

# Implementation with a Bounded Action Space

[Extended Abstract]\*

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## ABSTRACT

While traditional mechanism design typically assumes isomorphism between the agents' type- and action spaces, in many situations the agents face strict restrictions on their action space due to, e.g., technical, behavioral or regulatory reasons. We devise a general framework for the study of mechanism design in single-parameter environments with restricted action spaces. Our contribution is threefold. First, we characterize sufficient conditions under which the information-theoretically optimal social-choice rule can be implemented in dominant strategies, and prove that any multilinear social-choice rule is dominant-strategy implementable with no additional cost. Second, we identify necessary conditions for the optimality of action-bounded mechanisms, and fully characterize the optimal mechanisms and strategies in games with two players and two alternatives. Finally, we prove that for any multilinear social-choice rule, the optimal mechanism with  $k$  actions incurs an expected loss of  $O(\frac{1}{k^2})$  compared to the optimal mechanisms with unrestricted action spaces. Our results apply to various economic and computational settings, and we demonstrate their applicability to signaling games, public-good models and routing in networks.

## Categories and Subject Descriptors

J.4 [Social and Behavioral Sciences]: Economics; K.4.4 [Electronic Commerce]: Payment schemes

## General Terms

Economics, Design, Theory

## Keywords

Implementation, Mechanism Design, Single-Crossing Condition, Communication Complexity

\*A full version can be found on the web pages of the authors.

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## 1. INTRODUCTION

Mechanism design is a sub-field of game theory that studies how to design rules of games resulting in desirable outcomes, when the players are rational. In a standard setting, players hold some private information – their “types” – and choose “actions” from their action spaces to maximize their utilities. The social planner wishes to implement a social-choice function, which maps each possible state of the world (i.e., a profile of the players' types) to a single alternative. For example, a government that wishes to undertake a public-good project (e.g., building a bridge) only if the total benefit for the players exceeds its cost.

Much of the literature on mechanism design restricts attention to *direct revelation* mechanisms, in which a player's action space is identical to his type space. This focus is owing to the *revelation principle* that asserts that if some mechanism achieves a certain result in an equilibrium, the same result can be achieved in a truthful one – an equilibrium where each agent simply reports his private type [15].

Nonetheless, in many environments, direct-revelation mechanisms are not viable since the actions available for the players have a limited expressive power. Consider, for example, the well-studied “screening” model, where an insurance firm wishes to sell different types of policies to different drivers based on their caution levels, which is their private information. In this model, drivers may have a continuum of possible caution levels, but insurance companies offer only a few different policies since it might be either infeasible or illegal to advertise and sell more than few types of policies.

There are various reasons for such strict restrictions on the action spaces. In some situations, firms are not willing, or cannot, run a bidding process but prefer fixing a price for some product or service. The buyers in such environments face only two actions – to buy or not to buy – although they may have an infinite number of possible values for the item. In many similar settings, players might be also reluctant to reveal their accurate types, but willing to disclose partial information about them. For example, agents will typically be unwilling to reveal their types, even if it is beneficial for them in the short run, since it might harm them in future transactions. Agents may also not trust the mechanism to keep their valuations private [16], or not even know their exact type while computing it may be expensive [12]. Limitations on the action space can also be caused by technical constraints, such as severe restrictions on the communication lines [5] or from the need to perform quick transactions (e.g., discrete bidding in English auctions [9]).

Consider for example a public-good model: a social planner needs to decide whether to build a bridge. The two players in the game have some privately known benefits  $\theta_1, \theta_2 \in [0, 1]$  from using this bridge. The social planner aims to build the bridge only if the sum of these benefits exceeds the construction cost of the bridge. The social planner cannot access the private data of the players, and can only learn about it from the players' actions. When direct revelation is allowed, the social planner can run the well-known VCG mechanism, where the players have incentives to report their true data; hence, the planner can elicit the exact private information of the players and build the bridge only when it should be built. Assume now that the players cannot send their entire secret data, but can only choose an action out of two possible actions (e.g., "0" or "1"). Now, the social planner will clearly no longer be able to always build the bridge according to her objective function, due to the limited expressiveness of the players' messages. In this work we try to analyze what can be achieved in the presence of such restrictions.

Restrictions on the action space, for specific models, were studied in several earlier papers. The work of Blumrosen, Nisan and Segal [4, 6, 5] is the closest in spirit to this paper. They studied single-item auctions where bidders are allowed to send messages with severely bounded size. They characterized the optimal mechanisms under this restriction, and showed that nearly optimal results can be achieved even with very strict limitations on the action space. Other work studied similar models for the analysis of discrete-bid ascending auctions [9, 11, 8, 7], take-it-or-leave-it auctions [17], or for measuring the effect of discrete "priority classes" of buyers on the performance of electricity markets [19, 14]. Our work generalizes the main results of Blumrosen et al. to a general mechanism-design framework that can be applied to a multitude of models. We show that some main properties proved by Blumrosen et al. are preserved in more general frameworks (for example, that dominant-strategy equilibrium can be achieved with no additional cost, and that the loss diminishes with the number of possible actions in a similar rate), where some other properties do not always hold (for example, that asymmetric mechanisms are optimal and that players must always use all their action space).

A standard mechanism design setting is composed of agents with private information (their "types"), and a social planner, who wishes to implement a *social choice function*,  $c$  – a function that maps any profile of the agents' types into a chosen alternative. A classic result in this setting says that under some monotonicity assumption on the agents' preferences – the "single-crossing" assumption (see definition below) – a social-choice function is implementable in dominant strategies if and only if it is monotone in the players' types. However, in environments with restricted action spaces, the social planner cannot typically implement every social-choice function due to inherent informational constraints. That is, for some realizations of the players' types, the decision of the social planner will be incompatible with the social-choice function  $c$ . In order to quantitatively measure how well bounded-action mechanisms can approximate the original social-choice functions, we follow a standard assumption that the social choice function is derived from a *social-value* function,  $g$ , which assigns a real value for every alternative  $A$  and realization of the players' types. The social-choice function  $c$  will therefore choose an alterna-

tive that maximizes the social value function, given the type vector  $\vec{\theta} = (\theta_1, \dots, \theta_n)$ , i.e.,  $c(\vec{\theta}) = \operatorname{argmax}_A \{g(\vec{\theta}, A)\}$ . Observe that the social-value function is not necessarily the social *welfare* function – the social welfare function is a special case of  $g$  in which  $g$  is defined to be the sum of the players' valuations for the chosen alternative. Following are several simple examples of social-value functions:

- *Public goods.* A government wishes to build a bridge only if the sum of the benefits that agents gain from it exceeds its construction cost  $C$ . The social value functions in a 2-player game will therefore be:  
 $g(\theta_1, \theta_2, \text{"build"}) = \theta_1 + \theta_2 - C$  and  
 $g(\theta_1, \theta_2, \text{"do not build"}) = 0$ .
- *Routing in networks.* Consider a network that is composed of two links in parallel. Each link has a secret probability  $p_i$  of transferring a message successfully. A sender wishes to send his message through the network only if the probability of success is greater than, say, 90 percent – the known probability in an alternate network. That is,  
 $g(p_1, p_2, \text{"send in network"}) = 1 - (1 - p_1) \cdot (1 - p_2)$  and  
 $g(p_1, p_2, \text{"send in alternate network"}) = 0.9$ .
- *Single-item auctions.* Consider a 2-player auction, where the auctioneer wishes to allocate the item to the player who values it the most. The social choice function is given by:  $g(\theta_1, \theta_2, \text{"player 1 wins"}) = \theta_1$  and for the second alternative is  $g(\theta_1, \theta_2, \text{"player 2 wins"}) = \theta_2$ .

## 1.1 Our Contribution

In this paper, we present a general framework for the study of mechanism design in environments with a limited number of actions. We assume a Bayesian model where players have one-dimensional private types, independently distributed on some real interval.

The main question we ask is: when agents are only allowed to use  $k$  different actions, which mechanisms achieve the optimal expected social-value? Note that this question is actually composed of two separate questions. The first question is an information-theoretic question: what is the optimal result achievable when the players can only reveal information using these  $k$  actions (recall that their type space may be continuous). The other question involves game-theoretic considerations: what is the best result achievable with  $k$  actions, where this result should be achieved in a dominant-strategy equilibrium. These questions raise the question about the "*price of implementation*": can the optimal information-theoretic result always be implemented in a dominant-strategy equilibrium? And if not, to what extent does the dominant-strategy requirement degrade the optimal result? What we call "the price of implementation" was also explored in other contexts in game theory where computational restrictions apply: for example, is it always true that the optimal polynomial-time approximation ratio (for example, in combinatorial auctions) can be achieved in equilibrium? (The answer for this interesting problem is still unclear, see, e.g., [3, 2, 13].)

Our first contribution is the characterization of sufficient conditions for implementing the optimal information-theoretic social-choice rule in dominant strategies. We show that for the family of *multilinear* social-value functions (that

is, polynomials where each variable has a degree of at most one in each monomial) the dominant-strategy implementation incurs no additional cost.

**Theorem:** *Given any multilinear single-crossing social-value function, and for any number of alternatives and players, the social choice rule that is information-theoretically optimal is implementable in dominant strategies.*

Multilinear social-value functions capture many important and well-studied models, and include, for instance, the routing example given above, and any *social welfare* function in which the players’ valuations are linear in their types (such as public-goods and auctions).

The implementability of the information-theoretically optimal mechanisms enables us to use a standard routine in Mechanism Design and first determine the optimal social-choice rule, and then calculate the appropriate payments that ensure incentive compatibility. To show this result, we prove a useful lemma that gives another characterization for social-choice functions whose “price of implementation” is zero. We show that for any social-choice function, incentive compatibility in action-bounded mechanisms is equivalent to the property that the optimal expected social value is achieved with *non-decreasing* strategies (or *threshold strategies*).<sup>1</sup> In other words, this lemma implies that one can always implement, with dominant strategies, the best social-choice rule that is achievable with non-decreasing strategies.

Our second contribution is in characterizing the optimal action-bounded mechanisms. We identify some necessary conditions for the optimality of mechanisms in general, and using these conditions, we fully characterize the optimal mechanisms in environments with two players and two alternatives. The optimal mechanisms turn out to be “diagonal” – that is, in their matrix representation, one alternative will be chosen in, and only in, entries below one of the main diagonals (this term extends the concept of “Priority Games” used in [5] for bounded-communication auctions). We complete the characterization of the optimal mechanisms with the depiction of the optimal strategies – strategies that are “mutually maximizers”. Since the payments in a dominant-strategy implementation are uniquely defined by a monotone allocation and a profile of strategies, this also defines the payments in the mechanism. We give an intuitive proof for the optimality of such strategies, generalizing the concept of optimal “mutually-centered” strategies from [4]. Surprisingly, as opposed to the optimal auctions in [4], for some non-trivial social-value functions, the optimal “diagonal” mechanism may not utilize all the  $k$  available actions.

**Theorem:** *For any multilinear single-crossing social-value function over two alternatives, the informationally optimal 2-player  $k$ -action mechanism is diagonal, and the optimal dominant strategies are mutually-maximizers.*

Achieving a full characterization of the optimal action-bounded mechanism for multi-player or multi-alternative environments seems to be harder. To support this claim, we observe that the number of mechanisms that satisfy the necessary conditions above is growing exponentially in the number of players.

<sup>1</sup>The restriction to non-decreasing strategies is very common in the literature. One remarkable result by Athey [1] shows that when a non-decreasing strategy is a best response for any other profile of non-decreasing strategies, a pure Bayesian-Nash equilibrium must exist.

Our next result compares the expected social-value in  $k$ -action mechanisms to the optimal expected social value when the action space is unrestricted. For any number of players or alternatives, and for any profile of independent distribution functions, we construct mechanisms that are nearly optimal – up to an additive difference of  $O(\frac{1}{k^2})$ . This result is achieved in dominant strategies.

**Theorem:** *For any multilinear social-value function, the optimal  $k$ -action mechanism incurs an expected social loss of  $O(\frac{1}{k^2})$ .*

This is the same asymptotic rate proved for specific environments in [19, 9, 5]. Note that there are social-choice functions that can be implemented with  $k$  actions with no loss at all (for example, the rule “always choose alternative  $A$ ”). However, we know that in some settings (e.g., auctions [5]) the optimal loss may be proportional to  $\frac{1}{k^2}$ , thus a better general upper bound is impossible.

Finally, we present our results in the context of several natural applications. First, we give an explicit solution for a public-good game with  $k$ -actions. We show that the optimum is achieved in symmetric mechanisms (in contrast to action-bounded auctions [5]), and that the optimal allocation scheme depends on the value of the construction cost  $C$ . Then, we study the celebrated *signaling* model, in which potential employees send signals about their skills to potential employers by means of the education level they acquire. This is a natural application in our context since education levels are often discrete (e.g., B.A, M.A and PhD). Lastly, we present our results in the context of routing in networks, where it is reasonable to assume that links report whether they have low or high loss rates, but less reasonable to require them to report their accurate loss rates. The latter example illustrates how our results apply to settings where the goal of the social planner is not welfare maximization (nor variants of it like “affine maximizers”).

The rest of the paper is organized as follows: our model and notations are described in Section 2. We then describe our general results regarding implementation in multi-player and multi-alternative environments in Section 3, including the asymptotic analysis of the social-value loss. In Section 4, we fully characterize the optimal mechanisms for 2-player environments with two alternative. In Section 5, we conclude with applying our general results to several well-studied models. Due to lack of space, some of the proofs are missing and can be found in the full version that can be found on the authors’ web pages.

## 2. MODEL AND PRELIMINARIES

We first describe a standard mechanism-design model for players with one-dimensional types. Then, in Subsection 2.2, we impose limitation on the action space. The general model studies environments with  $n$  players and a set  $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$  of  $m$  alternatives. Each player has a privately known type  $\theta_i \in [\underline{\theta}_i, \bar{\theta}_i]$  (where  $\underline{\theta}_i, \bar{\theta}_i \in \mathbb{R}$ ,  $\underline{\theta}_i < \bar{\theta}_i$ ), and a type-dependent valuation function  $v_i(\theta_i, A)$  for each alternative  $A \in \mathcal{A}$ . In other words, player  $i$  with type  $\theta_i$  is willing to pay an amount of  $v_i(\theta_i, A)$  for alternative  $A$  to be chosen. Each type  $\theta_i$  is independently distributed according to a publicly known distribution  $F_i$ , with an always positive density function  $f_i$ . We denote the set of all possible type profiles by  $\Theta = \times_{i=1}^n [\underline{\theta}_i, \bar{\theta}_i]$ .

The social planner has a *social-choice function*  $c : \Theta \rightarrow \mathcal{A}$ , where the choice of alternatives is made in order to maximize a *social-value function*  $g(\vec{\theta}) : \Theta \times \mathcal{A} \rightarrow \mathbb{R}$ . That is,  $c(\vec{\theta}) \in \text{argmax}_{A \in \mathcal{A}} \{g(\vec{\theta}, A)\}$

We assume that for every alternative  $A \in \mathcal{A}$ , the function  $g(\cdot, A)$  is continuous and differentiable in every type. Since the players' types are private information, in order to choose the optimal alternative, the social planner needs to get the players' types as an input. The players reveal information about their types by choosing an *action*, from an action set  $B$ .

Each player uses a *strategy* for determining the action he plays for any possible type. A strategy for player  $i$  is therefore a function  $s_i : [\underline{\theta}_i, \bar{\theta}_i] \rightarrow B$ . We denote a profile of strategies by  $s = s_1, \dots, s_n$  and the set of the strategies of all players except  $i$  by  $s_{-i}$ . The utility of player  $i$  of type  $\theta_i$  from alternative  $A$  under the payment  $p_i$  is  $u_i = v_i(\theta_i, A) - p_i$ .

## 2.1 Dominant-Strategy Implementation

Following is a standard definition of a mechanism. The action space  $B$  is traditionally implicit, but we mention it explicitly since we later examine limitations on  $B$ .

DEFINITION 1. A mechanism with an action set  $B$  is a pair  $(t, p)$  where:

- $t : B^n \rightarrow \mathcal{A}$  is the allocation rule.<sup>2</sup>
- $p : B^n \rightarrow \mathbb{R}^n$  is the payment scheme (i.e.,  $p_i(b)$  is the payment to the  $i$ th player given a vector of actions  $b$ ).

The main goal of this paper is to optimize the expected social value (in action-bounded mechanisms) while preserving a dominant-strategy equilibrium.

We say that a strategy  $s_i$  is *dominant* for player  $i$  in mechanism  $(t, p)$  if player  $i$  cannot increase his utility by reporting a different action than  $s_i(\theta_i)$ , regardless of the actions of the other players  $b_{-i}$ .<sup>3</sup>

DEFINITION 2. We say that a social-choice function  $h$  is implementable with a set of actions  $B$  if there exists a mechanism  $(t, p)$  with a dominant-strategy equilibrium  $s_1, \dots, s_n$  (where for each  $i$ ,  $s_i : [\underline{\theta}_i, \bar{\theta}_i] \rightarrow B$ ) that always chooses an alternative according to  $h$ , i.e.,  $t(s_1(\theta_1), \dots, s_n(\theta_n)) = h(\vec{\theta})$ .

A fundamental result in the mechanism-design literature states that under the ‘‘single-crossing’’ condition, the monotonicity of the social-choice function is a sufficient and necessary condition for dominant-strategy implementability (in single-parameter environments). The single-crossing condition (also known as the Spence-Mirrlees condition) appears, very often implicitly, in almost every paper on mechanism design in one-dimensional domains. Without this assumption, general sufficient condition for implementability are unknown (for a survey on this topic see [10]). Throughout this paper, we assume that the valuation functions of the players are single-crossing, as defined below. A player's valuation function will be single-crossing if the effect of an increment in the player's type on the player's valuation for

<sup>2</sup>We will show that, w.l.o.g., we can focus on deterministic allocation schemes.

<sup>3</sup>That is, for every type  $\theta_i$  and every action  $b'_i$ , we have that  $v_i(\theta_i, t(s_i(\theta_i), b_{-i})) - p_i(s_i(\theta_i), b_{-i}) \geq v_i(\theta_i, t(b'_i, b_{-i})) - p_i(b'_i, b_{-i})$

two alternatives is always greater for one of these alternatives. The single-crossing condition on the players' preferences actually defines an order on the alternatives. For example, if the value of player  $i$  for alternative  $A$  increases more rapidly than his value for alternative  $B$ , we can denote it by  $A \succ_i B$ . Later on, we will use these orders for defining monotonicity of social-choice functions.

DEFINITION 3. A function  $h : \Theta \times \mathcal{A} \rightarrow \mathbb{R}$  is single crossing with respect to  $i$  if there is a weak order  $\succ_i$  on the alternatives, such that for any two alternatives  $A_j \succ_i A_l$  we have that for every  $\vec{\theta} \in \Theta$ ,

$$\frac{\partial h(\vec{\theta}, A_j)}{\partial \theta_i} > \frac{\partial h(\vec{\theta}, A_l)}{\partial \theta_i}$$

and if  $A_j \sim A_l$  (that is,  $A_l \succ_i A_j$  and  $A_j \succ_i A_l$ ) then  $h(\cdot, A_j) \equiv h(\cdot, A_l)$  (i.e., the functions are identical).

The definition of monotone social-choice functions also requires an order on the actions. This order is implicit in most of the standard settings where, for example, it is defined by the order on the real numbers (e.g., in direct revelation mechanisms where each type is drawn from a real interval). When the action space is discrete, the order can be determined by the names of the actions, for example, ‘‘0’’, ‘‘1’’, ‘‘...’’, ‘‘k-1’’ for  $k$ -action mechanisms. (We therefore describe this order with the standard relation on natural numbers  $<, >$ .)

DEFINITION 4. A deterministic mechanism is monotone if when player  $i$  raises his reported action, and fixing the actions of the other players, the mechanism never chooses an inferior alternative for  $i$ . That is, for any  $b_{-i} \in \{0, \dots, k-1\}^{n-1}$  if  $b'_i > b_i$  then  $t(b'_i, b_{-i}) \succeq_i t(b_i, b_{-i})$ .

Following is a classic result regarding the implementability of social-choice functions in single-parameter environments. Note, however, that this characterization does not hold when the action space is bounded.

PROPOSITION 1. Assume that the valuation functions  $v_i(\theta_i, A)$  are single crossing and that the action space is unrestricted. A social-choice function  $c$  is dominant-strategy implementable if and only if  $c$  is monotone.

## 2.2 Action-Bounded Mechanisms

The set of actions  $B$  is usually implicit in the literature, and it is assumed to be isomorphic to the type space. In this paper, we study environments where this assumption does not hold. We define a *k-action game* to be a game in which the number of possible actions for each player is  $k$ , i.e.,  $|B| = k$ . In  $k$ -action games, the social planner typically cannot always choose an alternative according to the social choice function  $c$  due to the informational constraints. Instead, we are interested in implementing a social-choice function that, with  $k$  actions, maximizes the *expected social value*:  $E_{\vec{\theta}} g(\vec{\theta}, t(s_1(\theta_1), \dots, s_n(\theta_n)))$ .

DEFINITION 5. We say that a social-choice function  $h : \Theta \rightarrow \mathcal{A}$  is informationally achievable with a set of actions  $B$  if there exists a profile of strategies  $s_1, \dots, s_n$  (where for each  $i$ ,  $s_i : [\underline{\theta}_i, \bar{\theta}_i] \rightarrow B$ ), and an allocation rule  $t : B^n \rightarrow \mathcal{A}$ , such that  $t$  chooses the same alternative as  $h$  for any type profile, i.e.,  $t(s_1(\theta_1), \dots, s_n(\theta_n)) = h(\vec{\theta})$ . If  $|B| = k$ , we say that  $h$  is  $k$ -action informationally achievable.

Note that this definition does not take into account strategic considerations. For example, consider an environment with two alternatives  $\mathcal{A} = \{A, B\}$ , and the following social-choice function:  $\tilde{c}(\theta_1, \theta_2) = A$  iff  $\{\theta_1 > 1/2 \text{ and } \theta_2 > 1/2\}$ .  $\tilde{c}$  is informationally achievable with two actions: if both players bid “0” when their value is greater than 1/2 and “1” otherwise, then the allocation rule “choose alternative  $A$  if both players report 1” derives exactly the same allocation for every profile of types. In contrast, it is easy to see that the function  $\hat{c}(\theta_1, \theta_2) = A$  iff  $\theta_1 + \theta_2 > 1/2$  is not informationally achievable with two actions.

We now define a social-choice rule that maximizes the social value under the information-theoretic constraints that are implied by the limitations on the number of actions.

**DEFINITION 6.** *A social-choice function is  $k$ -action informationally optimal with respect to the social-value function  $g$ , if it achieves the maximal expected social value among all the  $k$ -action informationally achievable social-choice functions.<sup>4</sup>*

Earlier in this section, we defined the single-crossing property for the players valuations. We now define a single-crossing property on the social-value function  $g$ . This property clearly ensures the monotonicity of the corresponding social choice rule, and we will later show that it is also useful for action-bounded environments.

**DEFINITION 7.** *We say that the social-choice rule  $g(\vec{\theta}, A)$  exhibits the single-crossing property if for every player  $i$ ,  $g$  exhibits the single-crossing property with respect to  $i$ .*

Note that the definition above requires that  $g$  will be single crossing with respect to every player  $i$ , given her individual order  $\succ_i$  on the alternatives. That is, the social value function  $g$  will be compatible in this sense with the single-crossing conditions on the players’ preferences.

Finally, we call attention to a natural set of strategies – “non-decreasing” strategies, where each player reports a higher action as her type increases. Equivalently, such strategies are *threshold strategies* – strategies where each player divides his type support into intervals, and simply reports the interval in which her type lies.

**DEFINITION 8.** *A real vector  $x = (x_0, x_1, \dots, x_k)$  is a vector of threshold values if  $x_0 \leq x_1 \leq \dots \leq x_k$ .*

**DEFINITION 9.** *A strategy  $s_i$  is a threshold strategy based on a vector of threshold values  $x = (x_0, x_1, \dots, x_k)$ , if for any action  $j$  it holds that  $s_i(\theta_i) = j$  iff  $\theta_i \in [x_j, x_{j+1}]$ . A strategy  $s_i$  is called a threshold strategy, if there exists a vector  $x$  of threshold values such that  $s_i$  is a threshold strategy based on  $x$ .*

### 3. IMPLEMENTATION WITH A LIMITED NUMBER OF ACTIONS

In this section, we study the general model of action-bounded mechanism design. Our first result is a sufficient and necessary condition for the implementability of the optimal solution achievable with  $k$  actions: this condition says that the optimal social-choice rule is achieved when all the

<sup>4</sup>For simplicity, we assume that a maximum is attained and thus the optimal function is well defined.

players use non-decreasing strategies. The basic idea is that with non-decreasing strategies (i.e., threshold strategies), we can apply the single-crossing property to show that when a player raises his reported action, the expected value for his high-priority alternatives increases faster; therefore, monotonicity must hold. The result holds for any number of players and alternatives, and for any profile of distribution functions on the players’ types, as long as they are statistically independent. (It is easy to illustrate that this result does not hold if the players’ types are dependent.)

**LEMMA 1.** *Consider a single-crossing social-value function  $g$ . The informationally optimal  $k$ -action social-choice function  $c^*$  (with respect to  $g$ ) is implementable if and only if  $c^*$  achieves its optimum when the players use non-decreasing strategies.*

Next, we show that for a wide family of social-value functions – multilinear functions – the “price of implementation” is zero. That is, the information-theoretically optimal rule is dominant-strategy implementable. This family of functions captures many common settings from the literature. In particular, it generalizes the auction setting studied by Blumrosen et al. [4, 6].

**DEFINITION 10.** *A multilinear function is a polynomial in which the degree of every variable in each monomial is at most 1.<sup>5</sup> We say that a social-choice rule  $g$  is multilinear, if  $g(\cdot, A)$  is multilinear for every alternative  $A \in \mathcal{A}$ .*

The basic idea behind the proof of the following theorem is as follows: for every player, we show that the expected social welfare when he chooses any action (fixing the strategies of the other players) is a linear function of his type. This is a result of the multilinearity of the social-value function and of the linearity of expectation. The maximum over a set of linear functions is a piecewise-linear function, hence the optimal social value is achieved when the player uses threshold strategies (the thresholds are the switching points). Since the optimum is achieved with threshold strategies, we can apply Lemma 1 to show the monotonicity of this social-choice rule. Note that in this argument we characterize the players’ strategies that maximize the social value, and not the players’ utilities.

**THEOREM 1.** *If the social-value function is multilinear and single crossing, the informationally optimal  $k$ -action social-choice function is implementable.*

**PROOF.** We will show that for any  $k$ -action mechanism, the optimal expected social value is achieved when all players use threshold strategies. This will be shown by proving that for any player  $i$  and for any action  $b_i$  of player  $i$ , the expected welfare when she chooses the action  $b_i$  is a linear function in player  $i$ ’s type  $\theta_i$ . Then, it will follow from Lemma 1 that the social choice function is implementable.

For every action  $b_i$  of player  $i$ , let  $q_A$  denote the probability that alternative  $A$  is allocated, i.e.,

$$q_A = Pr_{\vec{\theta}} [t(s(\vec{\theta})) = A | s_i(\theta_i) = b_i]$$

<sup>5</sup>For example,  $f(x, y, z) = xyz + 5xy + 7$ .

Due to the linearity of expectation, the expected social value when player  $i$  with type  $\theta_i$  reports  $b_i$  is:

$$\begin{aligned} & \sum_{A \in \mathcal{A}} q_A E_{\theta_{-i}} (g(\theta_i, \theta_{-i}, A) \mid t(b_i, s_{-i}(\theta_{-i})) = A) \quad (1) \\ &= \sum_{A \in \mathcal{A}} q_A \int_{\theta_{-i}} g(\theta_i, \theta_{-i}, A) f_{-i}^A(\theta_{-i}) d(\theta_{-i}) \quad (2) \end{aligned}$$

where  $f_{-i}^A(\theta_{-i})$  equals  $\frac{\prod_{j \neq i} f_j(\theta_j)}{q_A}$  for types profiles  $\theta_{-i}$  such that  $t(b_i, s_{-i}(\theta_{-i})) = A$ , and 0 otherwise.

Since  $g$  is multilinear, every function  $g(\theta_i, \theta_{-i}, A)$  is a linear function in  $\theta_i$ , where the coefficients depend on the values of  $\theta_{-i}$ . Denote this function by  $g(\theta_i, \theta_{-i}, A) = \lambda_{\theta_{-i}} \theta_i + \beta_{\theta_{-i}}$ . Thus, we can write Equation 2 as:

$$\begin{aligned} & \sum_{A \in \mathcal{A}} q_A \int_{\theta_{-i}} (\lambda_{\theta_{-i}} \theta_i + \beta_{\theta_{-i}}) f_{-i}^A(\theta_{-i}) d(\theta_{-i}) \\ &= \sum_{A \in \mathcal{A}} q_A \left( \theta_i \int_{\theta_{-i}} \lambda_{\theta_{-i}} f_{-i}^A(\theta_{-i}) d(\theta_{-i}) + \int_{\theta_{-i}} \beta_{\theta_{-i}} f_{-i}^A(\theta_{-i}) d(\theta_{-i}) \right) \end{aligned}$$

In this expression, each integral is a constant independent of  $\theta_i$  when the strategies of the other player are fixed. Therefore, each summand, thus the whole function, is a linear function in  $\theta_i$ . For achieving the optimal expected social value, the player must choose the action that maximizes the expected social value. A maximum of  $k$  linear functions is a piecewise-linear function with at most  $k - 1$  breaking points. These breaking points are the thresholds to be used by the player. For all types between subsequent thresholds, the optimum is clearly achieved by a single action; Since linear functions are single-crossing, every action will be maximal in at most one interval.

The same argument applies to all the players, and therefore the optimal social value is obtained with threshold strategies.  $\square$

Observe that the proof of Theorem 1 actually works for a more general setting. For proving that the information-theoretically optimal result is achieved with threshold strategies, it is sufficient to show that the social-choice function exhibits a *single-crossing condition on expectation*: given any allocation scheme, and fixing the behavior of the other players, the expected social value in any two actions (as a function of  $\theta_i$ ) is single crossing. Theorem 1 shows that this requirement holds for multilinear functions, but we were not able to give an exact characterization of this general class of functions.

The implementability of the information-theoretically optimal solution makes the characterization of the optimal incentive-compatible mechanisms significantly easier: we can apply the standard mechanism-design technique and first calculate the optimal allocation scheme and then find the “right” payments.

Observe that if the valuation functions of the players are linear and single crossing, then the social-welfare function (i.e., the sum of the players’ valuations) is multilinear and single-crossing. This holds since the single-crossing conditions on the valuations are defined with a similar order on the alternatives as in the social-value function. Therefore,

an immediate conclusion from Theorem 1 is that the optimal social welfare, which is achievable with  $k$  actions, is implementable when the valuations are linear.

**COROLLARY 1.** *If the valuation functions  $v_i(\cdot, A)$  are single crossing and linear in  $\theta_i$  for every player  $i$  and for every alternative, then the informationally optimal  $k$ -action social welfare function is implementable.*

### 3.1 Asymptotic Analysis

In this section we show that the social value loss of multilinear social-value rules diminishes quadratically with the number of possible actions,  $k$ . This is the same asymptotic ratio presented in the study of specific models in the same spirit [19, 5, 18, 9]. The main challenge here, compared to earlier results, is in dealing with the general mechanism-design framework, that allows a large family of social-value functions for any number of players and alternatives. As opposed to the specific models, the social-value function may be asymmetric with respect to the players’ types; for instance, the social-value loss may a-priori occur in any “entry” (i.e., profile of actions).

The basic intuition for the proof is that even for this general framework, we can construct mechanisms where the probability of having an allocation that is incompatible with the original social-choice function is  $O(\frac{1}{k})$ . (This fact holds for all single-crossing social-choice functions, not only for multilinear functions.) Then, we can use the multilinearity to show that the social-value loss will always be  $O(\frac{1}{k})$  in the mechanisms we construct. Taken together, the expected loss becomes  $O(\frac{1}{k^2})$ . Our proof is constructive – we present an explicit construction for a mechanism that exhibits the desired loss in dominant strategies. The additive expected social-value depends on the length of the support of the type space. Hence, we assume that the type space is normalized to  $[0, 1]$ , that is, for every player  $i$ ,  $\bar{\theta}_i = 0$  and  $\underline{\theta}_i = 1$ .

**THEOREM 2.** *Assume that the type spaces are normalized to  $[0, 1]$ . For any number of players and alternatives, and for any set of distribution functions of the players’ types, if the social-value function  $g$  is single crossing and multilinear, then the informationally optimal  $k$ -action social-choice function (with respect to  $g$ ) incurs an expected social-value loss of  $O(\frac{1}{k^2})$ .*

Moreover, as discussed in [4], this bound is asymptotically tight. That is, there exists a set of distribution functions for the players (the uniform distribution in particular) and there are social-value functions (e.g., auctions) for which *any* mechanism incurs a social-value loss of at least  $\Omega(\frac{1}{k^2})$ . In that sense, auctions are the hardest problems with respect to the incurred loss. Yet, note that this claim does not imply that the loss of *any* social-choice function will be proportional to  $\frac{1}{k^2}$ . For example, in the social choice function that chooses the same alternative for any type profile, no loss will be incurred (even with 0 actions).

## 4. OPTIMAL MECHANISMS FOR TWO PLAYERS AND TWO ALTERNATIVES

In this section, we present a full characterization of the optimal mechanisms in action-bounded environments with two players and two alternatives, where the social-choice functions are multilinear and single crossing.

Note that in this section, as in most parts of this paper, we characterize monotone mechanisms by their *allocation* scheme and by a profile of *strategies* for the players. Doing this, we completely describe which alternative is chosen for every profile of types of the players. It is well known that in monotone mechanisms for one dimensional environments, the allocation scheme uniquely defines the payments in the dominant-strategy implementation. We find this description, which does not explicitly mention the payments, easier for the presentation.

A key notion in our characterization of the optimal action-bounded mechanism, is the notion of *non-degenerate* mechanisms. In a degenerate mechanism, there are two actions for one of the players that are identical in their allocation. Intuitively, a degenerate mechanism does not utilize all the action space it is allowed to use, and therefore it cannot be optimal. Using this property, we then define “diagonal” mechanisms that turns out to exactly characterize the set of optimal mechanisms.

**DEFINITION 11.** *A mechanism is degenerate with respect to player  $i$  if there exist two actions  $b_i, b'_i$  for player  $i$  such that for all profiles  $b_{-i}$  of actions of the other players, the allocation scheme is identical whether player  $i$  reports  $b_i$  or  $b'_i$  (i.e.,  $\forall b_{-i}, t(b_i, b_{-i}) = t(b'_i, b_{-i})$ ).*

For example, a 2-player mechanism is degenerate with respect to the “rows” player, if there are two rows with identical allocation in the matrix representation of the game.

**DEFINITION 12.** *A 2-player 2-alternative mechanism with  $k$ -possible actions is called diagonal if it is monotone, and non-degenerate with respect to at least one of the players.*

The term “diagonal” originates from the matrix representation of these mechanisms, in which one of the diagonals determines the boundary between the choice of the two alternatives (see Figure 1). Simple combinatorial considerations show that diagonal mechanisms may come in very few forms. Interestingly, one of these forms is degenerate with respect to one of the players; that is, it can be described as a mechanism with  $k - 1$  actions for this player.

**PROPOSITION 2.** *Any diagonal 2-player mechanism has one of the following forms:*

1. *If both players favor the same alternative (w.l.o.g.,  $B \succ_i A$  for  $i = 1, 2$ ) then either*
  - (a).  $t(b_1, b_2) = B$  iff  $b_1 + b_2 \geq k - 1$
  - (b).  $t(b_1, b_2) = B$  iff  $b_1 + b_2 \geq k$ .
2. *If the two players have conflicting preferences (e.g.,  $A \succ_1 B$  and  $B \succ_2 A$ ) then either*
  - (a).  $t(b_1, b_2) = B$  iff  $b_1 \geq b_2$
  - (b).  $t(b_1, b_2) = B$  iff  $b_1 > b_2$ .

*In both cases, the optimal mechanism can also take the form of one of the possibilities described, except one of the players is not allowed to choose the “fixed allocation” action.*

To complete the description of the optimal allocation scheme, we now move to determine the optimal strategies in diagonal mechanisms. We define the notion of *mutually-maximizer* thresholds, and show that threshold strategies based on such thresholds are optimal. The reason why mutually-maximizer strategies maximize the expected social value in monotone mechanisms is intuitive: Consider some action  $i$  (“row” in the matrix representation) for player 1. In a monotone mechanism, the allocation in such a row will be of the form  $[A, A, \dots, B, B]$  (assuming that  $B \succ_2 A$ ). That is, the alternative  $A$  will be chosen for low actions of player 2, and the alternative  $B$  will be chosen for higher actions of player 2. By determining a threshold for player 2, the social planner actually determines the minimal type of player 2 from which the alternative  $B$  will be chosen. For optimizing the expected social value, this type for player 2 should clearly be the type for which the expected social value from  $A$  equals the expected social value from  $B$  (given that player 1 plays  $i$ ); for greater values of player 2, the single-crossing condition ensures that  $B$  will be preferred.

**DEFINITION 13.** *Consider a monotone 2-player mechanism  $g$  that is non-degenerate with respect to the two players, where the players use threshold strategies based on the threshold vectors  $x, y$ . We say that the threshold  $x_i$  of one player (w.l.o.g. player 1) is a maximizer if*

$$\begin{aligned} E_{\theta_2} ( g(x_i, \theta_2, A) \mid \theta_2 \in [y_j, y_{j+1}] ) = \\ E_{\theta_2} ( g(x_i, \theta_2, B) \mid \theta_2 \in [y_j, y_{j+1}] ) \end{aligned}$$

where  $j$  is the action of player 2 for which the mechanism swaps the chosen alternative exactly when player 1 plays  $i$ , i.e.,  $t(i, j) \neq t(i - 1, j)$  (we denote, w.l.o.g.,  $t(i, j) = A, t(i - 1, j) = B$ ).

*The threshold vectors  $x, y$  are called mutually maximizers if all their thresholds are maximizers (except the first and the last).*

It turns out that in 2-player, 2-alternative environments, where the social-choice rule is multilinear and single crossing, the optimal expected social value is achieved in diagonal mechanisms with mutually-maximizer strategies. In the proof, we start with a  $k \times k$  allocation matrix, and show that the mechanism cannot be degenerate with respect to one of the players (we show how to choose this player). If the player, w.l.o.g., the columns player, is degenerate, then there are two columns with an identical allocation. These two columns can be unified to a single action, and the mechanism can therefore be described as a  $k \times k - 1$  matrix. We then show that we can insert a new missing column, and an appropriately chosen threshold, and strictly increase the expected social value in the mechanism. Therefore, the original mechanism was not the optimal  $k$ -action mechanism.

**THEOREM 3.** *In environments with two alternatives and two players, if the social-value function is multilinear and single crossing, then the optimal  $k$ -action mechanism is diagonal, and the optimum is achieved with threshold strategies that are mutually maximizers.*

A corollary from the proof of Theorem 1 is that the optimal 2-player  $k$ -action mechanism may be degenerate for one of the players (that is, equivalent to a game where one of the players has only  $k - 1$  different actions). However, the proof identifies the following sufficient condition under which the

	0	1	2	3
0	A	A	A	B
1	A	A	B	B
2	A	B	B	B
3	B	B	B	B

	0	1	2	3
0	A	A	A	A
1	A	A	A	B
2	A	A	B	B
3	A	B	B	B

	0	1	2	3
0	B	B	B	B
1	A	B	B	B
2	A	A	B	B
3	A	A	A	B

	0	1	2	3
0	A	A	A	B
1	A	A	B	B
2	A	B	B	B

**Figure 1:** The three left tables show all possible diagonal allocation scheme with 4 possible actions for each player. The rightmost table show an example for a diagonal allocation scheme where one of the player has only  $k - 1$  possible actions.

optimal mechanism will be non-degenerate with respect to both players: if the players' preferences are correlated (e.g.,  $A \succ_1 B$  and  $A \succ_2 B$ ), then the optimal alternative must be the same under the profiles  $(\underline{\theta}_1, \underline{\theta}_2)$  and  $(\bar{\theta}_1, \underline{\theta}_2)$ . Similarly, if the players' preferences are conflicting (e.g.,  $A \succ_1 B$  and  $B \succ_2 A$ ), then the optimal alternative must be the same under the profiles  $(\underline{\theta}_1, \underline{\theta}_2)$  and  $(\bar{\theta}_1, \bar{\theta}_2)$ . Examples in which this condition holds are the public good model presented in section 5 and auctions [5].

We do not know how to give an exact characterization of the optimal mechanisms in multi-player and multi-alternative environments. The hardness stems from the fact that the necessary conditions we specified before for the optimality of the mechanisms (i.e., non-degenerate and monotone allocations) are not restrictive enough for the general model. In other words, for  $n > 2$  players, the number of monotone and non-degenerate mechanisms becomes exponential in  $n$ .

**PROPOSITION 3.** *The number of monotone non-degenerate  $k$ -action mechanisms in an  $n$ -player game is exponential in  $n$ , even if  $|\mathcal{A}| = 2$ .*

## 5. EXAMPLES

Our results apply to a variety of economic, computational and networked settings. In this section, we demonstrate the applicability of our results to public-good models, signaling games and routing applications.

### 5.1 Application 1: Public Goods

The public-good model deals with a social planner (e.g., government) that needs to decide whether to supply a public good, such as building a bridge. Let *Yes* and *No* denote the respective alternatives of building and not building the bridge.  $v = v_1, \dots, v_n$  is the vector of the players' types – the values they gain from using the bridge. The decision that maximizes the social welfare is to build the bridge if and only if  $\sum_i v_i$  is greater than its cost, denoted by  $C$ . If the bridge is built, the social welfare is  $\sum_i v_i - C$ , and zero otherwise; thus,  $g(v, Yes) = \sum_i v_i - C$ , and  $g(v, No) = 0$ . The utility of player  $i$  under payment  $p_i$  is  $u_i = v_i - p_i$  if the bridge is built, and 0 otherwise. It is well-known that under no restriction on the action space, it is possible to induce truthful revelation by VCG mechanisms, therefore full efficiency can be achieved. Obviously, when the action set is limited to  $k$  actions, we cannot achieve full efficiency due to the informational constraints. Yet, since  $g(v, Yes)$  and  $g(v, No)$  are multilinear and single crossing, we can directly apply Theorem 1. Hence, the information-theoretically optimal  $k$ -action mechanism is implementable in dominant strategies.

**COROLLARY 2.** *The  $k$ -action informationally optimal social welfare in the  $n$ -player public-good game is implementable in dominant strategies.*

Moreover, as Theorem 3 suggests, in the  $k$ -action 2-player public-good game, we can fully characterize the optimal mechanisms. In the proof of Theorem 3, we saw that when for both players  $g(\bar{\theta}_i, \underline{\theta}_i, A) = g(\bar{\theta}_i, \underline{\theta}_i, B)$ , the mechanism is non-degenerate with respect to both players.<sup>6</sup> This condition clearly holds here ( $1 + 0 - C = 0 + 1 - C$ ), therefore the optimal mechanisms will use all  $k$  actions.

**COROLLARY 3.** *The optimal expected welfare in a 2-player  $k$ -action public-good game is achieved with one of the following mechanisms:<sup>7</sup>*

1. Allocation: Build the bridge iff  $b_1 + b_2 \geq k$ .

*Strategies:* Threshold strategies based on the vectors  $\vec{x}, \vec{y}$  where for every  $1 \leq i \leq k-1$ ,

$$x_i = C - E[v_2 | v_2 \in [y_{k-i}, y_{k-i+1}]]$$

$$y_i = C - E[v_1 | v_1 \in [x_{k-i}, x_{k-i+1}]]$$

2. Allocation: Build the bridge iff  $b_1 + b_2 \geq k - 1$ .

*Strategies:* Threshold strategies based on the vectors  $\vec{x}, \vec{y}$  where for every  $1 \leq i \leq k-1$ :

$$x_i = C - E[v_2 | v_2 \in [y_{k-i-1}, y_{k-i}]]$$

$$y_i = C - E[v_1 | v_1 \in [x_{k-i-1}, x_{k-i}]]$$

Recall that we define the optimal mechanisms by their allocation scheme and by the optimal strategies for the players. It is well known, that the allocation scheme in monotone mechanisms uniquely defines the payments that ensure incentive-compatibility. In public-good games, these payments satisfy the rule that a player pays his lowest value for which the bridge is built, when the action of the other player is fixed. Therefore, the payments for the players 1 and 2 reporting the actions  $b_1$  and  $b_2$  are as follows: in mechanism 1 from Proposition 3,  $p_1 = x_{b_2}$  and  $p_2 = y_{b_1}$ ; in mechanism 2 from Proposition 3,  $p_1 = x_{b_2-1}$  and  $p_2 = y_{b_1-1}$ .

We now show a more specific example that assumes uniform distributions. The example shows how the optimal mechanism is determined by the cost  $C$ : for low costs, mechanism of type 1 is optimal, and for high costs the optimal mechanism is of type 2. An additional interesting feature of the optimal mechanisms in the example is that they are symmetric with respect to the players. This comes as opposed to the optimal mechanisms in the auction model [5] that are asymmetric (even when the players' values are drawn from identical distributions).

<sup>6</sup>More precisely, the condition for non-degeneracy when  $B \succ_1 A$  and  $B \succ_2 A$  is that  $sign(g(\underline{\theta}_i, \bar{\theta}_i, A) - g(\underline{\theta}_i, \bar{\theta}_i, B)) = sign(g(\bar{\theta}_i, \underline{\theta}_i, A) - g(\bar{\theta}_i, \underline{\theta}_i, B))$  (when  $sign(0)$  is considered both negative and positive).

<sup>7</sup>We denote  $x_0 = y_0 = 0$  and  $x_k = y_k = 1$ .



$C \leq 1$	0	1
0	No $p_1 = p_2 = 0$	No $p_1 = p_2 = 0$
1	No $p_1 = p_2 = 0$	Yes $p_1 = p_2 = \frac{2}{3}C - \frac{1}{3}$

$C \geq 1$	0	1
0	No $p_1 = p_2 = 0$	Yes $p_1 = 0; p_2 = \frac{2C}{3}$
1	Yes $p_1 = \frac{2C}{3}; p_2 = 0$	Yes $p_1 = p_2 = 0$

**Figure 2:** Optimal mechanisms in a 2-player, 2-alternative, 2-action public-goods game, when the types are uniformly distributed in  $[0, 1]$ . The mechanism on the left is optimal when  $C \leq 1$  and the other is optimal when  $C \geq 1$ .

EXAMPLE 1. Suppose that the types of both players are uniformly distributed on  $[0, 1]$ . Figure 2 illustrates the optimal mechanisms for  $k = 2$ , and shows how both the allocation scheme and the payments depend on the construction cost  $C$ . Then, the welfare-maximizing mechanisms are:

- If the cost of building is at least 1:  
Allocation: Build iff  $b_1 + b_2 \geq k$   
Strategies: The thresholds of both players are (for  $i = \{1, \dots, k-1\}$ ),  $x_i = \frac{2(k-i) \cdot C}{2k-1} - \frac{2k-4i+1}{2k-1}$
- If the cost of building is smaller than 1:  
Allocation: Build iff  $b_1 + b_2 \geq k-1$   
Strategies: The thresholds of both players are (for  $i = \{1, \dots, k-1\}$ ),  $x_i = \frac{2iC}{2k-1}$

## 5.2 Application 2: Signaling

We now study a signaling model in labor markets. In this model, the type of each worker,  $\theta_i \in [\underline{\theta}, \bar{\theta}]$ , describes the worker's productivity level. The firm wants to make her hiring decisions according to a decision function  $f(\vec{\theta})$ . For example, the firm may want to hire the most productive worker (like the auction model), or hire a group of workers only if their sum of productivities is greater than some threshold (similar to the public-good model). However, the worker's productivity is invisible to the firm; the firm only observes the worker's education level  $e$  that should convey signals about her productivity level. Note that the assumption here is that acquiring education, at any level, does not affect the productivity of the worker, but only signals about the worker's skills.

A main component in this model, is the fact that as the worker is more productive, it is easier for him to acquire high-level education. In addition, the cost of acquiring education increases with the education level. More formally, a continuous function  $C(e, \theta)$  describes the cost to a worker from acquiring each education level as a function of his productivity. The standard assumptions about the cost function are:  $\frac{\partial C}{\partial e} > 0$ ,  $\frac{\partial C}{\partial \theta} < 0$ ,  $\frac{\partial C}{\partial e \partial \theta} < 0$ , where the latter requirement is exactly equivalent to the single-crossing property (when  $C$  is differentiable in both variables). The utility of a worker is determined according to the education level he chooses and the wage  $w(e)$  attached to this education level, that is,  $u_i(e, \theta_i) = -C(\theta_i, e) + w(e)$ .

An action for a worker in this game is the education level he chooses to acquire. In standard models, this action space is continuous, and then a "fully separating equilibrium" exists (under the single-crossing conditions on the cost function). That is, there exists an equilibrium in which every type is mapped into a different education level; thus, the firm can induce the exact productivity levels of the workers

by this signaling mechanism. However, it is hard to imagine a world with a continuum of education levels. It is usually the case that there are only several discrete education levels (e.g., BSc, MSc, PhD).

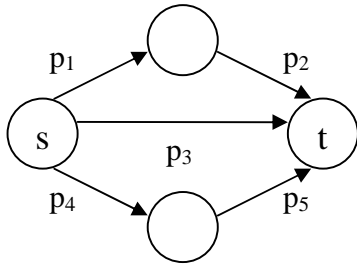
With  $k$  education levels, the firm may not be able to exactly follow the decision function  $f$ . For achieving the best result in  $k$  actions, the firm may want the workers to play according to specific threshold strategies. It turns out that the standard condition, the single-crossing condition on the cost function, suffices for ensuring that these threshold strategies will be dominant for the players. We can now apply Theorem 2, and show that if the decision function  $f$  of the firm is multilinear (i.e., the decisions are made to maximize a set of multilinear functions), then the firm can design the education system such that the expected loss will be  $O(\frac{1}{k^2})$ , with a dominant-strategy equilibrium. Note that while in the classic example of the job market it is not reasonable for each firm to select the education level, in other reasonable applications the social planners may be able to determine the thresholds, e.g., by fixing the levels of qualifying exams or other means for the players to demonstrate their skills.

COROLLARY 4. Consider a multilinear decision function  $f$ , and a single-crossing cost function for the players. With  $k$  education levels, the firm can implement in dominant strategies a decision function that incurs a loss of  $O(\frac{1}{k^2})$  compared with the decision function  $f$ .

## 5.3 Application 3: Routing

In our last example, we show the applicability of our results to routing in lossy networks. In such systems, a sender needs to decide through which network to transmit his message. It is natural to assume that the agents (i.e., links) may not be able to report their accurate probabilities of success, but only, e.g., whether these are "low", "intermediate", or "high". In this example, we focus on *parallel-path* networks.

Let  $N_1, N_2$  denote two networks, where each network is composed of multiple parallel paths with variable lengths from a given source to a given sink (an example for such a network appears in Figure 3). The edges in these networks are controlled by different selfish agents, and each edge appears only in one of the networks. Suppose that the sender, who wishes to send a message from the source to the sink, knows the topology of each network, but the probability of success on each link,  $p_i$ , is the link's private information. The problem of the sender is to decide whether to send a message through the network  $N_1$  or through an alternate network  $N_2$ . Obviously, the sender wishes to send the message through  $N_1$  only if the total probability of success in  $N_1$  is greater than the success probability in  $N_2$ . Let  $f^N(\vec{p})$  denote the probability of success in network  $N$  with a success-probability vector  $\vec{p}$ . The social choice function in this example is thus:  $c(\vec{p}) \in \operatorname{argmax}_{\{N_1, N_2\}} \{f^{N_1}(\vec{p}), f^{N_2}(\vec{p})\}$ .



**Figure 3:** An example for a parallel-path network, where each link has a probability  $p_i$  for transmission success. We show that the overall probability of success in such networks is multilinear in  $p_i$ , and thus the optimal  $k$ -action social-choice function is dominant-strategy implementable.

In this example, we assume that every agent has a single-crossing valuation function over the alternatives. That is, each player wishes that the message will be sent through his network, and his benefit is positively correlated with his secret data (e.g., the valuation of player  $i$  may be exactly  $p_i$ ). We would like to emphasize that the social planner in this example (the sender) does not aim to maximize the social welfare. That is, the social value is not the sum of the players' types nor any weighted sum of the types ("affine maximizer").

The success probability of sending a message through a parallel-path network is multilinear, since it can be expressed by the following multilinear formula (where  $\mathcal{P}$  denotes the set of all paths between the source and the sink):

$$1 - \prod_{P \in \mathcal{P}} (1 - \prod_{j \in P} p_j) \quad (3)$$

For example, in the network presented in figure 3, the probability of success is given by

$$f(\vec{p}) = 1 - (1 - p_1 p_2) \cdot (1 - p_3) \cdot (1 - p_4 p_5)$$

Thus, if all the candidate networks are parallel-path networks, the social-value function is multilinear, and we can apply Theorem 1 and get the following corollary. Note that for every link  $i$ , the partial derivative in  $p_i$  of the success probability written in Equation 3 is positive. In all the other networks, that do not contain link  $i$ , the partial derivative is clearly zero. Therefore, the social-value function is single crossing and our general results can be applied.

**COROLLARY 5.** *For any social-choice function that maximizes the success probability over parallel-path networks, the informationally optimal  $k$ -action social-choice function is implementable (for any  $k$ ).*

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