Pricing Multi-unit Markets

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Abstract. We study the power and limitations of posted prices in multi-unit markets, where agents arrive sequentially in an arbitrary order. We prove upper and lower bounds on the largest fraction of the optimal social welfare that can be guaranteed with posted prices, under a range of assumptions about the designer's information and agents' valuations. Our results provide insights about the relative power of uniform and non-uniform prices, the relative difficulty of different valuation classes, and the implications of different informational assumptions. Among other results, we prove constant-factor guarantees for agents with (symmetric) subadditive valuations, even in an incomplete-information setting and with uniform prices.

1 Introduction

We consider the problem of allocating identical items to agents to maximize the social welfare. More formally, there are m identical items, each agent $i \in [n]$ has a valuation function $v_i : [m] \to \mathbb{R}_{\geq 0}$ describing her value for a given number of items, and the goal is to compute nonnegative and integral quantities $q_1, \ldots, q_n$, with $\sum_{i=1}^{n} q_i \leq m$, to maximize the total value $\sum_{i=1}^{n} v_i(q_i)$ to the agents.

This problem underlies the design of *multi-unit auctions*, which have played a starring role in the fields of classical and algorithmic mechanism design, and in both theory and practice. As with any welfare-maximization problem, the problem can be solved in principle using the VCG mechanism. There has been extensive work on the design and analysis of more practical multi-unit auctions. There are indirect implementations of the VCG mechanism, most famously Ausubel's ascending *clinching auction* for downward-sloping (a.k.a. submodular) valuations [1]. Work in algorithmic mechanism design has identified mechanisms that retain the dominant-strategy incentive-compatibility of the VCG mechanism while running in time polynomial in n and $\log m$ (rather than polynomial in n and m), at the cost of a bounded loss in the social welfare. Indeed, Nisan [30] argues that the field of algorithmic mechanism design can be fruitfully viewed through the lens of multi-unit auctions.

The multi-unit auction formats used in practice typically sacrifice dominant-strategy incentive-compatibility in exchange for simplicity and equitability; a

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canonical example is the uniform-price auctions suggested by Milton Friedman (see [22]) and used (for example) by the U.S. Treasury to sell government securities. Uniform-price auctions do not always maximize the social welfare (e.g., because of demand reduction), but they do admit good "price-of-anarchy" guarantees [29], meaning that every equilibrium results in social welfare close to the maximum possible.

A key drawback of all of the mechanisms above is that they require all agents to participate simultaneously, in order to coordinate their allocations and respect the supply constraint. For example, in a uniform-price auction, all of the agents' bids are used to compute a market-clearing price-per-unit, which then determines the allocations of all of the agents. It is evident from our daily experience that, in many different markets, buyers arrive and depart asynchronously over time, making purchasing decisions as a function of their preferences and the current prices of the goods for sale.[1] The goal of this paper is to develop theory that explains the efficacy of such *posted prices* in markets where agents arrive sequentially rather than simultaneously, and that gives guidance on how to set prices to achieve an approximately welfare-maximizing outcome.

1.1 The Model

We consider a setting where a designer must post prices in advance, before the arrival of any agents. We assume that the supply m is known. The designer is given full or incomplete information about agents' valuations, and must then set a price for each item.[2] Agents then arrive in an arbitrary (worst-case) order, with each agent taking a utility-maximizing bundle (breaking ties arbitrarily), given the set of items that remain. These prices are *static*, in that they remain fixed throughout the entire process.

Example 1.1. *Suppose $m = 3$ and there are two agents, each with the valuation $v(1) = 5$, $v(2) = 9$, and $v(3) = 11$, and suppose a designer prices every item at 4. The first agent will choose either 1 or 2 items (breaking the tie arbitrarily). If the first agent chooses 2 items, the second agent will take the only item remaining; if the first agent chooses 1 item, then the second agent will take either 1 or 2 items.*

In general, we allow different items to receive different prices (as will be the case in the VCG mechanism for this problem, for example.) With identical items, however, it is natural to focus on *uniform prices*, where every item is given the same price. Generally speaking, we are most interested in positive results for uniform prices, and negative results for non-uniform prices.

[1] For examples involving identical items, think about general-admission concert tickets, pizzas at Una Pizza Napoletana (which shuts down for the night when the dough runs out), or shares in an IPO (other than Google [33]).

[2] No non-trivial guarantees are possible without at least partial knowledge about agents' valuations.

The overarching goal of this paper is to characterize the largest fraction of the optimal social welfare that can be guaranteed with posted prices, under a range of assumptions about the designer's information and agents' valuations. This goal is inherently quantitative, but our results also provide qualitative insights, for example about the relative power of uniform and non-uniform prices, the relative difficulty of different valuation classes, and the implications of different informational assumptions.

Table 1. Summary of results. All results are new to this paper unless indicated otherwise. Numbers in parentheses refer to the corresponding theorem or proposition number.

	Uniform prices	Non-uniform prices
Submodular	$\frac{1}{2}$ (4.6, 4.7, 4.8)	$\frac{2}{3}$ (4.1, 4.2) [2 items] $\geq \frac{5}{7} - \frac{1}{m}$ (4.3), ≤ 0.802 (4.4) [m items]
XOS	$\geq \frac{1}{2}$ (8.2)	$\leq 1 - \frac{1}{e}$ (5.1)
Subadditive	$\frac{1}{3}$ (6.1, 6.4) $\frac{2}{3}$ (6.2, 6.6) [2 identical buyers]	$\leq \frac{1}{2}$ (6.3) [even with 2 buyers] $\leq \frac{3}{4}$ (6.5) [even with 2 identical buyers]
General	$\frac{1}{m}$ (7.1)	$\frac{1}{m}$ (7.2)

(a) Full information

	Uniform prices	Non-uniform prices
XOS	$\frac{1}{2}$ (8.2)	$\frac{1}{2}$ [21]
Subadditive	$\geq \frac{1}{4}$ (8.4)	$\leq \frac{1}{2}$ (6.3) [even with 2 buyers] $\leq \frac{3}{4}$ (6.5) [even with 2 identical buyers]

(b) Incomplete information

1.2 Our Results

The majority of our results are summarized in Table 1; we highlight a subset of these next. First, consider the case of a Bayesian setting with XOS agent valuations (see Sect. 2 for definitions). That is, each agent's valuation is drawn independently from a known (possibly agent-specific) distribution over XOS valuations. Feldman et al. [21] show that, even with non-identical items, posted prices can always obtain expected welfare at least 1/2 times the maximum possible. This factor of 1/2 is tight, even for the special case of a single item and i.i.d. agents. The posted prices used by Feldman et al. [21] are non-uniform, even when the result is specialized to the case of identical items (the price of an item is based on its expected marginal contribution to an optimal allocation, which can vary across items). We prove in Theorem 8.2 that with identical items, and agents with independent (not necessarily identical) XOS valuations, uniform prices suffice to achieve the best-possible guarantee of half the optimal expected welfare. Moreover, this result extends to any class of valuations that is c-close to XOS valuations, with an additional loss of a factor of c (Theorem 8.3).

While the 1/2-approximation above is tight for an incomplete-information setting, this problem is already interesting in the full-information case where the buyers' valuations are known (with the order of arrival still worst-case). Can we improve over the approximation factor of 1/2 under this stronger informational assumption?

We prove that uniform prices cannot achieve an approximation factor better than 1/2, even for the more restrictive class of submodular valuations, and even with two agents (Proposition 4.7) or identical agents (Proposition 4.8). In contrast, with non-uniform prices (still for submodular valuations), we prove that an approximation of 2/3 is possible (Theorem 4.1). This is tight for the case of two items (Proposition 4.2), but in large markets (with $m \to \infty$) we show how to obtain an approximation guarantee of 5/7 (Theorem 4.3). In addition, if the order of arrival is known beforehand, we can extract the full optimal welfare (Theorem 4.5).

We next consider the family of subadditive valuations, which strictly generalize XOS valuations and are regarded as the most challenging class of valuations that forbid complements. For example, with non-identical items, it is not known whether or not posted prices can guarantee a constant fraction of the optimal social welfare. For identical items, we prove that this is indeed possible. In the incomplete-information setting (and identical items), we show that subadditive valuations are 2-close to XOS valuations (Sect. 3), which leads to an approximation factor of 1/4 (Theorem 8.4). We can also do better in the full-information setting: uniform prices can guarantee a 1/3 fraction of the optimal social welfare (Theorem 6.1), and the approximation is tight (Proposition 6.4), while even non-uniform prices cannot guarantee a factor bigger than 1/2, even with only two agents (Proposition 6.3). In the case of two identical agents, uniform prices can guarantee a 2/3 fraction of the optimal welfare (Theorem 6.2), and this is tight (Proposition 6.6).

With all these positive results, the reader might wonder whether constant factor guarantees can be provided for general valuations. Unfortunately, this is not the case. For general valuations, we show that even in the full-information setting and with non-uniform prices, and even when there are only two agents and the arrival order is known, posted prices can guarantee a $1/m$ fraction of the optimal social welfare, but not more (Proposition 7.1, Theorem 7.2). If the seller can control the arrival order, however, then even uniform prices can guarantee half of the optimal social welfare (Theorem 7.3). No better bound is possible, even for identical valuations and with non-uniform prices (Proposition 7.4).

1.3 Further Related Work

The design and analysis of simple mechanisms has been an active area of study in algorithmic mechanism design, particularly within the last decade. This focus is motivated in part by the observation that simple mechanisms are highly desired in practical scenarios. Examples of simple mechanisms that are used in practice are the generalized second price auctions (GSP) for online advertising [15,27,28,31,34], and simultaneous item auctions (where the agents bid separately and simultaneously on multiple items) [6,11,20,24]. These mechanisms are not truthful and are evaluated in equilibrium using the price of anarchy measure.

Posting prices is perhaps the most prevalent method for selling goods in practice. By simply publishing prices for individual items, posted price mechanisms are extremely easy to understand and participate in. It should therefore not come as a surprise that these mechanisms have been studied extensively for various objective functions (e.g., welfare, revenue, makespan), information structures of values (e.g., full-information, Bayesian, online), and valuation functions (e.g., unit-demand, submodular, XOS). For example, a long line of work has focused on sequential posted prices for revenue maximization and has shown, among other things, that a form of posted price mechanisms can achieve a constant fraction of the optimal revenue for agents with unit-demand valuations [8–10]. Revenue maximization with sequential posted prices has also been studied for a single item, both in large markets [7] and when the distributions are unknown [2], for additive valuations [4,5], and for a buyer with complements [17]. Dütting et al. [14] provides a general framework for posted price mechanisms. In several of these works, posted price mechanisms are allowed to discriminate between agents and set different prices for each of them. In contrast, in this work we do not consider discriminatory prices.

Another line of research relevant to our work considers market equilibria, for example those achieved by Walrasian prices. A result of Kelso and Crawford [25] states that for the class of gross-substitute valuations, there always exists a Walrasian equilibrium, meaning that one can assign prices to items so as to achieve the optimal social welfare. However, this result is based on the assumption that agents break ties in a particular way. As such, the existence of Walrasian prices does not carry over welfare guarantees to our setting, even for unit-demand valuations. We believe that the worst-case perspective that we take is more realistic in our setting, where we do not have control over how agents break ties.

In addition to the aforementioned works, a new line of research has considered dynamic posted prices in online settings such as for the k-server and parking problems [12]. Moreover, posted price mechanisms have been studied in the context of welfare maximization in matching markets, where prices are dynamic (i.e., can change over the course of the mechanism) but do not depend on the identity of the agents [13]. With static prices, it was recently shown that one can achieve strictly more than half of the welfare in the full information setting with binary unit-demand valuations [16].

The sequential arrival of agents considered in posted price mechanisms fits into the framework of online mechanisms, which deals with dynamic environments with multiple agents having private information [3,23,32]. Our work shows that for identical items and agents with subadditive valuations, posted prices can guarantee a constant fraction of the welfare even while setting the (uniform) prices up front.

2 Preliminaries

We consider a setting with a set M of m *identical* items, and a set N of n buyers. Each buyer has a valuation function $v_i : 2^M \rightarrow \mathbb{R}_{\geq 0}$ that indicates his value for every set of objects. Since items are identical, the valuation depends only on the number of items. We assume that valuations are monotone non-decreasing (i.e., $v_i(T) \leq v_i(S)$ for $T \subseteq S$) and normalized (i.e., $v_i(\emptyset) = 0$). We use $v_i(S|T) = v_i(S \cup T) - v_i(T)$ to denote the marginal value of bundle S *given* bundle T.

A buyer valuation profile is denoted by $\mathbf{v} = (v_1, \ldots, v_n)$. An *allocation* is a vector of disjoint sets $\mathbf{x} = (x_1, \ldots, x_n)$, where x_i denotes the bundle associated with buyer $i \in [n]$ (note that it is not required that all items are allocated). As with valuations, since we consider identical items, an allocation can be represented by the number of items allocated to each buyer. The *social welfare* of an allocation $\mathbf{x}$ is $\text{SW}(\mathbf{x}, \mathbf{v}) = \sum_{i=1}^{n} v_i(x_i)$, and the optimal social welfare is denoted by $\text{OPT}(\mathbf{v})$. When clear from the context we omit $\mathbf{v}$ and write OPT for the optimal social welfare.

For two valuation functions v, v', we say that $v \geq v'$ iff $v(S) \geq v'(S)$ for every set S. A hierarchy over complement-free valuations is given by Lehmann et al. [26].

Definition 2.1. *A valuation function v is*

- additive *if $v(S) = \sum_{i \in S} v(\{i\})$ for every set $S \subseteq M$.*
- submodular *if $v(\{i\}|S) \geq v(\{i\}|T)$ for every item $i \notin T$ and sets S, T such that $S \subseteq T \subseteq M$.*
- XOS *if there exist additive valuation functions $v^1, \ldots, v^k$ such that $v(S) = \max_{j=1,\ldots,k} v^j(S)$ for every set $S \subseteq M$.*
- subadditive *if $v(S) + v(T) \geq v(S \cup T)$ for any sets $S, T \subseteq M$.*

Since we assume throughout the paper that all items are identical, we only work with symmetric valuation functions.

Definition 2.2. *A valuation function v is* symmetric *if $v(S) = v(T)$ for every sets $S, T \subseteq M$ such that $|S| = |T|$. A symmetric valuation function can thus be represented by a monotone non-decreasing function $v : \{0, 1, \ldots, m\} \rightarrow \mathbb{R}_{\geq 0}$, which assigns a non-negative real value to any integer in $[m]$ (recall $v(0) = 0$ as we assume normalized functions).*

In what follows we adjust the definitions of additive, submodular, XOS, and subadditive functions in Definition 2.1 to the case of symmetric valuation functions. The simplified definition for XOS functions follows from the equivalence between XOS and fractional subadditivity [19].

Definition 2.3. *A symmetric valuation function v is said to be*

- additive *if $v(i) = a \cdot i$ for every integer $0 \leq i \leq m$ for some constant a.*
- submodular *if $v(i) - v(i-1) \geq v(i+1) - v(i)$ for every integer $1 \leq i \leq m-1$.*

– XOS *if* $v(i) \geq \frac{i}{j} \cdot v(j)$ *for any integers* $1 \leq i < j \leq m$*.*
– subadditive *if* $v(i) + v(j) \geq v(i + j)$ *for any integers* $1 \leq i, j \leq m$ *with* $i + j \leq m$*.*

We assume that the agents arrive sequentially. We will for the most part set static prices for the items; each arriving agent takes a bundle from the remaining items that maximizes her utility, with ties broken arbitrarily. For some results we will assume dynamic prices, i.e., the seller can set new prices for the remaining items for each iteration (but without knowing which agent will arrive next). If prices $\mathbf{p} = (p_1, \ldots, p_m)$ are set on the m items, and an agent buys a subset S of them, then her utility is given by $v(|S|) - \sum_{i \in S} p_i$. For most of the paper we will assume that the arrival order of the agents is unknown, but we will also consider settings where we know this order or where we even have control over the order. We are interested in the social welfare that we can obtain by setting prices in comparison to the optimal social welfare with respect to the worst case arrival order.

Due to space constraints, omitted results and proofs can be found in the full version of this paper [18].

3 Properties of Symmetric Functions

In this section, we consider properties of symmetric functions. In addition to being interesting in their own right, these properties will later help us establish welfare guarantees for posted prices (Theorem 8.4).

We are interested in approximating functions with “simpler” functions. Specifically, for two classes of functions $\mathcal{V}_1 \subseteq \mathcal{V}_2$, we want to determine the smallest constant c such that for any function $v \in \mathcal{V}_2$, there exists a function $\tilde{v} \in \mathcal{V}_1$ such that $v \leq \tilde{v} \leq cv$. We answer this question for each pair from the classes of subadditive, XOS, and submodular functions and show that the best constant is $c = 2$ for all of these pairs. (Note that since all three classes are closed under scalar multiplication, the inequality $v \leq \tilde{v} \leq cv$ above can also be replaced by $v/c \leq \tilde{v} \leq v$.) The details can be found in the full version of this paper [18].

4 Submodular Valuations

In this section we consider submodular valuations and establish bounds on the approximation ratio that can be obtained using different types of pricing.

4.1 Non-uniform Pricing

We first show that we can obtain 2/3 of the optimal welfare for submodular valuations if we are allowed to set non-uniform prices, and this bound is tight.

Theorem 4.1. *For every market with symmetric submodular valuations, there exists a static item pricing* $\mathbf{p}$ *that guarantees at least* $2/3$ *of the optimal social welfare.*

Proposition 4.2. *There exists a market with two items and two buyers with symmetric submodular valuations such that every static pricing can guarantee a social welfare of at most* $2/3$ *of the optimal social welfare.*

The negative result in Proposition 4.2 is obtained for a market with two items. In what follows we show that the guaranteed social welfare is higher when the number of items is large.

Theorem 4.3. *For every market of* m *items with symmetric submodular valuations, there exists a static item pricing* $\mathbf{p}$ *that guarantees at least* $5/7 - 1/m$ *of the optimal social welfare.*

The guarantee in Theorem 4.3 approaches $5/7 \approx 0.714$ as the number of items grows. The next theorem shows that this bound cannot exceed 0.802 even for an arbitrarily large number of items.

Theorem 4.4. *For every constant* c*, there exists a market with* $m > c$ *items with symmetric submodular valuations such that for any static item pricing* $\mathbf{p}$*, the social welfare guaranteed by the pricing is at most* 0.802 *of the optimal social welfare.*

The next result shows that if we know the order of the agents beforehand (while having no control over this order), then we can extract the full optimal welfare.

Theorem 4.5. *For every market with symmetric submodular valuations with a known order of arrival, there exists a static pricing* $\mathbf{p}$ *that guarantees the optimal social welfare.*

4.2 Uniform Pricing

We now show that if we restrict ourselves to using uniform pricing with submodular valuations, we can still guarantee 1/2 of the optimal welfare. This bound is also tight.

Theorem 4.6. *For every market with symmetric submodular valuations, there exists a static uniform pricing* p *that guarantees at least* $1/2$ *of the optimal social welfare.*

Proposition 4.7. *There exists a market with* m *items and two buyers with symmetric submodular valuations such that every uniform static pricing yields a social welfare of at most* $\frac{m}{2m-1}(\approx\frac{1}{2})$ *of the optimal social welfare.*

Proposition 4.8. *There exists a market with identical buyers with symmetric submodular valuation such that every uniform static pricing yields a social welfare of at most* $\frac{n+1}{2n}(\approx\frac{1}{2})$ *of the optimal social welfare.*

4.3 Dynamic Pricing

If we allow dynamic pricing, the following result shows that we can extract the full optimal welfare.

Theorem 4.9. *For every market with* n *agents with symmetric submodular valuations over* m *items, there exists a dynamic item pricing that guarantees the optimal social welfare.*

5 XOS Valuations

In this section we consider XOS valuations. We give upper bounds on the approximation ratio for both static and dynamic pricing.

Theorem 5.1. *There exists a market of* m *items and two agents with symmetric XOS valuations for which no static pricing yields more the* $1 - 1/e$ *of the optimal social welfare.*

Theorem 5.2. *There exists a market of three items and two agents with symmetric XOS valuations for which no dynamic pricing yields more the* $5/6$ *of the optimal social welfare.*

6 Subadditive Valuations

In this section we consider subadditive valuations. Our main result of this section is the existence of a uniform price that guarantees at least $1/3$ of the optimal welfare.

Theorem 6.1. *For every market of* m *items with symmetric subadditive valuations, there exists a uniform static item pricing* $\mathbf{p}$ *that guarantees at least* $1/3$ *of the optimal social welfare.*

If there are two identical agents, this bound can be improved to $2/3$.

Theorem 6.2. *For every market of* m *items and two identical agents with symmetric subadditive valuations, there exists a uniform static item pricing* $\mathbf{p}$ *that guarantees at least* $2/3$ *of the optimal social welfare.*

The next propositions show that the bound in Theorem 6.1 cannot be improved to more than $1/2$, and in the case of using only uniform pricing, cannot be improved to more than $1/3$. Hence, this bound is tight for uniform pricing.

Proposition 6.3. *There is a market with symmetric subadditive valuations with* m *items and two agents such that no static pricing* $\mathbf{p}$ *guarantees more than* $1/2$ *of the optimal social welfare.*

Proposition 6.4. *There is a market with symmetric subadditive valuations with* m *items and three agents such that no uniform static pricing* $\mathbf{p}$ *guarantees more than* $1/3$ *of the optimal social welfare.*

In the case of two identical agents, the approximation cannot be improved to more than $3/4$. In this special case, we can guarantee at least half of the social welfare by applying Theorem 7.3.

Proposition 6.5. *There exists a market of* m *items and two identical agents with a subadditive valuation such that no static pricing guarantees more the* $3/4$ *of the optimal social welfare.*

If we use uniform pricing, we cannot guarantee more than $2/3$ of the welfare for two identical agents. This means that the bound in Theorem 6.2 is tight.

Proposition 6.6. *There is a market with symmetric subadditive valuations with* m *items and two identical agents such that no uniform static pricing* $\mathbf{p}$ *guarantees more than* $2/3$ *of the optimal social welfare.*

7 General Valuations

In this section we consider general valuations. While the analysis assumes monotonicity, all results hold even for non-monotone valuations: simply do all calculations based on the monotone closure of the valuations.

7.1 Worst-Case Ordering

We first show that for general valuations, we cannot guarantee more than $1/m$ of the optimal welfare even if we know the order of arrival, and this is tight.

Proposition 7.1. *There is a market with symmetric valuations over* m *items and two agents such that no static pricing* $\mathbf{p}$ *guarantees more than* $1/m$ *of the optimal social welfare even for a known order of arrival.*

Theorem 7.2. *For every market of* m *items, there exists a uniform static item pricing* $\mathbf{p}$ *that guarantees at least* $1/m$ *of the optimal social welfare.*

7.2 Best-Case Ordering

Next, we show that if we can choose the order of arrival, then we can guarantee at least half of the optimal welfare. We remark that when agents are identical, the order of arrival does not matter, and therefore our result holds for the setting with identical agents as well. This bound is also tight.

Theorem 7.3. *For every market of* m *items, there exists a uniform static item pricing* $\mathbf{p}$ *along with an order of arrival that guarantees at least* $1/2$ *of the optimal social welfare.*

Proposition 7.4. *There exists a market of* m *items for which no static pricing and order of arrival yields more than* $1/2$ *of the optimal social welfare.*

8 Bayesian Setting

In this section, we consider the Bayesian setting, where the valuation function of each agent is drawn independently from a distribution which can be different for different agents.

8.1 XOS Valuations

Feldman et al. [21] showed that if agents' valuations are drawn independently from a distribution over XOS valuation functions, then there exist prices that yield expected welfare at least half of the expected optimal welfare. These posted prices are non-uniform, even when the result is specialized to the case of identical items. We first restate Feldman et al.'s result and then show that if the items are identical, then the same bound can be obtained using uniform prices.

Theorem 8.1 [21]**.** *Let $\mathcal{F} = F_1 \times \cdots \times F_n$ be a product distribution over XOS valuation functions. For every $\mathbf{v} = (v_1, \ldots, v_n) \in \mathcal{F}$, let $X^*(\mathbf{v}) = (X_1^*(\mathbf{v}), \ldots, X_n^*(\mathbf{v}))$ be any allocation that maximizes the social welfare. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be additive functions such that $v_i(S) \geq a_i(S)$ for any subset S of items, and $v_i(X_i^*) = a_i(X_i^*)$. When the items are offered at prices $p_j = E_{\mathbf{v}\in\mathcal{F}}[a_i(j)/2$ where $j \in X_i^*(\mathbf{v})]$, the expected social welfare is at least $OPT/2$.*

Theorem 8.2. *Let $\mathcal{F} = F_1 \times \cdots \times F_n$ be a product distribution over symmetric XOS valuation functions. Let OPT be the expected optimal social welfare. When all items are offered at the uniform price $OPT/(2m)$, the expected social welfare is at least $OPT/2$.*

8.2 Subadditive and General Valuations

We now define a notion that describes how close an arbitrary valuation function is to an XOS function and derive approximation results in terms of this closeness quantity. The proof of Theorem 8.3 follows the analysis presented by Feldman et al. [21].

Definition 8.1. *We say that a (not necessarily symmetric) valuation function v is* c-close *to XOS if there exists an XOS function $\tilde{v}$ such that for every set of item S, it holds that $v(S)/c \leq \tilde{v}(S) \leq v(S)$.*

Theorem 8.3. *For any product distribution $\mathcal{F}$ over (not necessarily symmetric) valuation functions that are c-close to XOS, there exist anonymous prices $\mathbf{p}$ that guarantee an expected social welfare of at least $1/(2c)$ of the optimal expected welfare.*

Since any symmetric subadditive function is 2-close to XOS (see Sect. 3), Theorem 8.3 implies that we can obtain at least 1/4 of the expected optimal welfare when the agents' valuations are drawn from a product distribution over subadditive valuations. In addition, using techniques similar to those in the proof of Theorem 8.2, we can achieve this with uniform prices.

Theorem 8.4. *Let $\mathcal{F} = F_1 \times \cdots \times F_n$ be a product distribution over symmetric subadditive valuation functions. Let OPT be the expected maximal social welfare. There exists a uniform price on the items for which the expected social welfare is at least $OPT/4$.*

Our results cease to hold for general valuations, even if we can control the order of arrival.

Proposition 8.5. *There is a market with n agents and $m = n^2$ items and a distribution over symmetric valuations such that no static pricing $\mathbf{p}$ yields expected welfare more than $\Theta(1/n)$ of the optimal expected welfare, even if we can control the arrival order.*

9 Discussion

In this paper, we study the fraction of the optimal social welfare that can be achieved via posted prices in markets with identical items under various assumptions on the designer's information and agents' valuations. We show that in the Bayesian setting, uniform posted prices can guarantee 1/2 and 1/4 of the optimal welfare for XOS and subadditive valuations, respectively. If the designer has full information on agents' valuations, then 1/3 of the optimal welfare can be obtained via uniform prices for subadditive valuations. For general valuations, we exhibit a tight bound of $1/m$ for both uniform and non-uniform prices; on the other hand, if the designer can control the arrival order, then 1/2 of the optimal welfare can be guaranteed for such valuations.

Our work sheds light on the power of uniform prices for settings with identical items. For submodular valuations in the full-information setting, there is a gap between the guarantee that can be obtained by uniform and non-uniform prices, while for XOS valuations in the Bayesian setting there is no gap. It would be interesting to determine whether such a gap exists for subadditive valuations, both for the full-information and the Bayesian setting. Finally, it also remains open whether the constant approximation guarantee provided here for subadditive valuations over identical items holds also for subadditive valuations over heterogeneous items. This problem has been raised by Feldman et al. [21] who provide a logarithmic (in m) bound for this setting.

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