0366.3267 Graph Theory

Fall Semester 2022

Homework assignment 2

Due: Monday December 18, 2022

Problem 1. Show that every two paths of maximum length in a connected graph have a vertex in common.

Problem 2. Prove: if for every edge e of a connected graph G there are two cycles C_1, C_2 in G such that $E(C_1) \cap E(C_2) = \{e\}$, then G is 3-edge-connected.

Problem 3. Let $k \geqslant 2$. Show that every k-connected graph with at least $2k$ vertices contains a cycle of length at least $2k$.

Problem 4. Let G be a graph in which every pair of vertices has an odd number of common neighbors. Prove that G is Eulerian.

Problem 5. Let d be a positive integer. Show that every $2d$-regular connected graph G with an even number of edges contains a spanning d-regular subgraph.

Problem 6. Let G be a connected graph with n vertices. Prove that G contains a path of length $\min\{2\delta(G), n-1\}$.

Problem 7. A *tournament* is a complete graph in which each edge uv is given a direction, either from u to v or from v to u. Show that a tournament must contain a Hamilton path, that is, a directed path through all the vertices. Does it necessarily contain a Hamilton cycle?

Problem 8. Let $t(n, H_n)$ be the maximum number of edges in a graph G on n vertices not containing a Hamilton cycle H_n. Prove: $t(n, H_n) = \binom{n-1}{2} + 1$. (You need to prove both lower and upper bounds for $t(n, H_n)$.)

The exercices below are for you to practice — please do NOT submit their written solutions:

Exercise 1. Let G be a graph and let $A \subseteq V(G)$. Let H be the graph obtained from G by adding to it a new vertex v with $N_H(v) = A$. Show that $\kappa(H) \geqslant \min\{|A|, \kappa(G)\}$.

Exercise 2. Let Q^d be the d-dimensional cube defined as follows: $V(Q^d) = \{0,1\}^d$, $\mathbf{x} = (x_1, \ldots, x_d), \mathbf{y} = (y_1, \ldots, y_d) \in V(Q^d)$ are connected by an edge in Q^d if and only if $\mathbf{x}$ and $\mathbf{y}$ differ in exactly one coordinate. Prove: $\kappa(Q^d) = \kappa'(Q^d) = d$.

Exercise 3. Let G be a graph with all degrees even. Prove that the edges of G can be oriented in such a way that every vertex of the resulting directed graph $\overrightarrow{G}$ has its outdegree equal to its indegree.

Exercise 4. Let G be a graph of connectivity $\kappa(G)$ and with independence number $\alpha(G)$. Assume $\kappa(G) \geqslant \alpha(G) - 1$. Show that G contains a Hamilton path.