Max-Min Greedy Matching

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CCS CONCEPTS

• Mathematics of computing → Matchings and factors; **• Theory of computation →** *Computational pricing and auctions.*

KEYWORDS

Online matching, Pricing mechanism, Markets

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There has been much recent interest in the online bipartite matching problem of Karp, Vazirani and Vazirani [2], and variations of it, due to its applicability to allocation problems in certain economic settings. A prominent example is online advertising; for more details, see the survey by Metha [3]. The new problems are both theoretically elegant and practically relevant.

Our Setting. We study a setting related to online bipartite matching, that we call *Max-Min Greedy matching*. Let G_n be the family of bipartite graphs with perfect matching of size n. Our setting is a game between a maximizing player and a minimizing player. The bipartite graph $G(U, V; E) \in G_n$ is given upfront. Upon seeing G the maximizing player chooses a permutation π over V. Upon seeing G and π, the minimizing player chooses a permutation σ over U. The combination of G, π and σ define a unique matching $M_G[\sigma, \pi]$ that we refer to as the greedy matching. It is the matching produced by the greedy matching algorithm in which vertices of U arrive in order σ and each vertex $u \in U$ is matched to the highest (under π) yet unmatched $v \in N(u)$ (or left unmatched, if all $N(u)$ is already matched).

Let $\rho[G] = \frac{1}{n} \max_\pi \min_\sigma [|M_G[\sigma, \pi]|]$, and let $\rho = \min_{G \in G_n} [\rho[G]]$. It is easy to see that $\rho \geq \frac{1}{2}$; since every greedy matching is a *maximal* matching, for *every* permutation π the obtained matching is of size at least $n/2$. The question we study in this work is whether the max player can ensure a matching of size strictly greater than $n/2$; that is, whether ρ is strictly greater than $\frac{1}{2}$. For an upper bound on ρ, it was observed by Cohen Addad et al. [1] that $\rho \leq 2/3$.

Our results. Our main result is the following:

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Theorem [main theorem]: It holds that $\rho \geq \frac{1}{2} + \frac{1}{86} > 0.51$. Moreover, there is a polynomial time algorithm that given $G(U, V; E)$ produces a permutation π over V satisfying the above bound.

We believe that further improvements are possible. A first attempt in proving such a result would be to check whether a random permutation π obtains the desired result (in expectation). Unfortunately, we show a graph G for which a random permutation matches no more than a fraction $1/2 + o(1)$ of the vertices. In contrast, we show that in the case of Hamiltonian graphs a random permutation guarantees a competitive ratio strictly greater than $1/2$. A similar proof approach applies to regular graphs as well.

We further establish lower and upper bounds for regular graphs.

Theorem [regular graphs]: For d-regulars bipartite graphs, $\rho \geq \frac{5}{9} - O(\frac{1}{\sqrt{d}})$. On the other hand, for every integer $d \geq 1$, there is a regular graph G_d of even degree $2d$ such that $\rho(G_d) \leq \frac{8}{9}$.

An additional natural problem is to find the best permutation π, given a graph G. For the special case of determining whether there is a *perfect* π (a permutation on V that for every permutation σ leads to a perfect matching), we give a polynomial time algorithm that outputs π.

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