets  $\mathcal{P}_1, \mathcal{P}_2$  of arbitrary (possibly different) cardinalities.

**Theorem 1.1.** *Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two sets of points in  $\mathbb{R}^2$  of cardinalities  $n$  and  $m$ , respectively, such that the points of  $\mathcal{P}_1$  all lie on a line  $\ell_1$ , the points of  $\mathcal{P}_2$  all lie on a line  $\ell_2$ , and the two lines are neither parallel nor orthogonal. Then the number of distinct distances between  $\mathcal{P}_1$  and  $\mathcal{P}_2$  is*

$$D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(\min\{n^{2/3}m^{2/3}, n^2, m^2\}).$$

Theorem 1.1 immediately implies the following improved lower bound for the balanced case.

**Corollary 1.2.** *Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two sets of points in  $\mathbb{R}^2$ , each of cardinality  $m$ , such that the points of  $\mathcal{P}_1$  all lie on a line  $\ell_1$ , the points of  $\mathcal{P}_2$  all lie on a line  $\ell_2$ , and the two lines are neither parallel nor orthogonal. Then  $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(m^{4/3})$ .*

Note that Theorem 1.1 also implies the result of [12], mentioned earlier, and slightly strengthens it, by providing the explicit lower bound  $\Omega(m^{1+2\epsilon/3})$ .

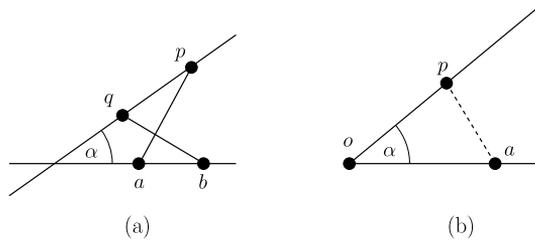


Fig. 2. (a) A quadruple  $(a, p, b, q)$  in  $Q$ . (b) By the law of cosines, we have  $\|ap\|^2 = \|oa\|^2 + \|op\|^2 - 2\|oa\|\|op\|\cos\alpha$ .

Even with the improved lower bound in Corollary 1.2 (over the lower bound in [3]), there is still a considerable gap to the near-quadratic upper bound in [3], and the prevailing belief is that the correct lower bound is indeed close to quadratic.

To obtain the improved bound, we use a double counting argument, applied to the number of quadruples  $(a, p, b, q)$  of points, with  $a, b \in \mathcal{P}_1$  and  $p, q \in \mathcal{P}_2$ , that satisfy  $\|ap\| = \|bq\|$ . The argument is similar to the one in the reduction devised by Elekes and presented in Elekes and Sharir [5]. For this double counting we use the same lower bound analysis as in [5], but replace the upper bound analysis by a considerably simpler one, in which the problem is reduced to that of bounding the number of incidences between certain points and hyperbolas in the plane. (In contrast, the original reduction in [5] is to incidences between points and lines in  $\mathbb{R}^3$ .)

2. The proof of Theorem 1.1

Without loss of generality, we may assume that the points of  $\mathcal{P}_1$  are on one side of the intersection point  $\ell_1 \cap \ell_2$ . Otherwise, we can partition  $\mathcal{P}_1$  into two subsets by splitting  $\ell_1$  at  $\ell_1 \cap \ell_2$ , and remove the subset that yields fewer distinct distances with the points of  $\mathcal{P}_2$ . At worst, this halves the number of distinct distances between the pairs in  $\mathcal{P}_1 \times \mathcal{P}_2$ . For the same reason, we may assume that the points of  $\mathcal{P}_2$  are also on one side of  $\ell_1 \cap \ell_2$ . Furthermore, without loss of generality, we may assume that  $n = |\mathcal{P}_1| \geq m = |\mathcal{P}_2|$ .

We rotate, translate, and possibly reflect the original plane, so that the origin  $o$  is  $\ell_1 \cap \ell_2$ ,  $\ell_1$  is the  $x$ -axis, the points of  $\mathcal{P}_1$  lie on the positive side of  $o$ , and the points of  $\mathcal{P}_2$  lie above  $\ell_1$ . We denote the angle between the two lines by  $\alpha$ . Since the two lines  $\ell_1$  and  $\ell_2$  are neither parallel nor orthogonal, we have  $\alpha \neq 0, \pi/2$ . We will also assume that  $o \notin \mathcal{P}_1 \cup \mathcal{P}_2$ , because the presence of  $o$  in either set can generate at most  $O(m+n)$  distinct distances.

We begin with a variant of the first part of the reduction from [5]. We set  $x = D(\mathcal{P}_1, \mathcal{P}_2)$  and denote the  $x$  distinct distances in  $\mathcal{P}_1 \times \mathcal{P}_2$  as  $\delta_1, \dots, \delta_x$ . For a pair of points  $u$  and  $v$ , we denote by  $\|uv\|$  the (Euclidean) distance between  $u$  and  $v$ . Let  $Q$  be the set of quadruples  $(a, p, b, q)$ , where  $a, b \in \mathcal{P}_1$  and  $p, q \in \mathcal{P}_2$ , such that  $\|ap\| = \|bq\| > 0$  and  $ap \neq bq$  (the two segments are allowed to share at most one endpoint); see Fig. 2(a). The quadruples are ordered, so that  $(a, p, b, q)$  and  $(b, q, a, p)$  are considered as two distinct elements of  $Q$ .

Let  $E_i = \{(a, p) \in \mathcal{P}_1 \times \mathcal{P}_2 \mid \|ap\| = \delta_i\}$ , for  $i = 1, \dots, x$ . Using the Cauchy–Schwarz inequality we have,

$$|Q| = 2 \sum_{i=1}^x \binom{|E_i|}{2} \geq \sum_{i=1}^x (|E_i| - 1)^2 \geq \frac{1}{x} \left( \sum_{i=1}^x (|E_i| - 1) \right)^2 = \frac{(mn - x)^2}{x}. \tag{1}$$

In the remainder of the proof we derive an upper bound on  $|Q|$ , showing that  $|Q| = O(m^{4/3}n^{4/3} + n^2)$ . Combined with (1) this yields the lower bound asserted in the theorem (under the assumption  $n \geq m$ ).

To obtain this upper bound, we re-interpret  $|Q|$  as an incidence count between certain points and hyperbolas in a suitable parametric plane, and then use standard machinery to bound this count. This replaces (and greatly simplifies) the second part of Elekes’s reduction, where  $|Q|$  is interpreted as an incidence count between points and lines in  $\mathbb{R}^3$ .

Let us consider a quadruple  $(a, p, b, q) \in (\mathcal{P}_1 \times \mathcal{P}_2)^2$ . By the law of cosines, we have  $\|ap\|^2 = \|oa\|^2 + \|op\|^2 - 2\|oa\|\|op\|\cos\alpha$  and  $\|bq\|^2 = \|ob\|^2 + \|oq\|^2 - 2\|ob\|\|oq\|\cos\alpha$  (see Fig. 2(b)). Thus, the quadruple  $(a, p, b, q)$  is in  $Q$  if and only if

$$\|oa\|^2 + \|op\|^2 - 2\|oa\|\|op\|\cos\alpha = \|ob\|^2 + \|oq\|^2 - 2\|ob\|\|oq\|\cos\alpha.$$

We represent each point  $u$  of  $\mathcal{P}_1$  or of  $\mathcal{P}_2$  by its distance  $\|ou\|$  from the origin  $o$ . Each of  $\mathcal{P}_1, \mathcal{P}_2$  is contained in a ray (with initial point  $o$ ) of the respective line  $\ell_1, \ell_2$ , we may assume that all these distances are all distinct in  $\mathcal{P}_1$  and are all distinct in  $\mathcal{P}_2$ .

In what follows, we will use  $u$  to denote both the point and its distance  $\|ou\|$  from  $o$ . Using the notation  $s = \cos\alpha$ , the above condition can be written as

$$a^2 - b^2 + p^2 - q^2 - 2s(ap - bq) = 0, \tag{2}$$

where  $s \neq 0, 1$ .

Let  $\mathcal{V}_1$  and  $\mathcal{V}_2$  denote the sets of ordered distinct pairs of  $\mathcal{P}_1$  and of  $\mathcal{P}_2$ , respectively. That is,

$$\mathcal{V}_i = \mathcal{P}_i \times \mathcal{P}_i \setminus \{(x, x) \mid x \in \mathcal{P}_i\} \quad \text{for } i = 1, 2.$$

For every  $i \in \{1, 2\}$  and  $(a, b) \in \mathcal{V}_i$ , there is a curve  $\gamma_{a,b}^{(i)}$ , corresponding to the pair  $(a, b)$ , given by the equation

$$a^2 - b^2 + x^2 - y^2 - 2s(ax - by) = 0. \tag{3}$$

This can also be written as

$$(x - sa)^2 - (y - sb)^2 = (1 - s^2)(b^2 - a^2).$$

Since  $s \neq 1$  and  $a \neq b$ , the curve  $\gamma_{a,b}^{(i)}$  is a hyperbola. Moreover, since  $s \neq 0$ , all the hyperbolas  $\gamma_{a,b}^{(1)}$  are distinct, and so are all the hyperbolas  $\gamma_{a,b}^{(2)}$ . Let  $\mathcal{C}_1$  denote the set of the  $m(m - 1)$  hyperbolas,  $\mathcal{C}_1 = \{\gamma_{a,b}^{(1)} \mid (a, b) \in \mathcal{V}_1\}$ . By construction, the hyperbola  $\gamma_{a,b}^{(1)} \in \mathcal{C}_1$  is incident to the point  $(p, q) \in \mathcal{V}_2$  if (and only if)  $a, b, p, q$  satisfy the condition in (2). That is, the number of quadruples  $(a, p, b, q)$  in  $Q$  for which  $a \neq b$  and  $p \neq q$  is at most the number of point-hyperbola incidences between  $\mathcal{V}_2$  and  $\mathcal{C}_1$ . The number of missing elements of  $Q$ , where either  $a = b$  or  $p = q$ , has the trivial upper bound  $4mn$ , as is easily verified.

Every pair of hyperbolas from  $\mathcal{C}_1$  intersect in at most two points in the real plane. Therefore, for every pair of hyperbolas from  $\mathcal{C}_1$  there are at most two points of  $\mathcal{V}_2$  that are incident to both hyperbolas. The roles of  $\mathcal{V}_1$  and  $\mathcal{V}_2$  in (3) are symmetric. In particular, every pair of hyperbolas from  $\mathcal{C}_2 = \{\gamma_{a,b}^{(2)} \mid (a, b) \in \mathcal{V}_2\}$  intersect in at most two points too. This implies that there are at most two hyperbolas of  $\mathcal{C}_1$  that pass through a given pair of points in  $\mathcal{V}_2$ .

We can therefore use the following result of Pach and Sharir [11].

**Theorem 2.1.** (See Pach and Sharir [11].) Consider a point set  $\mathcal{P}$ , a set of curves  $\mathcal{C}$ , and a constant positive integer  $s$ , such that

- (i) for every pair of points of  $\mathcal{P}$  there are at most  $s$  curves of  $\mathcal{C}$  that are incident to both points, and
- (ii) every pair of curves of  $\mathcal{C}$  intersect in at most  $s$  points.

Then the number of incidences between  $\mathcal{P}$  and  $\mathcal{C}$  is at most  $c(s)(|\mathcal{P}|^{2/3}|\mathcal{C}|^{2/3} + |\mathcal{P}| + |\mathcal{C}|)$ , where  $c(s)$  is a constant that depends on  $s$ .

We apply Theorem 2.1 to  $\mathcal{V}_2$  and  $\mathcal{C}_1$ , with  $s = 2$ , and obtain (adding the bound on the missing pairs in  $Q$ )

$$|Q| = O(m^{4/3}n^{4/3} + m^2 + n^2 + mn) = O(m^{4/3}n^{4/3} + n^2)$$

(recalling that we assume  $n \geq m$ ), as desired. As already noted, combining this bound with (1) implies

$$\frac{(mn - x)^2}{x} = O(m^{4/3}n^{4/3} + n^2),$$

or, as is easily checked,

$$x = \Omega(\min\{m^{2/3}n^{2/3}, m^2\}).$$

Combining this bound with its symmetric version when  $m \geq n$  yields the bound asserted in the theorem.

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