Large matchings and nearly spanning, nearly regular subgraphs of random subgraphs

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Abstract

Given a graph G and $p \in [0,1]$, the random subgraph G_p is obtained by retaining each edge of G independently with probability p. We show that for every $\epsilon > 0$, there exists a constant $C > 0$ such that the following holds. Let $d \geq C$ be an integer, let G be a d-regular graph and let $p \geq \frac{C}{d}$. Then, with probability tending to one as $|V(G)|$ tends to infinity, there exists a matching in G_p covering at least $(1-\epsilon)|V(G)|$ vertices.

We further show that for a wide family of d-regular graphs G, which includes the d-dimensional hypercube, for any $p \geq \frac{\log^5 d}{d}$ with probability tending to one as d tends to infinity, G_p contains an induced subgraph on at least $(1-o(1))|V(G)|$ vertices, whose degrees are tightly concentrated around the expected average degree dp.

1 Introduction

A classical result of Erdős and Rényi [10] states that $p = \frac{\log n}{n}$ is the threshold for the existence of a *perfect* matching (that is, a matching covering all but at most one vertex) in $G(n,p)$[1], which also coincides with the connectivity threshold (see also [4] for a hitting time result). Below this threshold, it is not hard to show that with probability tending to one as n tends to infinity a fixed proportion of the vertices are isolated and will not be covered in any matching. On the other hand, it follows from a celebrated result of Karp and Sipser [12] from 1981 that, when $p = \frac{C}{n}$ for a large enough constant C, $G(n,p)$ contains a matching on $(1-o_C(1))n$ vertices with probability tending to one as n tends to infinity. Subsequent work by Frieze [11] gave a precise estimate of the asymptotic proportion of vertices which are not covered by a largest matching in this regime.

The binomial random graph $G(n,p)$ is an instance of the model of *bond percolation*. Given a host graph G and a probability $p \in [0,1]$, we form the random subgraph $G_p \subseteq G$ by retaining each edge of G independently with probability p (indeed, $G(n,p)$ is equivalent to performing bond percolation on the complete graph K_n). In a qualitative sense, our first main result extends the result of Karp and Sipser [12] to any d-regular graph.

Theorem 1. *For every $\epsilon > 0$, there exists a constant $C > 0$ such that the following holds. Let $d \in \mathbb{N}, d \geq C$, let G be a d-regular graph on n vertices, and let $p \geq \frac{C}{d}$. Then, with probability tending to one as n tends to infinity, there exists a matching in G_p covering at least $(1-\epsilon)n$ vertices.*

In addition to the typical existence of large matchings in percolated d-regular graphs, we also explore typical structural properties of the hypercube under percolation.

In the setting of $G(n,p)$, finding a large k-regular subgraph when $p = \frac{C}{n}$ has been extensively studied (see, for example, [13, 15] and references therein). A natural variant is to try and find

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[1] In fact, Erdős and Rényi worked in the closely related *uniform* random graph model $G(n,m)$.

a *nearly regular*, nearly spanning subgraph (see [1] for an extremal variant of this question). We consider this question in the setting of random subgraphs of the hypercube. Recall that the d-dimensional binary hypercube is the graph whose vertex set is $\{0,1\}^d$, and where two vertices are connected if and only if their Hamming distance is one. In our second main result, we establish the typical existence of a nearly regular, nearly spanning induced subgraph of Q^d_p, whose degrees are tightly concentrated around the expected degree dp.

Theorem 2. *For every $\epsilon > 0$, there exists $d_0 \in \mathbb{N}$ such that the following holds for every integer $d \geq d_0$. Let $p \geq \frac{\log^5 d}{d}$. Then, with probability tending to one as d tends to infinity there exists an induced subgraph $H \subseteq Q^d_p$, such that $|V(H)| \geq (1-\epsilon)2^d$ and for every $v \in V(H)$, $\left|1 - \frac{d_H(v)}{dp}\right| \leq \epsilon$.*

We note that the proofs of the theorems are quite short. Further, let us remark that we have not tried to optimise the polylogarithmic dependence of p on d in Theorem 2. Finally, we note that we present the proof for Q^d, but, as will be expanded upon in the discussion section, Theorem 2 holds for a fairly wide family of graphs (see Theorem 7).

In Section 2 we prove Theorems 1 and 2. In Section 3 we discuss our results and avenues for future research.

2 Proofs of Theorems 1 and 2

Given a graph G and $v \in V(G)$, we denote by $d_G(v)$ the degree of v in G. For $k \in \mathbb{N}$, we denote by $N^k_G(v)$ the set of vertices at distance exactly k from v in G, where $N_G(v) \coloneqq N^1_G(v)$. When the graph G is clear from the context, we will omit the subscripts. Further, given $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by the vertices of S. Given $x, y, z \in \mathbb{R}$ we write $x = y \pm z$ as shorthand for $x \in [y-z, y+z]$. Throughout the paper, all logarithms are in the natural base.

We will make use of a typical Chernoff-type tail bound on the binomial distribution (see, for example, Appendix A in [2]).

Lemma 3. *Let $d \in \mathbb{N}$, let $p \in [0,1]$, and let $X \sim \mathrm{Bin}(d,p)$. Then for any $0 < t \leq \frac{dp}{2}$,*

$$\mathbb{P}\left[X \neq dp \pm t\right] \leq 2\exp\left\{-\frac{t^2}{3dp}\right\}.$$

We will also utilise a variant of the well-known Azuma-Hoeffding inequality (see, for example, Chapter 7 in [2]),

Lemma 4. *Let $m \in \mathbb{N}$ and let $p \in [0,1]$. Let $X = (X_1, X_2, \ldots, X_m)$ be a random vector with range $\Lambda = \{0,1\}^m$ with each X_i distributed according to independent Bernoulli(p). Let $f : \Lambda \to \mathbb{R}$ be a function such that there exists $K \in \mathbb{R}$ such that for every $x, x' \in \Lambda$ which differ only in one coordinate, $|f(x) - f(x')| \leq K$. Then, for every $t \geq 0$,*

$$\mathbb{P}\left[\big|f(X) - \mathbb{E}\left[f(X)\right]\big| \geq t\right] \leq 2\exp\left\{-\frac{t^2}{2K^2mp}\right\}.$$

2.1 Proof of Theorem 1

Fix $\epsilon > 0$. Let $\delta \coloneqq \delta(\epsilon) > 0$ and $C \coloneqq C(\delta, \epsilon) = C(\epsilon) > 0$ be constants satisfying that $C\exp\left\{-\frac{\delta^2 C}{16}\right\} \leq \frac{\epsilon}{4}$ and $\frac{C-\epsilon}{(1+\delta)C} \geq 1-\epsilon$. Note that by monotonicity, we may assume $p = \frac{C}{d}$. Let V_0 be the set of vertices of degree at least $(1+\delta)C$, and let E_0 be the set of edges touching V_0, that is,

$$V_0 \coloneqq \{v \in V(G) \colon d_{G_p}(v) \geq (1+\delta)C\} \quad \text{and} \quad E_0 \coloneqq \{uv \in E(G_p) : \{u,v\} \cap V_0 \neq \emptyset\}.$$

For any $e \in E(G)$ the probability that $e \in E(G_p)$ *and* at least one of its endpoints is in V_0 is at most $p \cdot 2\mathbb{P}\left(\mathrm{Bin}\left(d, \frac{C}{d}\right) \geq (1+\delta)C - 1\right)$, which is bounded above by $\frac{C}{d} \cdot 4\exp\left\{-\frac{\delta^2 C}{4}\right\}$ by Lemma 3 (where we assumed that $\delta C \geq 9$). Hence, using $|E(G)| = \frac{dn}{2}$, we have $\mathbb{E}\left[|E_0|\right] \leq 2C\exp\left\{-\frac{\delta^2 C}{4}\right\} n \leq \frac{\varepsilon}{8} n$. Now, note that adding/removing any edge can change $|V_0|$ by at most two, and thus $|E_0|$ by at most $2(1+\delta)C$. Hence, by Lemma 4 (with $t = \frac{\epsilon n}{8}$, $K = 2(1+\delta)C$, and $m = \frac{nd}{2}$), and using $\exp\left\{-\frac{\delta^2 C}{4}\right\} \leq \frac{\epsilon}{16C}$ we obtain

$$\mathbb{P}\left[|E_0| \geq \frac{\epsilon}{4} n\right] \leq 2\exp\left\{-\frac{(\epsilon n/8)^2}{2 \cdot 4(1+\delta)^2 C^2 \cdot nd/2 \cdot C/d}\right\} = \exp\left\{-\Omega(n)\right\} = o_n(1).$$

By similar (and simpler) arguments, with probability $1 - o_n(1)$, $|E(G_p)| \geq \frac{ndp}{2} - \frac{\epsilon n}{4} = \frac{Cn}{2} - \frac{\epsilon}{4} n$.

Let H be the subgraph of G_p induced by $V \setminus V_0$. Note that for every $v \in V(G)$, we have that $d_H(v) < (1+\delta)C$. Hence, by Vizing's theorem [16], there exists a proper colouring of H with $(1+\delta)C$ colours. Therefore, with probability $1 - o_n(1)$ there is a matching in H (and thus in G_p) covering at least

$$\frac{2|E(H)|}{(1+\delta)C} = \frac{2(|E(G_p)| - |E_0|)}{(1+\delta)C} \geq \frac{(C-\varepsilon)}{(1+\delta)C} n \geq (1-\epsilon)n$$

vertices, as required. □

2.2 Finding a nearly regular, nearly spanning subgraph

In this section, we prove Theorem 2. For ease of presentation, we write $G := Q^d$, so that $G_p = Q^d_p$. Throughout the section, we assume $d \in \mathbb{N}$ is sufficiently large, and all asymptotic notation in this section will be with respect to the parameter d. Further, throughout this section, we let $\delta := \delta(d)$ be a function tending to 0 arbitrarily slowly as d tends to infinity. We recall our assumption that $dp \geq \log^5 d$, and in particular, $dp = \omega(1)$ (we will use this fact at various points in the proof).

We will analyse the following 'pruning' process on G_p. For each $t \in \mathbb{N}$ let

$$\delta_t := \frac{t \cdot \delta}{\lfloor \log d \rfloor}.$$

Let $\tau := \lfloor \log d \rfloor$ so that $\delta_\tau = \delta$. Let $H_1 := G_p$, and let

$$A_1 := \left\{v \in V(H_1) \colon d_{H_1}(v) \neq (1 \pm \delta_1)dp\right\}. \tag{1}$$

We then proceed as follows: At the t-th iteration, for $t \in [2,\tau] \cap \mathbb{N}$, we let $H_t := G_p\left[V(G) \setminus \bigcup_{i=1}^{t-1} A_i\right]$, and let

$$A_t := \left\{v \in V(H_t) \colon d_{H_t}(v) < (1 - \delta_t)dp\right\}.$$

We run this process until $t = \tau$, and let

$$H := H_\tau \quad \text{and} \quad A := \bigcup_{t=1}^{\tau} A_t.$$

Note that if for some $1 \leq t \leq \tau$, we have $A_t = \varnothing$, then the process stabilises, that is, for every $t' \geq t$, we would have that $A_{t'} = \varnothing$ and $H_{t'} = H_t$.

We will make use of the following observations:

(i) The sets $\{A_1, \ldots, A_\tau\}$ are pairwise disjoint;

(ii) If $A_\tau = \varnothing$, then for every $v \in V(G) \setminus A$, $d_H(v) = (1 \pm \delta)dp$.

The first one is apparent by construction. To see (ii) assume that $A_\tau = \varnothing$. Then, every $v \in (V(G) \setminus A) = \left(V(G) \setminus \bigcup_{t=1}^{\tau-1} A_t\right) = V(H_\tau)$ satisfies $d_H(v) \geq (1-\delta_\tau)dp = (1-\delta)dp$. On the other hand, since $v \in (V(G) \setminus A) \subseteq (V(G) \setminus A_1)$, it follows that $d_H(v) \leq d_{G_p}(v) \leq (1+\delta_1)dp \leq (1+\delta)dp$.

In particular, by (ii), in order to prove Theorem 2, it suffices to show that **whp**[2]

$$A_\tau = \varnothing \quad \text{and} \quad |A| = o(2^d).$$

Key to the proof is the following lemma, which gives a simple-to-analyse condition for when $v \in A_t$ for $t \in [2, \tau] \cap \mathbb{N}$.

Lemma 5. *Let $t \in [2,\tau] \cap \mathbb{N}$. If $v \in A_t$, then there is a set X of at least $\left(\frac{\delta dp}{2t \log d}\right)^{t-1}$ vertices at distance exactly $t-1$ from v, such for all $x \in X$,*

$$d_{G_p}(x) \neq (1 \pm \delta_1)dp.$$

Proof. We will use the following observation about the structure of Q^d, which is easy to verify:

$$\text{If } k \leq d \text{ and } v \text{ and } w \text{ are at distance } k \text{ in } Q^d, \text{ then } w \text{ has precisely } d-k \text{ neighbours at distance } k+1 \text{ from } v \text{ and } k \text{ neighbours at distance } k-1 \text{ from } v. \tag{2}$$

The lemma will follow from iteratively applying the next fairly simple claim.

Claim 6. *Let $t \in [2, \tau] \cap \mathbb{N}$ and $k \in [1,t] \cap \mathbb{N}$, let $v \in V(G)$, and let $S \subseteq \left(A_t \cap N^k(v)\right)$. Then, there exists a set $X \subseteq \left(A_{t-1} \cap N^{k+1}(v)\right)$ with $|X| \geq |S| \frac{\delta dp}{2(k+1)\log d}$.*

Proof. Since each $s \in S$ is in A_t, $d_{H_t}(s) < (1-\delta_t)dp$. On the other hand, since A_t and A_{t-1} are disjoint, each $s \in S$ is not in A_{t-1} and consequently $d_{H_{t-1}}(s) \geq (1-\delta_{t-1})dp$. Thus, there are at least $(\delta_t - \delta_{t-1})dp = \frac{\delta dp}{\lfloor \log d \rfloor} \geq \frac{\delta dp}{\log d}$ neighbours of s which are in A_{t-1}, let us denote them by Y_s.

If we let $X_s := \left(Y_s \cap N^{k+1}(v)\right) \subseteq \left(A_{t-1} \cap N^{k+1}(v)\right)$, then by (2), $|X_s| \geq \frac{\delta dp}{\log d} - k$. On the other hand, since $X := \bigcup_{s \in S} X_s \subseteq N^{k+1}(v)$ and $S \subseteq N^k(v)$, again by (2) each $x \in X$ lies in at most $k+1$ sets X_s. It follows by a simple double counting argument that

$$|X| \geq \frac{|S|}{k+1}\left(\frac{\delta dp}{\log d} - k\right) \geq |S| \frac{\delta dp}{2(k+1)\log d},$$

where we used that $k \leq \tau = \lfloor \log d \rfloor \leq \log d$ and $dp \geq \log^5 d$. □

To complete the proof of Lemma 5, note that if $v \in A_t$, then $v \notin A_{t-1}$, and by the same argument as above, there exists $S_1 \subseteq (N(v) \cap A_{t-1})$ with $|S_1| \geq \frac{\delta dp}{2 \log d}$.

Iteratively applying Claim 6 to the sets S_i, for $i \in [1, t-2] \cap \mathbb{N}$, we obtain a sequence of sets $S_{i+1} \subseteq \left(N^{i+1}(v) \cap A_{t-i+1}\right)$ with $|S_{i+1}| \geq |S_i| \frac{\delta dp}{2(i+1)\log d}$. It follows that

$$X := S_{t-1} \subseteq \left(N^{t-1}(v) \cap A_1\right)$$

has size at least

$$\prod_{i=1}^{t-1} \frac{\delta dp}{2(i+1)\log d} = \frac{1}{t!}\left(\frac{\delta dp}{2 \log d}\right)^{t-1} \geq \left(\frac{\delta dp}{2t \log d}\right)^{t-1},$$

where we used $t! \leq t^{t-1}$. Since $X \subseteq A_1$, by (1) X satisfies the assertion of the lemma. □

[2] With high probability, that is, with probability tending to one as d tends to infinity.

With these lemmas at hand, we are now ready to prove Theorem 2.

Proof of Theorem 2. It suffices to show that **whp** $A_\tau = \varnothing$ and $|A| = o(2^d)$.

Fix $v \in V(G)$ and $t \in [2, \tau] \cap \mathbb{N}$. We start by showing

$$\mathbb{P}[v \in A_t] \leq \exp\left\{-\left(\frac{\delta dp}{2t\log d}\right)^{t-1}\right\}. \tag{3}$$

Indeed, by Lemma 5, if $v \in A_t$, then there is a set X of at least $\left(\frac{\delta dp}{2t\log d}\right)^{t-1}$ vertices at distance $t-1$ from v, such that $d_{G_p}(x) \neq (1 \pm \delta_1)dp$ for all $x \in X$. Furthermore, since every $x \in X$ is at distance exactly $t-1$ from v, they have the same parity, and thus X is an independent set in G. For each $x \in X$, $d_{G_p}(x) \sim \text{Bin}(d,p)$ and so by Lemma 3, we have that

$$\mathbb{P}\left[d_{G_p}(x) \neq (1 \pm \delta_1)dp)\right] \leq 2\exp\left\{-\frac{\delta_1^2 d^2p^2}{3dp}\right\} \leq \exp\left\{-\frac{\delta^2 dp}{4\log^2 d}\right\}.$$

Since $X \subseteq N^{t-1}(v)$, there are at most $\binom{|N^{t-1}(v)|}{|X|}$ possible choices for X. Note that $|N^{t-1}(v)| = \binom{d}{t-1}$ (choosing $t-1$ of the d coordinates to obtain a vertex at distance $t-1$ from v). Recalling that $d_{G_p}(x)$ is independent for each $x \in X$ and that $|X| = r \geq \left(\frac{\delta dp}{2t\log d}\right)^{t-1}$, by a union bound we obtain

$$\begin{aligned}
\mathbb{P}[v \in A_t] &\leq \binom{\binom{d}{t-1}}{r}\exp\left\{-r\cdot\frac{\delta^2 dp}{4\log^2 d}\right\} \leq \left((ed)^{t-1}\exp\left\{-3\log^2 d\right\}\right)^r \\
&\leq \exp\left\{r\left(2\log^2 d - 3\log^2 d\right)\right\} \leq \exp\left\{-2r\right\} \\
&\leq \exp\left\{-\left(\frac{\delta dp}{2t\log d}\right)^{t-1}\right\},
\end{aligned}$$

where the third inequality follows since $t \leq \tau = \lfloor \log d \rfloor \leq \log d$. Note that in this estimation, we used (very generously) our assumption of $p \geq \frac{\log^5 d}{d}$, that is, that the numerator is polylogarithmic in d.

We first note that (3) implies that for each $t \in [2, \tau] \cap \mathbb{N}$,

$$\mathbb{E}\left[|A_t|\right] \leq 2^d \exp\left\{-\left(\frac{\delta dp}{2t\log d}\right)^{t-1}\right\} \leq 2^d \exp\left\{-\log^2 d\right\}.$$

Also, by a similar application of Lemma 3, we have

$$\mathbb{E}[|A_1|] \leq 2^d \exp\left\{-\frac{\delta^2 dp}{4\log^2 d}\right\} \leq 2^d \exp\left\{-\log^2 d\right\}.$$

Recalling that $A := \bigcup_{t=1}^{\tau} A_t$, it follows that $\mathbb{E}[|A|] \leq 2^d \tau \exp\left\{-\log^2 d\right\} = o(2^d)$, and so by Markov's inequality, **whp** $|A| = o(2^d)$.

Secondly, by (3) we obtain

$$\begin{aligned}
\mathbb{P}[A_\tau \neq \emptyset] &\leq \sum_{v \in V(G)} \mathbb{P}[v \in A_\tau] \leq 2^d \exp\left\{-\left(\frac{\delta dp}{5\log^2 d}\right)^{\tau-1}\right\} \\
&\leq 2^d \exp\left\{-\left(\log^2 d\right)^{\lfloor \log d \rfloor - 1}\right\} \leq 2^d \exp\{-d\} = o(1).
\end{aligned}$$

□

3 Discussion

We showed that for any d-regular graph G on n vertices, for every $\epsilon > 0$ there exists a constant $C > 0$ such that if $d \geq C$ and $p \geq \frac{C}{d}$, then there typically is a matching on at least $(1-\epsilon)n$ vertices in G_p. Further, we showed that when $p \geq \frac{\log^5}{d}$, Q^d_p typically contains an induced nearly spanning subgraph, whose degrees are tightly concentrated around the expected degree dp. To find this subgraph, we employed a fairly simple pruning process.

In recent years there has been an interest in the universality of properties of $G(n,p)$, and in particular in extending results on the quantitative similarity of the structure of $G(n,p)$ and Q^d_p to broader classes of *high-dimensional* graphs. For example, the typical emergence of the giant component, its uniqueness and its asymptotic properties have been considered [5, 6, 7, 8, 14]. Moreover, Diskin and Geisler [9] extended the result of Bollobás [3] to these settings as well, roughly showing that for any d-regular high-dimensional product graph, the hitting times of minimum degree one, connectivity, and perfect matching are **whp** the same. It is not hard to verify that the proof laid out in this paper generalises, almost verbatim, to d-regular high-dimensional Cartesian product graphs (specifically, the t-dimensional product of regular graphs of bounded order). In fact, one can even further relax the assumptions.

Theorem 7. *Let G be a d-regular graph with $d = \omega(1)$. Suppose that for every $v \in V(G)$, every $k \leq \log d$, and every $u \in N^k(v)$, we have that $\left|N(u) \cap \bigcup_{i=1}^{k} N^i(v)\right| = O(\log d)$. Let $p \geq \frac{\log^5 d}{d}$. Then, **whp** G_p contains an induced subgraph H such that $|V(H)| = (1-o_d(1))|V(G)|$ and for every $v \in V(H)$, $d_H(v) = (1 \pm o_d(1))dp$.*

This raises the following more general question.

Question 8. *Let G be a d-regular graph with $d = \omega(1)$. What 'minimal' assumptions on G and p suffice to have: **whp** G_p has an induced nearly spanning, nearly regular subgraph with all degrees concentrated around dp?*

As an application of Theorem 1 we have that when $p \geq \frac{C}{d}$, **whp** a largest matching in Q^d_p contains $(1-o_C(1))2^d$ vertices, that is, the first order term is 2^d whereas the second order term is bounded by $o_C(2^d)$. Much more precise results are known in the setting of $G(n,p)$ [11]. It would be interesting to determine more precisely the typical size of a largest matching in Q^d_p when $p = \frac{C}{d}$ (for $d > C$). A first step to answer this would be to determine the second order term — there, it is perhaps natural to conjecture that the size of the 'defect set' (that is, the set of vertices that are not covered by a largest matching in Q^d_p) is dominated by the number of isolated vertices and thus **whp** a largest matching typically covers $2^d - (2(1-p))^d + o_C((2(1-p))^d)$ vertices (indeed, this would resonate with the known hitting time results [3]).

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