# Advanced Topics in Computational and Combinatorial Geometry

# Assignment 2 (short answers and hints)

# Problem 1

**Complexity of a single cell in an arrangement of *n* rays is *O(n)*.** The proof will be similar to the proof given in class for an arrangement of segments and it will use similar notations. We walk along the edges of the single cell so that the cell is always to the left of us. When we walk along the edge in the direction of ray heading to infinity we’ll assign a positive symbol to the edge, otherwise a negative symbol.

**b**

walk order

**a**

**c**

**d**

d -

a -

a +

d -

c -

c +

d +

d -

c +

a +

b -

b +

a +

The sequence: d-, a-, a+, d-, c-, c+, d-, d+, c+, a+, b-, b+, a+

We claim that the sequence of positive symbols is a *DS(n, 2)* sequence. The same will hold for negative symbols, symmetrically – if we reverse the direction of the walk, the signs of the symbols switch.

Steps in the proof:

* prove that “a+, a+” cannot appear
* prove that each *“a+ … b+ … a+”* alternation implies an intersection point between ray *a* and ray *b* that lies between the two occurrences of *a+* on ray *a* and after the occurrence of *b+* on ray *b*.

**Complexity of outer zone of any closed convex curve in an arrangement of *n* lines.** Each line intersects a closed convex curve C at two or zero points in general. When the line intersects the curve at two points it is split into 3 parts – two rays outside C and a segment inside C. The outer zone of C consists now of lines (which didn’t intersect C) and rays. We can split each of the untouched lines into 2 rays as well and then we have that the outer zone is a single cell in the arrangement of *2n* rays. So its complexity is *O(n)*.

# Problem 2

It is easy to see that the zone of a line in the arrangement of circles has at most *2n+1* cells. We choose a point from each such cell – denote this set *P*, |*P*| = *2n+1*. Complexity of the zone of the line in the arrangement of upper half-circles is *O(n α(n))* since each pair of upper half-circles intersect in at most one point. The total complexity of all cells containing points of *P* is also *O(n α(n))* since they are the same cells as the cells of the zone. Same argument is symmetrically valid for the arrangement of the lower half-circles. Using the combination lemma we conclude that the complexity of all cells containing *P* in the combined arrangement (i.e. the arrangement of the full circles) is
 *O(n α(n) + n α(n) + 2n + 1) = O(n α(n))*, and so is the complexity of the zone of a line.

In the case the circles have arbitrary radii, the complexity of the zone is bounded by *O(λ4(n))*: Cutting and slightly opening the circles at their intersections with the line, turns the zone into a single cell in an arrangement of O(n) circular arcs, each pair intersecting at most twice.

# Problem 3

First note that the separation distance satisfies the triangle inequality (easy argument).

1. Let *p* be a point located at separation distance *k* from *x*. By definition there are *k* lines crossing *px*. Let *l* be a line with closest intersection point to *x*, and let *q* be the intersection point. Line *l* has *n-1* intersection points with other lines from *L*. Let’s take *k* closest intersection points to *q*. The separation distance between all these points and *q* is at most *k*. Separation distance between *q* and *x* is 0, thus it follows from the triangle inequality that the separation distance of these *k* points to *x* is at most *k*. I.e. line *l* contributed *k* points from the arrangement with separation distance up to *k*. Applying the same argument to the next line, will show that it contributes *k – 1* points with separation distance up to *k*. Finally, all *k* lines together will contribute
 *k + (k – 1) + (k – 2) + … + 1 = Ω(k2)* points.
2. Hint: Denote by *d* the minimal separation distance between any pair of points in *P*. Define *Si* to be the set of vertices of the arrangement that are at distance at most $k=\left⌊\frac{d-1}{2}\right⌋$ from the i'th point. Clearly |*S1*|+…|*Sm*| ≤ *n2*, so by (a): $mck^{2}\leq n^{2} ⟹ k\leq c\frac{n}{\sqrt{m}} ⟹ d=O\left(\frac{n}{\sqrt{m}}\right)$.
3. **SpanningTree**(*L, P*):
4. Find a pair of points *p,q*∈*P* with minimal separation distance.
5. T = **SpanningTree**(*L, P \ {p}*)
6. return T ∪ {*(p, q)*}

The total number of crossings between lines of L and edges of T is

 $O(\frac{n}{\sqrt{m}}+\frac{n}{\sqrt{m-1}}+…)≈O\left(n\sqrt{m}\right).$

# Problem 4

Fix $p\_{i},p\_{j}\in P$, and let $b\_{ij}$ denote the bisector line of $p\_{i}p\_{j}$. For each $q\in P∖\{p\_{i},p\_{j}\}$ define $φ\_{q}^{+}(t)$ (resp., $φ\_{q}^{-}(t)$) to be the distance along $b\_{ij}$, at time t, to the center $c\_{ij}$ of the circum circle of $p\_{i}p\_{j}q$, if q lies in the *right* (resp. *left)* of the direction line $\vec{p\_{i}p\_{j}}$, and $+\infty $ (resp., $-\infty $) otherwise. (We measure the distance, for simplicity, from the midpoint of $p\_{i}p\_{j}$; the distance is positive to the right, negative to the left.)



It is now an easy observation that, at time t, $p\_{i}$ and $p\_{j}$ are neighbors iff

$$\max\_{q}φ\_{q}^{-}\left(t\right)<\min\_{q}φ\_{q}^{+}\left(t\right).$$

If this holds, the Voronoi edge on $b\_{ij}$ is delimited by the Voronoi vertices that correspond to the two values $\max\_{q}φ\_{q}^{-}\left(t\right)$, $\min\_{q}φ\_{q}^{+}\left(t\right)$. Note that the functions$φ\_{q}^{-}$, $φ\_{q}^{+}$ are partially define (they start/stop being defined when $q$ crosses the line $p\_{i}p\_{j}$), but this happens only $O(1)$ times for each $q$. Altogether, each of the above envelopes has complexity $\leq λ\_{s}(n)$ for some suitable $s$, and the sandwich region between them has complexity $O(λ\_{s}\left(n\right))$. Multiplying by $\left(\genfrac{}{}{0pt}{}{n}{2}\right)$ pairs $p\_{i}, p\_{j}$ completes the analysis.