

# Biased Games On Random Boards

Asaf Ferber\* Roman Glebov† Michael Krivelevich‡ Alon Naor §

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## Abstract

In this paper we analyze biased Maker-Breaker games and Avoider-Enforcer games, both played on the edge set of a random board  $G \sim G(n, p)$ . In Maker-Breaker games there are two players, denoted by Maker and Breaker. In each round, Maker claims one previously unclaimed edge of  $G$  and Breaker responds by claiming  $b$  previously unclaimed edges. We consider the Hamiltonicity game, the perfect matching game and the  $k$ -vertex-connectivity game, where Maker's goal is to build a graph which possesses the relevant property. Avoider-Enforcer games are the reverse analogue of Maker-Breaker games with a slight modification, where the two players claim at least 1 and at least  $b$  previously unclaimed edges per move, respectively, and Avoider aims to avoid building a graph which possesses the relevant property.

Maker-Breaker games are known to be “bias-monotone”, that is, if Maker wins the  $(1, b)$  game, he also wins the  $(1, b - 1)$  game. Therefore, it makes sense to define the *critical bias* of a game,  $b^*$ , to be the “breaking point” of the game. That is, Maker wins the  $(1, b)$  game whenever  $b < b^*$  and loses otherwise. An analogous definition of the critical bias exists for Avoider-Enforcer games: here, the critical bias of a game  $b^*$  is such that Avoider wins the  $(1, b)$  game for every  $b \geq b^*$ , and loses otherwise.

We prove that, for every  $p = \omega\left(\frac{\ln n}{n}\right)$ ,  $G \sim G(n, p)$  is typically such that the critical bias for all the aforementioned Maker-Breaker games is asymptotically  $b^* = \frac{np}{\ln n}$ . We also prove that in the case  $p = \Theta\left(\frac{\ln n}{n}\right)$ , the critical bias is  $b^* = \Theta\left(\frac{np}{\ln n}\right)$ . These results settle a conjecture of Stojaković and Szabó. For Avoider-Enforcer games, we prove that for  $p = \Omega\left(\frac{\ln n}{n}\right)$ , the critical bias for all the aforementioned games is  $b^* = \Theta\left(\frac{np}{\ln n}\right)$ .

## 1 Introduction

Let  $X$  be a finite set and let  $\mathcal{F} \subseteq 2^X$ . In an  $(a, b)$  Maker-Breaker game  $\mathcal{F}$ , the two players – Maker and Breaker – alternately claim  $a$  and  $b$  previously unclaimed elements of *the board*  $X$ ,

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\*School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: ferberas@post.tau.ac.il.

†Institut für Mathematik, Freie Universität Berlin, Arnimallee 3-5, D-14195 Berlin, Germany. Email: glebov@math.fu-berlin.de. Research supported by DFG within the research training group “Methods for Discrete Structures”.

‡School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: krivelev@post.tau.ac.il. Research supported in part by USA-Israel BSF Grant 2010115 and by grants 1063/08, 912/12 from Israel Science Foundation.

§School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: alonnaor@post.tau.ac.il.

respectively. Maker's goal is to claim all the elements of some *target set*  $F \in \mathcal{F}$ . If Maker does not fully claim any target set by the time all board elements are claimed, then Breaker wins the game. When  $a = b = 1$ , the game is called *unbiased*, otherwise it is called *biased*. It is easy to see that being the first player is never a disadvantage in a Maker-Breaker game. Indeed, suppose the first player has some strategy as the second player. He can play arbitrarily in his first move and pretend that he didn't make this move and he now starts a new game as a second player; whenever his strategy tells him to claim some edge he had previously claimed he just claims arbitrarily some edge. So, in order to prove that Maker wins a certain game, it is enough to prove that he can win as a second player. Throughout the paper we assume that Maker is the *second* player to move.

In an  $(a, b)$  Avoider-Enforcer game played on a hypergraph  $\mathcal{F} \subseteq 2^X$ , the two players are called Avoider and Enforcer, alternately claim *at least*  $a$  and *at least*  $b$  previously unclaimed elements of the board  $X$  per move, respectively. Avoider loses the game if at some point during the game he fully claims all the elements of some target set  $F \in \mathcal{F}$ . Otherwise, Avoider wins.

In both Maker-Breaker games and Avoider-Enforcer games (also referred to as *Positional Games*), we may assume that there are no  $F_1, F_2 \in \mathcal{F}$  such that  $F_1 \subset F_2$ , since in this case Maker wins (or Avoider loses) once he claims all the elements in  $F_1$ , and so the two  $(a, b)$  games  $\mathcal{F}$  and  $\mathcal{F} \setminus \{F_2\}$  are identical.

It is natural to play Positional Games on the edge set of a graph  $G$ . In this case, the board is  $X = E(G)$ , and the target sets are all the edge sets of subgraphs  $H \subseteq G$  which possess some given monotone increasing graph property  $\mathcal{P}$ . In the *connectivity* game  $\mathcal{C}(G)$ , the target sets are all edge sets of spanning trees of  $G$ . In the *perfect matching* game  $\mathcal{M}(G)$  the target sets are all sets of  $\lfloor |V(G)|/2 \rfloor$  independent edges of  $G$ . Note that if  $n$  is odd, then such a matching covers all vertices of  $G$  but one. In the *Hamiltonicity* game  $\mathcal{H}(G)$  the target sets are all edge sets of Hamilton cycles of  $G$ . Given a positive integer  $k$ , in the  *$k$ -connectivity* game  $\mathcal{C}^k(G)$  the target sets are all edge sets of  $k$ -vertex-connected spanning subgraphs of  $G$ .

Maker-Breaker games played on the edge set of the complete graph  $K_n$  are well studied. In this case, many natural unbiased games are drastically in favor of Maker (see, e.g., [19], [12], [16], [8]). Hence, in order to even out the odds, it is natural to give Breaker more power by increasing his *bias* (that is, to play a  $(1, b)$  game instead of a  $(1, 1)$  game), and/or to play on different types of boards.

Maker-Breaker games are *bias monotone*. That means that if Maker wins some game with bias  $(a, b)$ , he also wins this game with bias  $(a', b')$ , for every  $a' \geq a, b' \leq b$ . Similarly, if Breaker wins a game with bias  $(a, b)$ , he also wins this game with bias  $(a', b')$ , for every  $a' \leq a, b' \geq b$ . Avoider-Enforcer games are also bias monotone in the version considered in this paper (this version is called the *monotone* version, as opposed to the *strict* version, where Avoider and Enforcer claim exactly  $a$  and  $b$  elements per move, respectively. The strict version is not bias monotone.). It means that if Avoider wins some game with bias  $(a, b)$ , he also wins this game with bias  $(a', b')$ , for every  $a' \leq a, b' \geq b$ , and that if Enforcer wins a game with bias  $(a, b)$ , he also wins this game with bias  $(a', b')$ , for every  $a' \geq a, b' \leq b$ .

This bias monotonicity of Maker-Breaker games allows us to define the *critical bias* (also referred to as the *threshold bias*): for a given Maker-Breaker game  $\mathcal{F}$ , the critical bias  $b^*$  is the value for which Maker wins the game  $\mathcal{F}$  with bias  $(1, b)$  for every  $b < b^*$ , and Breaker wins the game  $\mathcal{F}$  with bias  $(1, b)$  for every  $b \geq b^*$ . Similarly, in an Avoider-Enforcer game, the critical

bias  $b'$  (which is in general different from the critical bias in the Maker-Breaker game) is the value for which Avoider wins the game  $\mathcal{F}$  with bias  $(1, b)$  for every  $b \geq b'$ , and Enforcer wins the game  $\mathcal{F}$  with bias  $(1, b)$  for every  $b < b'$ .

In their seminal paper [6], Chvatál and Erdős proved that playing the  $(1, b)$  connectivity game on the edge set of the complete graph  $K_n$ , for every  $\varepsilon > 0$  and  $n$  sufficiently large, Breaker wins for every  $b \geq \frac{(1+\varepsilon)n}{\ln n}$ , and Maker wins for every  $b \leq \frac{n}{(4+\varepsilon)\ln n}$ . They stated that the threshold bias for this game ought to come around  $b = \frac{n}{\ln n}$ , and Beck [3] later explicitly conjectured that this is indeed asymptotically the case. Gebauer and Szabó [11] proved the conjecture. Later on, Krivelevich proved in [17] that  $b = \frac{n}{\ln n}$  is also the threshold bias for the Hamiltonicity game.

Stojaković and Szabó suggested in [21] to play Maker-Breaker games on the edge set of a random board  $G \sim G(n, p)$ . In this well known and well studied model, the graph  $G$  consists of  $n$  labeled vertices, and each pair of vertices is chosen to be an edge in the graph independently with probability  $p$ . They examined some games on this board such as the connectivity game, the perfect matching game, the Hamiltonicity game and “building a  $k$ -clique” game. Since then, much progress has been made in understanding Maker-Breaker games played on  $G \sim G(n, p)$ . For example, it was proved in [13] and [4] that for  $p = \frac{(1+o(1))\ln n}{n}$ ,  $G \sim G(n, p)$  is typically (i.e. with probability tending to 1 as  $n$  tends to infinity) such that Maker wins the  $(1, 1)$  games  $\mathcal{M}(G)$ ,  $\mathcal{H}(G)$  and  $\mathcal{C}_k(G)$ . Moreover, the proofs in [4] are of a “hitting time” type. It means that in the random graph process (see [5]), typically at the moment the graph reaches the needed minimum degree for Maker to win the desired game, Maker indeed wins this game. Later on, in [7], fast winning strategies for Maker in various games played on  $G \sim G(n, p)$  were considered, and in [20] a hitting time result was established for the “building a triangle” game, and it was proved that the threshold probability for the property “Maker can build a  $k$ -clique” game is  $p = \Theta(n^{-2/(k+1)})$ .

In [21], Stojaković and Szabó conjectured the following:

**Conjecture 1.1 ([21], Conjecture 1)** *There exists a constant  $C$  such that for every  $p \geq \frac{C \ln n}{n}$ , a random graph  $G \sim G(n, p)$  is typically such that the threshold bias for the game  $\mathcal{H}(G)$  is  $b^* = \Theta\left(\frac{np}{\ln n}\right)$ .*

In this paper we prove Conjecture 1.1, and in fact, for  $p = \omega\left(\frac{\ln n}{n}\right)$  we prove the following stronger statement:

**Theorem 1.2** *Let  $p = \omega\left(\frac{\ln n}{n}\right)$ . Then  $G \sim G(n, p)$  is typically such that  $\frac{np}{\ln n}$  is the asymptotic threshold bias for the games  $\mathcal{M}(G)$ ,  $\mathcal{H}(G)$  and  $\mathcal{C}_k(G)$ .*

In order to prove Theorem 1.2 we prove the following two theorems:

**Theorem 1.3** *Let  $0 \leq p \leq 1$ ,  $\varepsilon > 0$  and  $b \geq (1 + \varepsilon)\frac{np}{\ln n}$ . Then  $G \sim G(n, p)$  is typically such that in the  $(1, b)$  Maker-Breaker game played on  $E(G)$ , Breaker has a strategy to isolate a vertex in Maker’s graph, as a first or a second player.*

**Theorem 1.4** *Let  $p = \omega\left(\frac{\ln n}{n}\right)$ ,  $\varepsilon > 0$  and  $b = (1 - \varepsilon)\frac{np}{\ln n}$ . Then  $G \sim G(n, p)$  is typically such that Maker has a winning strategy in the  $(1, b)$  games  $\mathcal{M}(G)$ ,  $\mathcal{H}(G)$ , and  $\mathcal{C}_k(G)$  for a fixed positive integer  $k$ , as a first or a second player.*

In the case  $p = \Theta\left(\frac{\ln n}{n}\right)$  we establish two non-trivial bounds for the critical bias  $b^*$ . This also settles Conjecture 1.1 for this case but does not determine the exact value of  $b^*$  (notice that in this case,  $b^*$  is a constant!).

**Theorem 1.5** *Let  $p = \frac{K \ln n}{n}$ , where  $K > 10^5$  and let  $\varepsilon > 0$ . Then  $G \sim G(n, p)$  is typically such that the threshold bias for the games  $\mathcal{M}(G)$ ,  $\mathcal{H}(G)$  and  $\mathcal{C}_k(G)$  lies between  $K/10$  and  $K + \varepsilon$ .*

**Remark:** In the terms of Theorem 1.5, if  $1 < K \leq 10^5$ , we get by Theorem 1.3 that  $b^* \leq c + \varepsilon$ , and by the main result of [4] that  $b^* > 1$ , so indeed  $b^* = \Theta\left(\frac{np}{\ln n}\right)$  in this case as well.

We also consider the analogous Avoider-Enforcer games played on the edge set of a random board  $G \sim G(n, p)$ . Here Avoider aims to avoid claiming all the edges of a graph which contains a perfect matching, a Hamilton cycle, or that is  $k$ -connected (according to the game), and Enforcer tries to force him claiming all the edges of such a subgraph. We prove the following analog of Conjecture 1.1:

**Theorem 1.6** *Let  $\frac{70000 \ln n}{n} \leq p \leq 1$ . A random graph  $G \sim G(n, p)$  is typically such that the asymptotic threshold bias for the  $(1, b)$  Avoider-Enforcer games  $\mathcal{M}(G)$ ,  $\mathcal{H}(G)$  and  $\mathcal{C}_k(G)$  (for a fixed positive integer  $k$ ) is  $b^* = \Theta\left(\frac{np}{\ln n}\right)$ .*

As in the Maker-Breaker case, we divide our result into two separate theorems, one which establishes Avoider's win, and one which establishes Enforcer's win:

**Theorem 1.7** *Let  $0 \leq p \leq 1$  and  $b \geq \frac{26np}{\ln n}$ . Then  $G \sim G(n, p)$  is typically such that in the  $(1, b)$  Avoider-Enforcer game played on  $E(G)$ , Avoider has a strategy to isolate a vertex in his graph, as a first or a second player.*

**Theorem 1.8** *Let  $\frac{70000 \ln n}{n} \leq p \leq 1$  and  $b \leq \frac{np}{20000 \ln n}$ . Then  $G \sim G(n, p)$  is typically such that Enforcer has a winning strategy in the  $(1, b)$  games  $\mathcal{M}(G)$ ,  $\mathcal{H}(G)$  and  $\mathcal{C}_k(G)$  (for every positive integer  $k$ ), as a first or a second player.*

## 1.1 Notation and terminology

Our graph-theoretic notation is standard and follows that of [22]. In particular, we use the following:

For a graph  $G$ , let  $V = V(G)$  and  $E = E(G)$  denote its sets of vertices and edges, respectively. For subsets  $U, W \subseteq V$ , and for a vertex  $v \in V$ , we denote by  $E_G(U)$  all the edges with both endpoints in  $U$ , by  $E_G(U, W)$  all the edges  $e$  with both endpoints in  $U \cup W$ , for which  $e \cap U \neq \emptyset$  and  $e \cap W \neq \emptyset$ , and by  $E_G(v, U)$  all the edges with one endpoint being  $v$  and the other endpoint in  $U$ . We further denote  $e_G(U) := |E(U)|$ ,  $e_G(U, W) := |E(U, W)|$  and  $e_G(v, U) := |E(v, U)|$ .

For a subset  $U \subseteq V(G)$  we denote by  $N_G(U)$  the *external* neighborhood of  $U$ , that is:  $N_G(U) := \{v \in V \setminus U : \exists u \in U \text{ s.t. } uv \in E\}$ . For simplicity of notation, whenever the underlying graph is clear from the context we omit the graph from the index.

Assume that some Maker-Breaker game, played on the edge set of some graph  $G$ , is in progress. At any given moment during the game, we denote the graph formed by Maker's edges by  $M$ , the graph formed by Breaker's edges by  $B$ , and the edges of  $G \setminus (M \cup B)$  by  $F$ . For any vertex  $v \in V$ ,  $d_M(v)$  and  $d_B(v)$  denote the degree of  $v$  in  $M$  and in  $B$ , respectively. The edges of  $G \setminus (M \cup B)$  are called *free edges*, and  $d_F(v)$  denotes the number of free edges incident to  $v$ , for any  $v \in V$ .

Whenever we say that  $G \sim G(n, p)$  *typically* has some property, we mean that  $G$  has that property with probability tending to 1 as  $n$  tends to infinity.

We use the following notation throughout this paper:

$$f(n) := \frac{np}{\ln n}.$$

For the sake of simplicity and clarity of presentation, and in order to shorten some of our proofs, no real effort has been made here to optimize the constants appearing in our results. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that  $n$  is sufficiently large.

## 2 Auxiliary results

In this section we present some auxiliary results that will be used throughout the paper.

### 2.1 Binomial distribution bounds

We use extensively the following well known bound on the lower and the upper tails of the Binomial distribution due to Chernoff (see, e.g., [1]):

**Lemma 2.1** *If  $X \sim \text{Bin}(n, p)$ , then*

- $\Pr(X < (1 - a)np) < \exp\left(-\frac{a^2 np}{2}\right)$  for every  $a > 0$ .
- $\Pr(X > (1 + a)np) < \exp\left(-\frac{a^2 np}{3}\right)$  for every  $0 < a < 1$ .

The following is a trivial yet useful bound:

**Lemma 2.2** *Let  $X \sim \text{Bin}(n, p)$  and  $k \in \mathbb{N}$ . Then*

$$\Pr(X \geq k) \leq \left(\frac{enp}{k}\right)^k.$$

**Proof.**  $\Pr(X \geq k) \leq \binom{n}{k} p^k \leq \left(\frac{enp}{k}\right)^k$ . □

## 2.2 Basic positional games results

### 2.2.1 Maker-Breaker games

The following fundamental theorem, due to Beck [2], is a useful sufficient condition for Breaker's win in the  $(a, b)$  game  $(X, \mathcal{F})$ . It will be used in the proof of Theorem 1.4.

**Theorem 2.3** ([2], **Theorem 20.1**) *Let  $X$  be a finite set and let  $\mathcal{F} \subseteq 2^X$ . Breaker, as a first or a second player, has a winning strategy in the  $(a, b)$  game  $(X, \mathcal{F})$ , provided that:*

$$\sum_{F \in \mathcal{F}} (1+b)^{-|F|/a} < \frac{1}{1+b}.$$

While Theorem 2.3 simply shows that Breaker can win certain games, the following lemma shows that Maker can win certain games quickly (see [2]):

**Lemma 2.4 (Trick of fake moves)** *Let  $X$  be a finite set and let  $\mathcal{F} \subseteq 2^X$ . Let  $b' < b$  be positive integers. If Maker has a winning strategy for the  $(1, b)$  game  $(X, \mathcal{F})$ , then he has a strategy to win the  $(1, b')$  game  $(X, \mathcal{F})$  within  $\left\lceil \frac{|X|}{b+1} \right\rceil$  moves.*

The main idea of the proof of Lemma 2.4 is that, in every move of the  $(1, b')$  game  $(X, \mathcal{F})$ , Maker (in his mind) gives Breaker  $b - b'$  additional board elements. The straightforward details can be found in [2].

Recall the classic *box game* which was first introduced by Chvátal and Erdős in [6]. In the Box Game  $Box(m, \ell, b)$  there are  $m$  pairwise disjoint boxes  $A_1, \dots, A_m$ , each of size  $\ell$ . In every round, the first player, called *BoxMaker*, claims  $b$  elements of  $\bigcup_{i=1}^m A_i$  and then the second player, called *BoxBreaker*, destroys one box. BoxMaker wins the game  $Box(m, \ell, b)$  if and only if he is able to claim all elements of some box before it is destroyed. We use the following theorem which was proved in [6]:

**Theorem 2.5** *Let  $m, \ell$  be two integers. Then, BoxMaker wins the game  $Box(m, \ell, b)$  for every  $b > \frac{\ell}{\ln m} + 1$ .*

### 2.2.2 Avoider-Enforcer games

Similarly to Theorem 2.3, we have the following sufficient condition for Avoider's win, which was proved in [14]:

**Lemma 2.6** ([14], **Theorem 1.1**) *Let  $X$  be a finite set and let  $\mathcal{F} \subseteq 2^X$ . Avoider, as a first or a second player, has a winning strategy in the  $(a, b)$  game  $(X, \mathcal{F})$ , provided that:*

$$\sum_{F \in \mathcal{F}} \left(1 + \frac{1}{a}\right)^{-|F|} < \left(1 + \frac{1}{a}\right)^{-a}.$$

In the proof of Theorem 1.7 we use the Avoider-Enforcer version of the box game – *monotone- $r$ Box*  $(b_1, \dots, b_n, (p, q))$  which was analyzed in [9]. In this game there are  $n$  disjoint boxes of sizes  $1 \leq b_1 \leq \dots \leq b_n$ , Avoider claims at least  $p$  elements per move, Enforcer claims at least  $q$  elements per move, and Avoider loses if and only if he claims all the elements in some box by the end of the game. The following lemma can be easily derived from Theorem 1.7 and Remark 3.2 in [9]:

**Lemma 2.7** *Let  $b, k$  be positive integers. For every integer  $n \geq 2e^{k/b}$  and for every sequence of integers  $1 \leq b_1 \leq \dots \leq b_n \leq k$ , Enforcer wins the game *monotone- $r$ Box*  $(b_1, \dots, b_n, (b, 1))$  as a first or a second player.*

### 2.3 $(R, c)$ -Expanders

**Definition 2.8** *For every  $c > 0$  and every positive integer  $R$  we say that a graph  $G = (V, E)$  is an  $(R, c)$ -expander if  $|N(U)| \geq c|U|$  for every subset of vertices  $U \subseteq V$  such that  $|U| \leq R$ .*

In the proof of Theorem 1.4 Maker builds an expander and then he turns it into a Hamiltonian graph. In order to describe the relevant connection between Hamiltonicity and  $(R, c)$ -expanders, we need the notion of *boosters*.

Given a graph  $G$ , we denote by  $\ell(G)$  the maximum length of a path in  $G$ .

**Definition 2.9** *For every non-Hamiltonian graph  $G$ , we say that a non-edge  $uv \notin E(G)$  is a booster with respect to  $G$ , if either  $G \cup \{uv\}$  is Hamiltonian or  $\ell(G \cup \{uv\}) > \ell(G)$ . We denote by  $\mathcal{B}_G$  the set of boosters with respect to  $G$ .*

The following is a well-known property of  $(R, 2)$ -expanders (see e.g. [10]).

**Lemma 2.10** *If  $G$  is a connected non-Hamiltonian  $(R, 2)$ -expander, then  $|\mathcal{B}_G| \geq R^2/2$ .*

Our goal is to show that during a game on an appropriate graph  $G$ , assuming Maker can build a subgraph of  $G$  which is an  $(R, 2)$ -expander, he can also claim sufficiently many such boosters, so that his  $(R, 2)$ -expander becomes Hamiltonian. In order to do so, we need the following lemma:

**Lemma 2.11** *Let  $a > 0$  and  $p > \frac{50000a \ln n}{n}$ . Then  $G \sim G(n, p)$  is typically such that every subgraph  $\Gamma \subseteq G$  which is a non-Hamiltonian  $(n/5, 2)$ -expander with  $\frac{an \ln n}{2 \ln \ln n} \leq |E(\Gamma)| \leq \frac{100an \ln n}{\ln \ln n}$  satisfies  $|E(G) \cap \mathcal{B}_\Gamma| > \frac{n^2 p}{100}$ .*

**Proof.** First, notice that any  $(n/5, 2)$ -expander is connected. Indeed, let  $C$  be a connected component of  $G$ . If  $|C| \leq n/5$  then clearly  $C$  has neighbors outside, a contradiction. Otherwise, since  $G$  is an  $(n/5, 2)$ -expander,  $C$  must be of size at least  $3n/5 > n/2$ . Hence there is exactly one such component and  $G$  is connected. Now, fix a non-Hamiltonian  $(n/5, 2)$ -expander  $\Gamma$  in the complete graph  $K_n$ . Then clearly  $\Pr(\Gamma \subseteq G) = p^{|E(\Gamma)|}$ . By definition, the set of boosters of  $\Gamma$ ,  $\mathcal{B}_\Gamma$ , is a subset of the potential edges of  $G$ . Therefore,  $|E(G) \cap \mathcal{B}_\Gamma| \sim \text{Bin}(|\mathcal{B}_\Gamma|, p)$  and the

expected number of boosters is  $|\mathcal{B}_\Gamma|p \geq \frac{n^2p}{50}$  by Lemma 2.10. Now, by Lemma 2.1 we get that  $\Pr\left(|E(G) \cap \mathcal{B}_\Gamma| \leq \frac{n^2p}{100}\right) \leq \exp\left(-\frac{n^2p}{8}\right)$ . Running over all choices of  $\Gamma$  with  $\frac{an \ln n}{2 \ln \ln n} \leq |E(\Gamma)| \leq \frac{100an \ln n}{\ln \ln n}$  and using the union bound we get

$$\begin{aligned} & \Pr\left(\exists \Gamma \text{ such that } \Gamma \subseteq G, \frac{an \ln n}{2 \ln \ln n} \leq |E(\Gamma)| \leq \frac{100an \ln n}{\ln \ln n}, \text{ and } |E(G) \cap \mathcal{B}_\Gamma| \leq \frac{n^2p}{100}\right) \\ & \leq \sum_{m=\frac{an \ln n}{2 \ln \ln n}}^{\frac{100an \ln n}{\ln \ln n}} \binom{\binom{n}{2}}{m} p^m \exp\left(-\frac{n^2p}{400}\right) \\ & \leq \sum_{m=\frac{an \ln n}{2 \ln \ln n}}^{\frac{100an \ln n}{\ln \ln n}} \left(\frac{en^2p}{2m}\right)^m \exp\left(-\frac{n^2p}{400}\right) \\ & \leq \sum_{m=\frac{an \ln n}{2 \ln \ln n}}^{\frac{100an \ln n}{\ln \ln n}} \exp\left(m \ln\left(\frac{en^2p}{2m}\right) - \frac{n^2p}{400}\right) = \heartsuit \end{aligned}$$

To complete the proof we should show that  $\heartsuit = o(1)$ . For that goal we consider each of the cases  $np = \omega(\ln^2 n)$  and  $np = O(\ln^2 n)$  separately. For the former we have that

$$\heartsuit \leq \sum_{m=\frac{an \ln n}{2 \ln \ln n}}^{\frac{100an \ln n}{\ln \ln n}} \exp\left(n \ln^2 n - \frac{n^2p}{400}\right) = o(1);$$

and for the latter, recalling that  $p > \frac{50000a \ln n}{n}$ , we have

$$\begin{aligned} \heartsuit & \leq \sum_{m=\frac{an \ln n}{2 \ln \ln n}}^{\frac{100an \ln n}{\ln \ln n}} \exp\left(\frac{100an \ln n}{\ln \ln n} \ln\left(\frac{enp \ln \ln n}{a \ln n}\right) - \frac{n^2p}{400}\right) \\ & \leq \sum_{m=\frac{an \ln n}{2 \ln \ln n}}^{\frac{100an \ln n}{\ln \ln n}} \exp\left(\frac{100an \ln n}{\ln \ln n} \ln(C \ln n \ln \ln n) - \frac{n^2p}{400}\right) \\ & = \sum_{m=\frac{an \ln n}{2 \ln \ln n}}^{\frac{100an \ln n}{\ln \ln n}} \exp\left((1 + o(1)) 100an \ln n - \frac{n^2p}{400}\right) \\ & < \frac{100an \ln n}{\ln \ln n} \exp(-24an \ln n) = o(1). \end{aligned}$$

This completes the proof.  $\square$

The following lemma shows that an  $(R, c)$ -expander with the appropriate parameters is also  $k$ -vertex-connected.

**Lemma 2.12** ([4], Lemma 5.1) *For every positive integer  $k$ , if  $G = (V, E)$  is an  $(R, c)$ -expander such that  $c \geq k$ , and  $Rc \geq \frac{1}{2}(|V| + k)$ , then  $G$  is  $k$ -vertex-connected.*



## 2.4 Properties of $G \sim G(n, p)$

Throughout this paper we use the following properties of  $G \sim G(n, p)$ :

**Theorem 2.13** *Let  $p \geq \frac{\ln n}{n}$  and recall our notation  $f(n) := \frac{np}{\ln n}$ . A random graph  $G \sim G(n, p)$  is typically such that the following properties hold:*

(P1) *For every  $v \in V$ ,  $d(v) \leq 4np$ . For every  $\alpha > 0$  there are only  $o(n)$  vertices with degree at least  $(1 + \alpha)np$ .*

*If  $f(n) = \omega(1)$  then for every  $0 < \alpha < 1$  and for every  $v \in V$ ,*

$$(1 - \alpha)np \leq d(v) \leq (1 + \alpha)np.$$

(P2) *For every subset  $U \subseteq V$ ,  $e(U) \leq \max\{3|U|\ln n, 3|U|^2p\}$ .*

(P3) *For every subset  $U \subseteq V$  of size  $|U| \leq \frac{3n \ln \ln n}{\ln n}$ ,  $e(U) \leq 100|U|f(n) \ln \ln n$ .*

(P4) *Let  $\varepsilon > 0$ . For every constant  $\alpha > 0$  and for every subset  $U \subseteq V$  where  $1 \leq |U| \leq \frac{\alpha}{p}$ ,*

$$|N(U)| \geq \beta|U|np, \text{ for } \beta = \frac{1 - \sqrt{\frac{(2+\varepsilon)(\alpha+1)}{f(n)}}}{\alpha+1}.$$

(P5) *For every  $U \subseteq V$ ,  $\frac{1}{p} \leq |U| \leq \frac{n}{\ln n}$ ,  $|N(U)| \geq n/4$ .*

(P6) *Let  $\varepsilon > 0$ . For every  $\alpha \geq \sqrt{\frac{4}{f(n)}} + \varepsilon$  and for every set  $U \subseteq V$ , the number of edges between the set and its complement  $U^c$  satisfies:*

$$e(U, U^c) \geq (1 - \alpha)|U|(n - |U|)p.$$

(P7) *Let  $\varepsilon, \alpha$  be two positive constants which satisfy  $\alpha^2 \varepsilon f(n) > 4$ , and denote  $m := \frac{\varepsilon n \ln \ln n}{\ln n}$ . For every two disjoint subsets  $A, B \subseteq V$  with  $|A| = |B| = m$ ,  $e(A, B) \geq (1 - \alpha)m^2p$ .*

(P8)  *$e(A, B) \geq (1 - \alpha)|A||B|p$  for every two disjoint subsets  $A, B \subseteq V$  with  $|A| = \frac{10000n}{\ln \ln n}$ ,  $|B| = n/10$  and for every  $\alpha > 0$ .*

(P9) *For every subset  $U \subseteq V$  such that  $1 \leq |U| \leq \frac{n}{\ln^2 n}$ , and for every  $\varepsilon > 0$ ,  $|\{v \in V \setminus U : d(v, U) \leq \frac{\varepsilon np}{\ln n}\}| = (1 - o(1))n$ .*

**Proof.** For the proofs of (P4), (P5) below we will use the following:

Let  $U \subseteq V$ . For every vertex  $v \in V \setminus U$  we have that  $\Pr(v \in N(U)) = 1 - (1 - p)^{|U|}$  independently of all other vertices. Therefore  $|N(U)| \sim \text{Bin}(n - |U|, 1 - (1 - p)^{|U|})$ . Notice that for any  $0 < p < 1$  (all the properties above trivially hold for  $p = 1$ ) and for any positive integer  $k$  we have the following variation of Bernoulli's inequality:  $(1 - p)^{-k} \geq 1 + kp$ . Therefore,  $(1 - (1 - p)^{|U|}) \geq \left(1 - \frac{1}{1 + |U|p}\right) = \frac{|U|p}{1 + |U|p}$ . It follows that:

$$\mathbb{E}(|N(U)|) = (n - |U|) \left(1 - (1 - p)^{|U|}\right) \geq \frac{(n - |U|)|U|p}{1 + |U|p}. \quad (1)$$

(P1) For every  $v \in V$ , since  $d(v) \sim \text{Bin}(n-1, p)$ , it follows by Lemma 2.2 that

$$\Pr(d(v) \geq 4np) \leq \left(\frac{enp}{4np}\right)^{4np} < e^{-1.2np} \leq e^{-1.2 \ln n} = n^{-1.2}.$$

Applying the union bound we get that

$$\Pr(\exists v \in V \text{ with } d(v) \geq 4np) \leq n \cdot n^{-1.2} = o(1).$$

Now let  $\alpha > 0$ . By Lemma 2.1 we get that for every  $v \in V$ :

$$\Pr(d(v) > (1 + \alpha)np) \leq \exp(-\alpha'np) \leq n^{-\alpha'},$$

for some constant  $\alpha'$ . Denote by  $S$  the set of all vertices with such degree.  $\mathbb{E}(|S|) \leq n^{1-\alpha'}$ .  $|S|$  is a nonnegative random variable, so by Markov's inequality we get that:

$$\Pr(|S| > n^{1-\frac{\alpha'}{2}}) \leq \frac{n^{1-\alpha'}}{n^{1-\frac{\alpha'}{2}}} = n^{-\frac{\alpha'}{2}} = o(1).$$

Therefore, w.h.p.  $|S| \leq n^{1-\frac{\alpha'}{2}} = o(n)$ .

Assume now that  $f(n) = \omega(1)$ , and let  $0 < \alpha < 1$  be a constant. By Lemma 2.1 and the union bound we get that

$$\begin{aligned} \Pr(\exists v \in V \text{ with } d(v) \geq (1 + \alpha)np) &\leq n \exp\left(-\frac{\alpha^2}{3}np\right) \\ &= n \exp\left(-\frac{\alpha^2}{3}f(n) \ln n\right) = n^{-\omega(1)} = o(1). \end{aligned}$$

The lower bound is achieved in a similar way.

(P2) Since  $e(U) \sim \text{Bin}\left(\binom{|U|}{2}, p\right)$ , using Lemma 2.2 and the union bound we get that:

$$\begin{aligned} &\Pr(\exists U \subseteq V \text{ with } e(U) > \max\{3|U| \ln n, 3|U|^2 p\}) \\ &\leq \sum_{t=1}^{\frac{\ln n}{p}} \binom{n}{t} \left(\frac{e\binom{t}{2}p}{3t \ln n}\right)^{3t \ln n} + \sum_{t=\frac{\ln n}{p}}^n \binom{n}{t} \left(\frac{e\binom{t}{2}p}{3t^2 p}\right)^{3t^2 p} \\ &\leq \sum_{t=1}^{\frac{\ln n}{p}} \left[ n \left(\frac{tp}{2 \ln n}\right)^{3 \ln n} \right]^t + \sum_{t=\frac{\ln n}{p}}^n \left[ n \left(\frac{1}{2}\right)^{3tp} \right]^t \\ &\leq \sum_{t=1}^n \left(\frac{e}{8}\right)^{t \ln n} \leq \sum_{t=1}^n n^{-t} = o(1). \end{aligned}$$

(P3) Let  $U \subset V$  be a subset of size at most  $\frac{3n \ln \ln n}{\ln n}$ . Since  $e(U) \sim \text{Bin}\left(\binom{|U|}{2}, p\right)$ , by Lemma 2.2 we get that

$$\Pr(e(U) \geq 10|U|f(n) \ln \ln n) \leq \left(\frac{e|U|^2 p}{20|U|f(n) \ln \ln n}\right)^{10|U|f(n) \ln \ln n}.$$

Applying the union bound we get that

$$\begin{aligned}
& \Pr \left( \exists U \text{ such that } |U| \leq \frac{3n \ln \ln n}{\ln n} \text{ with } e(U) \geq 10|U|f(n) \ln \ln n \right) \\
& \leq \sum_{k=1}^{\frac{3n \ln \ln n}{\ln n}} \binom{n}{k} \left( \frac{ek^2p}{20kf(n) \ln \ln n} \right)^{10kf(n) \ln \ln n} \\
& \leq \sum_{k=1}^{\frac{3n \ln \ln n}{\ln n}} \left[ \frac{en}{k} \left( \frac{ekp}{20f(n) \ln \ln n} \right)^{10f(n) \ln \ln n} \right]^k \\
& = \sum_{k=1}^{\frac{3n \ln \ln n}{\ln n}} \left[ \frac{e^2np}{20f(n) \ln \ln n} \left( \frac{ekp}{20f(n) \ln \ln n} \right)^{10f(n) \ln \ln n - 1} \right]^k \\
& \leq \sum_{k=1}^{\frac{3n \ln \ln n}{\ln n}} \left[ \frac{e^2 \ln n}{20 \ln \ln n} \left( \frac{3enp \ln \ln n}{20f(n) \ln n \ln \ln n} \right)^{10f(n) \ln \ln n - 1} \right]^k \\
& \leq \sum_{k=1}^{\frac{3n \ln \ln n}{\ln n}} \left[ \frac{e^2 \ln n}{20 \ln \ln n} \left( \frac{3e}{20} \right)^{10f(n) \ln \ln n - 1} \right]^k \\
& = o(1).
\end{aligned}$$

(P4) Since  $n - |U| = (1 - o(1))n$  in this range, by (1) we have that:

$$\mathbb{E}(|N(U)|) \geq \frac{(n - |U|)|U|p}{1 + |U|p} \geq (1 - o(1)) \frac{|U|np}{\alpha + 1}.$$

By Lemma 2.1 we have that for any  $\delta > 0$ :

$$\Pr(|N(U)| < (1 - \delta) \mathbb{E}(|N(U)|)) \leq e^{-\frac{\delta^2}{2} \mathbb{E}(|N(U)|)} \leq e^{-\alpha' |U|np},$$

where  $\alpha' = \frac{\delta^2}{(2+o(1))(\alpha+1)}$ . Now, by taking  $\delta = \sqrt{\frac{(2+\varepsilon)(\alpha+1)}{f(n)}}$  (for some  $\varepsilon > 0$ ) we get that  $\alpha' f(n) > 1 + \frac{\varepsilon}{3}$ , and so by applying the union bound we get that:

$$\Pr \left( \exists U \subseteq V, |U| \leq \frac{\alpha}{p}, |N(U)| < (1 - \delta) \mathbb{E}(|N(U)|) \right) \leq \sum_{k=1}^{\alpha/p} \binom{n}{k} e^{-\alpha' knp} \leq \sum_{k=1}^{\alpha/p} \left[ ne^{-\alpha' f(n) \ln n} \right]^k = o(1).$$

Therefore, w.h.p. for every  $U$  of size at most  $\alpha/p$ , we obtain  $|N(U)| \geq (1 - \delta) \mathbb{E}(|N(U)|) \geq \beta |U|np$  with  $\beta = \frac{1 - \sqrt{\frac{(2+\varepsilon)(\alpha+1)}{f(n)}}}{\alpha+1}$ .

(P5) Let  $\frac{1}{p} \leq |U| \leq \frac{n}{\ln n}$ . By (1),  $\mathbb{E}(|N(U)|) \geq \frac{(n - |U|)|U|p}{1 + |U|p} \geq n/3$ .

By Lemma 2.1 we have that  $\Pr(|N(U)| \leq n/4) \leq e^{-0.01n}$ .

Applying the union bound we get that

$$\Pr(\exists \text{ such } U) \leq \sum_{k=1/p}^{n/\ln n} \binom{n}{k} e^{-0.01n} \leq n \binom{n}{n/\ln n} e^{-0.01n}$$

$$\leq n (e \ln n)^{\frac{n}{\ln n}} e^{-0.01n} = n \exp\left(\frac{n}{\ln n} \ln(e \ln n) - 0.01n\right) = o(1).$$

(P6) Assume first that  $|U| \leq n/2$ , otherwise switch the roles of  $U$  and  $U^c$ . Since every edge between  $U$  and  $U^c$  is chosen independently,  $e(U, U^c) \sim \text{Bin}(|U||U^c|, p)$ . By Lemma 2.1 we have that for given  $\alpha > 0$  and  $U \subseteq V$ :

$$\begin{aligned} \Pr(e(U, U^c) < (1 - \alpha)|U|(n - |U|)p) &\leq \exp\left(-\frac{\alpha^2}{2}|U|(n - |U|)p\right) \\ &\leq \exp\left(-\frac{\alpha^2}{4}|U|np\right) \leq \exp\left(-\left(\frac{1}{f(n)} + \delta\right)|U|np\right) \\ &= \exp(-|U|(\ln n + \delta np)), \end{aligned}$$

for some  $\delta = \delta(\varepsilon) > 0$ . By the union bound we get that:

$$\begin{aligned} \Pr(\exists \text{ such } U) &\leq \sum_{k=1}^{n/2} \binom{n}{k} \exp(-k(\ln n + \delta np)) \leq \sum_{k=1}^{n/2} [n \exp(-\ln n - \delta np)]^k \\ &= \sum_{k=1}^{n/2} \left(n^{-\delta f(n)}\right)^k = o(1). \end{aligned}$$

(P7) Similarly to (P6), given  $A, B \subset V$ ,  $|A| = |B| = m$ ,  $e(A, B) \sim \text{Bin}(m^2, p)$ . Therefore, by Lemma 2.1 we have that:

$$\Pr(e(A, B) \leq (1 - \alpha)m^2p) \leq \exp\left(-\frac{\alpha^2}{2}m^2p\right).$$

Applying the union bound we get that:

$$\begin{aligned} \Pr(\exists \text{ such } A, B) &\leq \binom{n}{m}^2 \exp\left(-\frac{\alpha^2}{2}m^2p\right) \leq \left[\left(\frac{en}{m}\right)^2 \exp\left(-\frac{\alpha^2}{2}mp\right)\right]^m \\ &= \left[\left(\frac{e \ln n}{\varepsilon \ln \ln n}\right)^2 \exp\left(-\frac{\alpha^2}{2}\varepsilon f(n) \ln \ln n\right)\right]^m < \left(\frac{e}{\varepsilon \ln \ln n}\right)^{2m} = o(1). \end{aligned}$$

(P8) Given subsets  $A, B \subseteq V$  as described, since  $e(A, B) \sim \text{Bin}(|A||B|, p)$ , by Lemma 2.1 we get that

$$\Pr(e(A, B) \leq (1 - \alpha)|A||B|p) \leq \exp\left(-\frac{\alpha^2}{2}|A||B|p\right) = \exp\left(-\frac{\alpha' n^2 p}{\ln \ln n}\right),$$

for some constant  $\alpha'$ . Applying the union bound we get that:

$$\Pr(\exists \text{ such } A, B) \leq \binom{n}{\frac{10000n}{\ln \ln n}} \binom{n}{n/10} \exp\left(-\frac{\alpha' n^2 p}{\ln \ln n}\right) \leq 4^n \exp(-\omega(n)) = o(1).$$

(P9) Suppose towards a contradiction that there exists a subset  $U \subseteq V$  such that  $1 \leq |U| \leq \frac{n}{\ln^2 n}$  and that there are  $\Theta(n)$  vertices  $v \in V \setminus U$  with  $d(v, U) \geq \frac{\varepsilon np}{\ln n}$ . Therefore, the average degree of the vertices in  $U$  is at least  $\Omega\left(n \cdot \frac{np}{\ln n} \cdot \frac{1}{|U|}\right) = \Omega(np \ln n)$ . But by (P1),  $d(v) \leq 4np$  for every  $v \in V$  — a contradiction. Hence,  $|\{v \in V \setminus U : d(v, U) \leq \frac{\varepsilon np}{\ln n}\}| = o(n)$ .  $\square$

The following two lemmas may seem somewhat unnatural, but they will be crucial for our purposes. The first one will be useful in the proof of Theorem 1.8:

**Lemma 2.14** *Let  $p \geq \frac{80 \ln n}{n}$ . A random graph  $G \sim G(n, p)$  is typically such that for every set  $U \subseteq V$  of size  $\frac{80}{p} \leq |U| \leq \frac{n}{\ln n}$ , and for every set  $W \subseteq N(U)$  of size  $|W| = \frac{1}{2}|N(U)|$ , the following holds:*

$$e(U, W) \geq \frac{1}{50}|U|np.$$

**Proof.** First, we prove the following claim:

**Claim 2.15** *For  $p \geq \frac{80 \ln n}{n}$ ,  $G \sim G(n, p)$  is typically such that for every two disjoint sets  $U, W \subseteq V$  such that  $\frac{80}{p} \leq |U| \leq \frac{n}{\ln n}$  and  $|W| = \frac{n}{100}$ ,  $e(U, W) \leq 1.5|U||W|p$ .*

**Proof of Claim 2.15.** Let  $U, W \subseteq V$  as described above. Since  $e(U, W) \sim \text{Bin}(|U||W|, p)$ , by Lemma 2.1 we have that:

$$\Pr\left(e(U, W) > 1.5|U|\frac{n}{100}p\right) \leq e^{-\frac{1}{1200}|U|np}.$$

By the union bound we get that:

$$\begin{aligned} \Pr(\exists \text{ such } U, W) &\leq \sum_{k=\frac{80}{p}}^{\frac{n}{\ln n}} \binom{n}{k} \binom{n}{n/100} e^{-\frac{1}{1200}knp} \\ &\leq \sum_{k=\frac{80}{p}}^{\frac{n}{\ln n}} \left[ \left(\frac{en}{k}\right) (100e)^{n/100k} e^{-np/1200} \right]^k \leq \sum_{k=\frac{80}{p}}^{\frac{n}{\ln n}} \left[ \left(\frac{enp}{80}\right) (e^6)^{np/8000} e^{-np/1200} \right]^k \\ &\leq \sum_{k=\frac{80}{p}}^{\frac{n}{\ln n}} \left[ np \exp\left(-\frac{np}{12000}\right) \right]^k = o(1). \end{aligned}$$

$\square$

Now we return to the proof of Lemma 2.14, and we assume that  $G$  satisfies the properties of Theorem 2.13 and Claim 2.15. Let  $U \subseteq V$ ,  $\frac{80}{p} \leq |U| \leq \frac{n}{\ln n}$ , fix some  $W \subseteq N(U)$  such that  $|W| = \frac{1}{2}|N(U)|$ , and denote  $W' = N(U) \setminus W$ . Denote by  $E_1$  the number of edges between  $U$  and  $W$ , and by  $E_2$  the number of edges between  $U$  and  $W'$ . Notice that for these sizes of  $U$ ,  $n - |U| = (1 - o(1))n$ , and that  $\sqrt{\frac{4}{80}} < 0.23$ , so by (P6) of Theorem 2.13 we have that  $E_1 + E_2 \geq 0.77|U|np$ .

Now partition  $W'$  into subsets of size  $\frac{n}{100}$  (the last one may be of smaller size). Since  $|W'| \leq \frac{n}{2}$ , 50 such subsets suffice. By Claim 2.15, there are at most  $\frac{1.5}{100}|U|np$  edges between  $U$  and each of the subsets, so  $E_2 \leq 0.75|U|np$ . Putting the two inequalities together, we get that  $E_1 \geq \frac{1}{50}|U|np$ .  $\square$

The second lemma will be a key ingredient in the main proof of the next subsection.

**Lemma 2.16** *Let  $p = \omega\left(\frac{\ln n}{n}\right)$ . Then  $G \sim G(n, p)$  is typically such that the following holds:*

*For every subset  $J_N = \{v_1, \dots, v_N\} \subseteq V$  we have that*

$$\sum_{j=1}^N \frac{e(v_j, J_j)}{j} = o(np)$$

*where  $N = \frac{n}{\ln^3 n}$  and  $J_j = \{v_1, \dots, v_j\}$ ,  $1 \leq j \leq N$ .*

**Proof.** Let  $t = \lfloor \log_2(N+1) \rfloor$ . We have that:

$$\sum_{j=1}^N \frac{e(v_j, J_j)}{j} \leq \sum_{i=0}^t \sum_{j=2^i}^{\min(N, 2^{i+1}-1)} \frac{e(v_j, J_j)}{2^i} \leq \sum_{i=1}^t \frac{e(J_{2^{i+1}})}{2^i} = \spadesuit.$$

Now, note that  $|J_j| = j$  for every  $1 \leq j \leq N$  and distinguish between the following two cases:

(i)  $pn = \omega(\ln^2 n)$ . In this case, using Property (P2) of Theorem 2.13 we have that:

$$\begin{aligned} \spadesuit &\leq \sum_{i=1}^{\log_2\left(\frac{\ln n}{p}\right)} \frac{3|J_{2^{i+1}}| \ln n}{2^i} + \sum_{i=\log_2\left(\frac{\ln n}{p}\right)}^t \frac{3|J_{2^{i+1}}|^2 p}{2^i} \\ &\leq \sum_{i=1}^{\log_2\left(\frac{\ln n}{p}\right)} \frac{3 \cdot 2^{i+1} \ln n}{2^i} + \sum_{i=\log_2\left(\frac{\ln n}{p}\right)}^t \frac{3 \cdot 2^{2i+2} p}{2^i} \\ &\leq \log_2\left(\frac{\ln n}{p}\right) 6 \ln n + 12p \cdot \frac{2^{t+1} - 1}{2 - 1} \leq c_1 \ln^2 n + c_2 Np, \end{aligned}$$

for some positive constants  $c_1$  and  $c_2$ . This is clearly  $o(np)$  as desired.

(ii)  $pn = O(\ln^2 n)$ . In this case we need a more careful calculation. First, we prove the following claim:

**Claim 2.17** *If  $pn = O(\ln^2 n)$ , then for every  $c > 3$ ,  $G \sim G(n, p)$  is typically such that  $e(X) \leq c|X|$  for every subset  $X \subseteq V$  of size  $|X| \leq \frac{n}{\ln^3 n}$ .*

**Proof of Claim 2.17.** Let  $X \subset V$  be a subset of size at most  $\frac{n}{\ln^3 n}$ . Since  $e(X) \sim \text{Bin}\left(\binom{|X|}{2}, p\right)$ , by Lemma 2.2 we get that  $\Pr(e(X) \geq c|X|) \leq \left(\frac{e|X|^2 p}{c|X|}\right)^{c|X|}$ . Applying the union bound we get that

$$\begin{aligned} & \Pr\left(\exists X \text{ such that } |X| \leq \frac{n}{\ln^3 n} \text{ and } e(X) \geq c|X|\right) \\ & \leq \sum_{k=1}^{\frac{n}{\ln^3 n}} \binom{n}{k} \left(\frac{ek^2 p}{ck}\right)^{ck} \leq \sum_{k=1}^{\frac{n}{\ln^3 n}} \left[\frac{en}{k} \left(\frac{ekp}{c}\right)^{c-1}\right]^k \\ & = \sum_{k=1}^{\frac{n}{\ln^3 n}} \left[\frac{e^2 np}{c} \left(\frac{ekp}{c}\right)^{c-1}\right]^k \leq \sum_{k=1}^{\frac{n}{\ln^3 n}} \left[O(\ln^2 n) O(\ln^{-1} n)^{c-1}\right]^k \\ & = \sum_{k=1}^{\frac{n}{\ln^3 n}} \left[O(\ln n)^{3-c}\right]^k = o(1). \end{aligned}$$

□

Now, applying Claim 2.17 with  $c = 4$ , we get that

$$\spadesuit \leq \sum_{i=1}^t \frac{4|J_{2^{i+1}}|}{2^i} \leq \sum_{i=1}^t 8 = O(\ln n) = o(np).$$

This completes the proof of Lemma 2.16. □

## 2.5 The minimum degree game

In the proof of Theorem 1.4, Maker has to build a suitable expander which possesses some relevant properties. The first step towards creating a good expander is to create a spanning subgraph with a large enough minimum degree. The following theorem was proved in [11]:

**Theorem 2.18 ([11], Theorem 1.3)** *Let  $\varepsilon > 0$  be a constant. Maker has a strategy to build a graph with minimum degree at least  $\frac{\varepsilon}{3(1-\varepsilon)} \ln n$  while playing against Breaker's bias of  $(1-\varepsilon) \frac{n}{\ln n}$  on  $E(K_n)$ .*

In fact, the following theorem can be derived immediately from the proof of Theorem 2.18:

**Theorem 2.19** *Let  $\varepsilon > 0$  be a constant. Maker has a strategy to build a graph with minimum degree at least  $c = c(n) = \frac{\varepsilon}{3(1-\varepsilon)} \ln n$  while playing against a Breaker's bias of  $(1-\varepsilon) \frac{n}{\ln n}$  on  $E(K_n)$ . Moreover, Maker can do so within  $cn$  moves and in such a way that for every vertex  $v \in V(K_n)$ , at the same moment  $v$  becomes of degree  $c$  in Maker's graph, there are still  $\Theta(n)$  free edges incident with  $v$ .*

Using Theorem 2.19, the third author of this paper proved in [17] that Maker has a strategy to build a good expander. Here, we wish to prove an analog of Theorem 2.19 for  $G(n, p)$ :

**Theorem 2.20** Let  $p = \omega\left(\frac{\ln n}{n}\right)$ ,  $\varepsilon > 0$  and let  $b = (1 - \varepsilon)\frac{np}{\ln n}$ . Then  $G \sim G(n, p)$  is typically such that in the  $(1, b)$  Maker-Breaker game played on  $E(G)$ , Maker has a strategy to build a graph with minimum degree  $c = c(n) \leq \frac{\varepsilon}{6} \ln n$ . Moreover, Maker can do so within  $cn$  moves and in such a way that for every vertex  $v \in V(G)$ , at the same moment that  $v$  becomes of degree  $c$  in Maker's graph, at least  $\varepsilon np/3$  edges incident with  $v$  are free.

**Proof of Theorem 2.20.** The proof is very similar to the proof of Theorem 2.18 so we omit some of the calculations (for more details, the reader is referred to [11]). Since claiming an extra edge is never a disadvantage for any of the players, we can assume that Breaker is the first player to move. A vertex  $v \in V$  is called *dangerous* if  $d_M(v) < c$ . The game ends at the first moment in which either none of the vertices is dangerous (and Maker won), or there exists a dangerous vertex  $v \in V$  with less than  $\varepsilon np/3$  free edges incident to it (and Breaker won). For every vertex  $v \in V$  let  $\text{dang}(v) := d_B(v) - 2b \cdot d_M(v)$  be the *danger value* of  $v$ . For a subset  $X \subseteq V$ , we define  $\overline{\text{dang}}(X) = \frac{\sum_{v \in X} \text{dang}(v)}{|X|}$  (the average danger of vertices in  $X$ ).

The strategy proposed to Maker is the following one:

**Maker's strategy  $S_M$ :** As long as there is a vertex of degree less than  $c$  in Maker's graph, Maker claims a free edge  $vu$  for some  $v$  which satisfies  $\text{dang}(v) = \max\{\text{dang}(u) : u \in V\}$  (ties are broken arbitrarily).

Suppose towards a contradiction that Breaker has a strategy  $S_B$  to win against Maker who plays according to the strategy  $S_M$  as suggested above. Let  $g$  be the length of this game and let  $I = \{v_1, \dots, v_g\}$  be the multi-set which defines Maker's game, i.e., in his  $i$ th move, Maker plays at  $v_i$  (in fact, according to the assumption Maker does not make his  $g$ th move, so let  $v_g$  be the vertex which made him lose). For every  $0 \leq i \leq g-1$ , let  $I_i = \{v_{g-i}, \dots, v_g\}$ . Following the notation of [11], let  $\text{dang}_{B_i}(v)$  and  $\text{dang}_{M_i}(v)$  denote the danger value of a vertex  $v \in V$  directly before Breaker's and Maker's  $i$ th move, respectively. Notice that in his last move, Breaker claims  $b$  edges to decrease the minimum degree of the free graph to at most  $\varepsilon np/3$ . In order to be able to do that, directly before Breaker's last move  $B_g$ , there must be a dangerous vertex  $v_g$  with  $d_M(v_g) \leq c-1$  and  $d_F(v_g) \leq \frac{\varepsilon np}{3} + b$ . By (P1) of Theorem 2.13 we can assume that  $\delta(G) \geq (1 - \frac{\varepsilon}{12})np$ . Therefore we have that  $\text{dang}_{B_g}(v_g) \geq (1 - \frac{\varepsilon}{12} - \frac{\varepsilon}{3})np - (c-1) - b - 2b(c-1) = (1 - \frac{5}{12}\varepsilon)np - (c-1) - b(2c-1) > (1 - \frac{3}{4}\varepsilon)np$ .

Analogously to the proof of Theorem 2.18 in [11], we state the following lemmas which estimate the change of the average danger after each move of any of the players. In the first lemma we estimate the change after Maker's move:

**Lemma 2.21** Let  $i$  be an integer,  $1 \leq i \leq g-1$ . Then

(i) if  $I_i \neq I_{i-1}$ , then  $\overline{\text{dang}}_{M_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \geq 0$ , and

(ii) if  $I_i = I_{i-1}$ , then  $\overline{\text{dang}}_{M_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \geq \frac{2b}{|I_i|}$ .

In the second lemma we estimate the change of the average danger during Breaker's moves:

**Lemma 2.22** Let  $i$  be an integer,  $1 \leq i \leq g-1$ . Then

(i)  $\overline{\text{dang}}_{M_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i}}(I_i) \leq \frac{2b}{|I_i|}$ , and



(ii)  $\overline{\text{dang}}_{M_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i}}(I_i) \leq \frac{b+e(v_{g-i}, I_i) + a(i-1) - a(i)}{|I_i|}$ , where  $a(i)$  denotes the number of edges spanned by  $I_i$  which Breaker claimed in the first  $g-i-1$  rounds.

Combining Lemmas 2.21 and 2.22, we get the following corollary which estimates the change of the average danger after a full round:

**Corollary 2.23** *Let  $i$  be an integer,  $1 \leq i \leq g-1$ . Then*

(i) *if  $I_i = I_{i-1}$ , then  $\overline{\text{dang}}_{B_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \geq 0$ ,*

(ii) *if  $I_i \neq I_{i-1}$ , then  $\overline{\text{dang}}_{B_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \geq -\frac{2b}{|I_i|}$ , and*

(iii) *if  $I_i \neq I_{i-1}$ , then  $\overline{\text{dang}}_{B_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \geq -\frac{b+e(v_{g-i}, I_i) + a(i-1) - a(i)}{|I_i|}$ , where  $a(i)$  denotes the number of edges spanned by  $I_i$  which Breaker took in the first  $g-i-1$  rounds.*

In order to complete the proof, we prove that before Breaker's first move,  $\overline{\text{dang}}_{B_1}(I_{g-1}) > 0$ , thus obtaining a contradiction.

Let  $N := \frac{n}{\ln^3 n}$ . For the analysis, we split the game into two parts: the main game, and the end game which starts when  $|I_i| \leq N$ .

Let  $|I_{g-1}| = r$  and let  $i_1 < \dots < i_{r-1}$  be those indices for which  $I_{i_j} \neq I_{i_{j-1}}$ . Note that  $|I_{i_j}| = j+1$ . Note also that since  $I_{i_{j-1}} = I_{i_j-1}$  and  $i_{j-1} \leq i_j - 1$ ,  $a(i_j - 1) \leq a(i_{j-1})$ .

Recall that the danger value of  $v_g$  directly before  $B_g$  is at least

$$\text{dang}_{B_g}(v_g) > \left(1 - \frac{3}{4}\varepsilon\right) np. \quad (2)$$

Assume first that  $r < N$ .

$$\begin{aligned} \overline{\text{dang}}_{B_1}(I_{g-1}) &= \overline{\text{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1} \left( \overline{\text{dang}}_{B_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \right) \\ &\geq \overline{\text{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1} \left( \overline{\text{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\text{dang}}_{B_{g-i_{j+1}}}(I_{i_{j-1}}) \right) \quad [\text{by Corollary 2.23}(i)] \\ &\geq \overline{\text{dang}}_{B_g}(I_0) - \sum_{j=1}^{r-1} \frac{b + e(v_{g-i_j}, I_{i_j}) + a(i_j - 1) - a(i_j)}{j+1} \quad [\text{by Corollary 2.23}(iii)] \\ &\geq \overline{\text{dang}}_{B_g}(I_0) - b \ln r - \sum_{j=1}^{r-1} \frac{e(v_{g-i_j}, I_{i_j})}{j+1} - \frac{a(0)}{2} + \sum_{j=2}^{r-1} \frac{a(i_{j-1})}{(j+1)j} + \frac{a(i_{r-1})}{r} \\ &\geq \overline{\text{dang}}_{B_g}(I_0) - b \ln r - o(np) \quad [\text{by Lemma 2.16}] \\ &> \left(1 - \frac{3}{4}\varepsilon\right) np - (1 - \varepsilon + o(1)) np \quad [\text{by (2)}] \\ &> 0. \end{aligned} \quad (3)$$

Assume now that  $r \geq N$ .

$$\begin{aligned}
\overline{\text{dang}}_{B_1}(I_{g-1}) &= \overline{\text{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1} \left( \overline{\text{dang}}_{B_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \right) \\
&\geq \overline{\text{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1} \left( \overline{\text{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\text{dang}}_{B_{g-i_{j+1}}}(I_{i_{j-1}}) \right) \quad [\text{by Corollary 2.23}(i)] \\
&= \overline{\text{dang}}_{B_g}(I_0) + \sum_{j=1}^{N-1} \left( \overline{\text{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\text{dang}}_{B_{g-i_{j+1}}}(I_{i_{j-1}}) \right) + \\
&\quad \sum_{j=N}^{r-1} \left( \overline{\text{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\text{dang}}_{B_{g-i_{j+1}}}(I_{i_{j-1}}) \right) \\
&\geq \text{dang}_{B_g}(v_g) - \sum_{j=1}^{N-1} \frac{b}{j+1} - o(np) - \sum_{j=N}^{r-1} \frac{2b}{j+1} \quad [\text{by Corollary 2.23}(ii) \text{ and (3)}] \\
&\geq \left( 1 - \frac{3}{4}\varepsilon \right) np - b \ln n - o(np) - 2b \left( \ln n - \ln \frac{n}{\ln^3 n} \right) \quad [\text{by (2)}] \\
&= \left( 1 - \frac{3}{4}\varepsilon \right) np - (1 - \varepsilon) np - o(np) - 6b \ln \ln n \\
&= \frac{\varepsilon}{4} np - o(np) \\
&> 0.
\end{aligned}$$

This completes the proof. □

### 3 Maker-Breaker games on $G(n, p)$

#### 3.1 Breaker's win

In this subsection we prove Theorem 1.3.

Chvátal and Erdős proved in [6] that playing on the edge set of the complete graph  $K_n$ , if Breaker's bias is  $b = (1 + \varepsilon) \frac{n}{\ln n}$ , then Breaker is able to isolate a vertex in his graph and thus to win a lot of natural games such as the perfect matching game, the Hamiltonicity game and the  $k$ -connectivity game.

In their proof, Breaker wins by creating a large clique which is disjoint of Maker's graph and then playing the box game on the stars centered in this clique. Our proof is based on the same idea.

**Proof of Theorem 1.3:** First, we may assume that  $p \geq \frac{\ln n}{n}$ , since otherwise  $G \sim G(n, p)$  typically contains isolated vertices and Breaker wins no matter how he plays. Now we introduce a strategy for Breaker and then we prove it is a winning strategy. At any point during the game, if Breaker cannot follow the proposed strategy then he forfeits the game. Breaker's strategy is divided into the following two stages:

**Stage I:** Throughout this stage Breaker maintains a subset  $C \subseteq V$  which satisfies the following properties:

- (i)  $E_G(C) = E_B(C)$ .
- (ii)  $d_M(v) = 0$  for every  $v \in C$ .
- (iii)  $d_G(v) \leq (1 + \varepsilon/2)np$  for every  $v \in C$ .

Initially,  $C = \emptyset$ . In every move, Breaker increases the size of  $C$  by at least one. This stage ends after the first move in which  $|C| \geq \frac{n}{\ln^2 n}$ .

**Stage II:** For every  $v \in C$ , let  $A_v = \{vu \in E(G) : vu \notin E(B)\}$ . In this stage, Breaker claims all the elements of one of these sets.

It is evident that if Breaker can follow the proposed strategy, then he isolates a vertex in Maker's graph and wins the game. It thus remains to prove that Breaker can follow the proposed strategy. We consider each stage separately.

**Stage I:** Notice that in every move Maker can decrease the size of  $C$  by at most one. Hence, it is enough to prove that in every move Breaker is able to find at least two vertices which are isolated in Maker's graph and have bounded degree as required, and to claim all the free edges between them and  $C$ , as well as the edge between the two vertices, if it exists in  $G$ . For this it is enough to prove that Breaker can always find two vertices  $u, v \in V \setminus C$  which have the proper degree in  $G$  and are isolated in Maker's graph, and such that  $e(u, C), e(v, C) \leq \frac{b-1}{2}$ . Since this stage lasts  $o(n)$  moves, and the number of vertices with too high degree in  $G$  is  $o(n)$  by property (P1) of Theorem 2.13, the existence of such vertices is trivial by property (P9) of Theorem 2.13.

**Stage II:** Notice that  $|C| \geq \frac{n}{\ln^2 n}$  and that  $A_v \cap A_u = \emptyset$  for every two vertices  $u \neq v$  in  $C$ . In addition, by the way Breaker chooses his vertices we have that  $|A_v| \leq (1 + \varepsilon/2)np$  for every  $v \in C$ . Recall that  $b = (1 + \varepsilon) \frac{np}{\ln n} > \frac{(1 + \varepsilon/2)np}{\ln |C|} + 1$ . Therefore, by Theorem 2.5 Breaker (as BoxMaker) wins the Box Game on these sets.

This completes the proof. □

## 3.2 Maker's win

In this subsection we prove Theorems 1.4 and 1.5. We start with providing Maker with a winning strategy in the Hamiltonicity game for each case (which implies the perfect matching game) and then we sketch the changes which need to be done in order to turn it into a winning strategy in the  $k$ -connectivity game as well.

**Proof of Theorem 1.4.** First we describe a strategy for Maker and then prove it is a winning strategy.

At any point during the game, if Maker is unable to follow the proposed strategy (including the time limits), then he forfeits the game. Maker's strategy is divided into the following three stages:

**Stage I:** Maker builds an  $(\frac{10000n}{\ln \ln n}, 2)$ -expander within  $\frac{100n \ln n}{\ln \ln n}$  moves.

**Stage II:** Maker makes his graph an  $(n/5, 2)$ -expander within additional  $\frac{300n \ln n}{\ln \ln n}$  moves.

**Stage III:** Maker makes his graph Hamiltonian by adding at most  $n$  boosters.

It is evident that if Maker can follow the proposed strategy without forfeiting the game he wins. It thus suffices to prove that indeed Maker can follow the proposed strategy. We consider each stage separately.

**Stage I:** In his first  $\frac{100n \ln n}{\ln \ln n}$  moves, Maker creates a graph with minimum degree  $c = c(n) = \frac{100 \ln n}{\ln \ln n}$ . Maker plays according to the strategy proposed in Theorem 2.20 except of the seemingly minor but crucial change that in every move, when Maker needs to claim an edge incident with a vertex  $v$ , Maker randomly chooses such a free edge. We prove that, with a positive probability, this non-deterministic strategy ensures that Maker's graph is an  $(\frac{10000n}{\ln \ln n}, 2)$ -expander and then, since our game is a perfect information game, we conclude that indeed there exists a deterministic such strategy for Maker. Recall that according to the strategy proposed in Theorem 2.20, at any move Maker claims a free edge  $vu$  with  $\text{dang}(v) = \max \{ \text{dang}(u) : u \in V \}$ . In this case we say that the edge  $vu$  is *chosen* by  $v$ . We wish to show that the probability of having a subset  $A \subseteq V$  with  $|A| \leq \frac{10000n}{\ln \ln n}$  and  $|N_M(A)| \leq 2|A| - 1$  is  $o(1)$ . To that end, we can assume that  $G$  satisfies all the properties listed in Theorem 2.13 and Theorem 2.20.

Assume that there exists a subset  $A \subset V$  of size  $|A| \leq \frac{10000n}{\ln \ln n}$  such that after this stage  $N_M(A)$  is contained in a set  $B$  of size at most  $2|A| - 1$ . This implies that

$$|E_M(A, A \cup B)| \geq c|A|/2 = \frac{50|A| \ln n}{\ln \ln n}.$$

Recall that  $f(n) := \frac{np}{\ln n}$ . We distinguish between the following two cases:

**Case I:** At least  $c|A|/4$  edges of Maker which are incident to  $A$  were chosen by vertices from  $A$ .

Notice that if  $|A| \leq \frac{n \ln \ln n}{\ln n}$ , then there are at most  $o(|A|)$  vertices  $v \in A$  such that  $e(v, A \cup B) = \Omega(f(n)(\ln \ln n)^2)$ , since otherwise we have that  $e(A \cup B) = \Omega(f(n)(\ln \ln n)^2|A|)$  which contradicts (P3) of Theorem 2.13. Furthermore, if  $\frac{n \ln \ln n}{\ln n} < |A| \leq \frac{10000n}{\ln \ln n}$ , then there are at most  $o(|A|)$  vertices  $v \in A$  such that  $e(v, A \cup B) = \Omega(np)$  (follows from (P2) of Theorem 2.13). Consider an edge  $e = ab$  with  $a \in A$  and  $b \in A \cup B$  and assume that  $e$  has been chosen by  $a$ . Notice that by Theorem 2.20, when Maker chose  $e$ , the vertex  $a$  had at least  $\varepsilon np/3$  free neighbors. Therefore, for at least  $(1 - o(1))|A|$  such vertices  $a \in A$ , the probability that Maker chose an edge with a second endpoint in  $A \cup B$  is at most  $\left(\frac{f(n)(\ln \ln n)^2}{\varepsilon np/3}\right) = \frac{3(\ln \ln n)^2}{\varepsilon \ln n}$  when  $|A| \leq \frac{n \ln \ln n}{\ln n}$  or an arbitrarily small constant  $\delta > 0$  when  $\frac{n \ln \ln n}{\ln n} < |A| \leq \frac{10000n}{\ln \ln n}$ . Therefore, the probability that all of Maker's edges incident to  $A$  were chosen in  $A \cup B$  is at most  $\left(\frac{3(\ln \ln n)^2}{\varepsilon \ln n}\right)^{(1-o(1))c|A|/4}$  for  $|A| \leq \frac{n \ln \ln n}{\ln n}$  and at most  $\delta^{(1-o(1))c|A|/4}$  otherwise. Applying the union bound we get that the probability that there exists such  $A$  (with  $|N_M(A)| \leq 2|A| - 1$  and at least  $c|A|/4$  edges chosen by  $A$ ) is at most

$$\sum_{|A| < \frac{n \ln \ln n}{\ln n}} \binom{n}{|A|} \binom{n}{2|A| - 1} \left(\frac{3(\ln \ln n)^2}{\varepsilon \ln n}\right)^{(1-o(1))c|A|/4} + \sum_{|A| = \frac{n \ln \ln n}{\ln n}}^{\frac{10000n}{\ln \ln n}} \binom{n}{|A|} \binom{n}{2|A| - 1} \delta^{(1-o(1))c|A|/4}$$

$$\begin{aligned}
&\leq \sum_{|A| < \frac{n \ln \ln n}{\ln n}} n^{3|A|} \left( \frac{3(\ln \ln n)^2}{\varepsilon \ln n} \right)^{\frac{24|A| \ln n}{\ln \ln n}} + \sum_{|A| = \frac{n \ln \ln n}{\ln n}} \left( \frac{e^3 n^3}{4|A|^3} \right)^{|A|} \delta^{\frac{24|A| \ln n}{\ln \ln n}} \\
&\leq \sum_{|A| < \frac{n \ln \ln n}{\ln n}} \left[ n^3 \exp \left( \frac{24 \ln n}{\ln \ln n} \ln \left( \frac{3(\ln \ln n)^2}{\varepsilon \ln n} \right) \right) \right]^{|A|} + \sum_{|A| = \frac{n \ln \ln n}{\ln n}} \left( \alpha \frac{\ln^3 n}{(\ln \ln n)^3} \delta^{\frac{24 \ln n}{\ln \ln n}} \right)^{|A|} \\
&\leq \sum_{|A| < \frac{n \ln \ln n}{\ln n}} \left[ n^3 \exp \left( -(1 - o(1)) 24 \ln n \right) \right]^{|A|} + o(1) = o(1).
\end{aligned}$$

**Case II:** At least  $c|A|/4$  edges of Maker which are incident to  $A$  were chosen by vertices from  $B$ .

As in Case I, notice that there are at most  $o(|B|)$  vertices  $v \in B$  such that  $e(v, A) \geq f(n) (\ln \ln n)^2$  when  $|B| \leq \frac{2n \ln \ln n}{\ln n}$  and at most  $o(|B|)$  vertices  $v \in B$  such that  $e(v, A) = \Omega(np)$  when  $\frac{2n \ln \ln n}{\ln n} \leq |B| \leq \frac{20000n}{\ln \ln n}$ . Similar to the previous case, with the only difference being that not all the edges which were chosen by vertices from  $B$  have to touch  $A$ , we get that the probability that all Maker's edges incident to  $A$  were chosen in  $A \cup B$  is at most

$$\binom{c|B|}{c|A|/4} \left( \frac{3(\ln \ln n)^2}{\varepsilon \ln n} \right)^{(1-o(1))c|A|/4}$$

or

$$\binom{c|B|}{c|A|/4} \delta^{(1-o(1))c|A|/4},$$

for an arbitrarily small  $\delta$ , for  $|B| \leq \frac{2n \ln \ln n}{\ln n}$  or  $\frac{2n \ln \ln n}{\ln n} \leq |B| \leq \frac{20000n}{\ln \ln n}$ , respectively (the binomial coefficient corresponds to the number of possible choices of edges from  $E_M(A, B)$  out of all edges chosen by vertices from  $B$ ). Applying the union bound, similar to the computations in Case I, we get that the probability that there exists such  $A$  (with  $|N_M(A)| \leq 2|A| - 1$  and at least  $c|A|/4$  edges incident to  $A$  chosen by  $N(A)$ ) is  $o(1)$ .

This completes the proof that Maker can build a  $(\frac{10000n}{\ln \ln n}, 2)$ -expander fast and thus is able to follow Stage I of the proposed strategy.

**Stage II:** It is enough to prove that Maker has a strategy to ensure that  $E_M(A, B) \neq \emptyset$  for every two disjoint subsets  $A, B \subseteq V$  of sizes  $|A| = \frac{10000n}{\ln \ln n}$  and  $|B| = n/10$ . Indeed, if there exists a subset  $X \subseteq V$  of size  $\frac{10000n}{\ln \ln n} \leq |X| \leq n/5$  such that  $|X \cup N(X)| < 3|X|$ , then there exist two subsets  $A \subseteq X$  and  $B \subseteq V \setminus (X \cup N(X))$  with  $|A| = \frac{10000n}{\ln \ln n}$  and  $|B| = n/10$  such that  $E_M(A, B) = \emptyset$ .

Recall that by Property (P8) of Theorem 2.13,  $G \sim G(n, p)$  is typically such that for every two such subsets  $A, B \subseteq V$  and for sufficiently small  $\alpha > 0$  we have that

$$e_G(A, B) \geq (1 - \alpha) |A| |B| p \geq \frac{999n^2 p}{\ln \ln n}.$$

To achieve his goal for this stage, Maker can use the trick of fake moves and to play as  $\mathcal{F}$ -Breaker in the  $(\frac{np \ln \ln n}{100 \ln n}, 1)$  Maker-Breaker game where the winning sets are

$$\mathcal{F} = \left\{ E_F(A, B) : A, B \subseteq V, A \cap B = \emptyset, |A| = \frac{10000n}{\ln \ln n} \text{ and } |B| = n/10 \right\}.$$

Notice that since so far Breaker has played at most  $\frac{100n \ln n}{\ln \ln n}$  rounds, we get that  $e_F(A, B) \geq \frac{899n^2 p}{\ln \ln n}$  for every  $A, B \subset V(G)$  of sizes  $|A| = \frac{10000n}{\ln \ln n}$  and  $|B| = n/10$ . Finally, since the following inequality holds

$$\binom{n}{\frac{10000n}{\ln \ln n}} \binom{n}{n/10} 2^{-89900n \ln n / (\ln \ln n)^2} \leq 4^n 2^{-\omega(n)} = o(1)$$

it follows by Theorems 2.3 and 2.4 that indeed Maker (or, as we called him in this stage,  $\mathcal{F}$ -Breaker) can achieve his goals for this stage within  $\frac{e(G)}{np \ln \ln n / 100 \ln n} < \frac{300n \ln n}{\ln \ln n}$  moves (recall that  $e(G) \leq 3n^2 p$ ).

**Stage III:** So far Maker has played at most  $\frac{400n \ln n}{\ln \ln n}$  moves (and at least  $\frac{50n \ln n}{\ln \ln n}$  moves) and his graph is an  $(n/5, 2)$ -expander. Notice that for the choice  $a = 2$  Lemma 2.11 holds. In addition, Maker and Breaker together claimed  $o(n^2 p)$  edges of  $G$ . Therefore, there are still  $\Theta(n^2 p)$  free boosters in  $G$ , so Maker can easily claim  $n$  boosters and to turn his graph into a Hamiltonian graph.

This completes the proof that Maker wins the game  $\mathcal{H}(G)$  (and of course also the game  $\mathcal{M}(G)$ ).  $\square$

Now, we briefly sketch the proof of Theorem 1.5.

**Sketch of proof of Theorem 1.5.** Let  $K > 10^5$ ,  $p = \frac{Kn}{\ln n}$  and  $G \sim G(n, p)$ . The upper bound on  $b^*$  is obtained immediately from Theorem 1.3. We wish to show that  $G$  is typically such that given  $b \leq K/10$ , Maker has a winning strategy in the  $(1, b)$  game  $\mathcal{H}(G)$ . First, we make the following modifications to Theorem 2.20:

- In the statement of the theorem,  $p = \frac{Kn}{\ln n}$ ,  $b \leq \frac{np}{10 \ln n} = \frac{K}{10}$ , and  $\varepsilon$  is some positive constant.
- By similar calculations to those in (P1) of Theorem 2.13, we can assume that  $\delta(G) \geq \frac{1}{2}np$ .
- We conclude that  $\text{dang}_{B_g}(v_g) \geq \left(\frac{1}{2} - \frac{\varepsilon}{3}\right)np - (c-1) - b(2c-1) > \left(\frac{1}{2} - \frac{\varepsilon}{3} - \frac{\varepsilon}{60}\right)np$ .
- Finally, we use the following calculation:

$$\begin{aligned} \overline{\text{dang}}_{B_1}(I_{g-1}) &\geq \overline{\text{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1} \left( \overline{\text{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\text{dang}}_{B_{g-i_{j+1}}}(I_{i_{j+1}}) \right) \\ &\geq \overline{\text{dang}}_{B_g}(I_0) - \sum_{j=1}^{r-1} \frac{2b}{j+1} \\ &\geq \overline{\text{dang}}_{B_g}(I_0) - 2b \ln n \\ &\geq \left( \frac{1}{2} - \frac{\varepsilon}{3} - \frac{\varepsilon}{60} \right) np - \frac{1}{5} np \\ &> 0 \end{aligned}$$

to get a contradiction (for sufficiently small  $\varepsilon$ ).

With this variant of Theorem 2.20, adapted to the case  $p = \Theta\left(\frac{\ln n}{n}\right)$ , the proof of Theorem 1.5 goes the same as the proof of Theorem 1.4, mutatis mutandis.  $\square$

**Remark:** To win the  $k$ -connectivity game, Maker follows Stages I and II of the proposed strategy  $S_M$  with the following parameter changes:

- In Stage I, Maker creates an  $\left(\frac{10000n}{\ln \ln n}, k\right)$ -expander by creating a graph with minimum degree at least  $c = 100k \ln n / \ln \ln n$ . The calculations are almost identical to these appear in the proof of Theorem 1.4, Stage I.
- In Stage II, Maker makes his graph an  $\left(\frac{n+k}{2k}, k\right)$ -expander by claiming an edge between any two disjoint subsets  $A, B \subseteq V$  such that  $|A| = \frac{10000n}{\ln \ln n}$ ,  $|B| = \frac{n}{10k}$ , in a similar way as in Stage II of Theorem 1.4.

Then, by Lemma 2.12, Maker's graph is  $k$ -connected and he wins the game. We omit the straightforward details and the calculations, which are almost identical to those of the Hamiltonicity game.

## 4 Avoider-Enforcer games on $G(n, p)$

### 4.1 Avoider's win

In this subsection we prove Theorem 1.7.

**Proof.** In order to isolate a vertex in his graph Avoider does the following: before his first move (regardless of the identity of the first player) Avoider identifies a set  $U \subseteq V$  of size  $\sqrt{\frac{n}{\ln n}}$  with  $e(U) \leq \frac{np}{\ln n}$  (he can find such a set very easily since this number is twice larger than the expected number of edges inside a set of this size). Assume  $|U| \equiv 0 \pmod{3}$  (otherwise Avoider removes from  $U$  a vertex or two and everything works the same). Then, in his first move, Avoider claims all the edges not incident to  $U$ . He then ignores – until he can no longer do so – all the edges inside  $U$  and pretends he is Enforcer in the following reverse box game: he partitions the vertices of  $U$  into triplets – each triplet is a box. The elements in each box are all the edges between the three vertices of the box and  $V \setminus U$ . For better readability, we will now call him Box-Enforcer.

Box-Enforcer does not claim edges inside  $U$  unless he has to. The former Enforcer, however, in his new role as Box-Avoider in the reverse box game, may claim occasionally edges inside  $U$ . However, since his bias is too big, in each move he must make at least  $b - e(U) \geq \frac{25np}{\ln n}$  of his steps “in the boxes” (between  $U$  and its complement).

We may assume that  $p \geq \frac{\ln n}{n}$ , since otherwise  $G \sim G(n, p)$  typically contains isolated vertices and Box-Enforcer wins no matter how he plays. Therefore, by (P1) we can bound from above the degree of every vertex in the graph by  $4np$ , and so the size of each box is bounded from above by  $12np$ . The number of boxes in this game is  $\frac{1}{3}\sqrt{\frac{n}{\ln n}}$ . Box-Avoider's bias is 1 and Box-Enforcer's bias is at least  $\frac{25np}{\ln n}$ . Putting it all together in the terms of Lemma 2.7 we get

that, as

$$2 \exp\left(\frac{12np}{\frac{25np}{\ln n}}\right) < \exp(0.49 \ln n) = o\left(\frac{1}{3}\sqrt{\frac{n}{\ln n}}\right),$$

Box-Enforcer wins this game, i.e. Box-Avoider is forced to claim all the elements in one of the boxes.

Now let's go back to the original game. By what we have just shown, as long as Avoider (or, as we called him, Box-Enforcer) does not claim edges inside  $U$ , he has at least three isolated vertices in his graph. So if he can avoid claiming edges inside  $U$  throughout the game, he wins. If he is forced to claim an edge inside  $U$ , it means that all the remaining free edges on the board are inside  $U$ . By claiming one edge he will touch at most two of his isolated vertices, and Enforcer (as Box-Avoider in this game) in his next move will be forced to claim all the remaining edges on the board, leaving at least one isolated vertex in Avoider's graph.  $\square$

## 4.2 Enforcer's win

In this subsection we prove Theorem 1.8. For this proof we would like to use similar techniques to those used by Krivelevich and Szabó in [18]. We use the following Hamiltonicity criterion by Hefetz et al:

**Lemma 4.1** ([15], **Theorem 1.1**) *Let  $12 \leq d \leq e^{\sqrt[3]{\ln n}}$  and let  $G$  be a graph on  $n$  vertices satisfying properties (R1), (R2) below:*

(R1) *For every  $S \subseteq V$ , if  $|S| \leq k_1(n, d) := \frac{n \ln \ln n \ln d}{d \ln n \ln \ln n}$  then  $|N(S)| \geq d|S|$ .*

(R2) *There is an edge in  $G$  between any two disjoint subsets  $A, B \subseteq V$  such that  $|A|, |B| \geq k_2(n, d) := \frac{n \ln \ln n \ln d}{4130 \ln n \ln \ln n}$ .*

*Then  $G$  is Hamiltonian, for sufficiently large  $n$ .*

Clearly, if by the end of the game Avoider's graph is Hamiltonian, it also contains a perfect matching. In addition, the proof of Theorem 6 in [18] shows that, in the terms of Lemma 4.1, if  $G$  satisfies (R1) and (R2) then  $G$  is  $d$ -connected. In particular, if  $d = \omega(1)$ , then  $G$  is  $k$ -connected for any fixed  $k$ . Theorem 1.8 is now an immediate corollary of Lemma 4.1 and the following theorem:

**Theorem 4.2** *Let  $p \geq \frac{70000 \ln n}{n}$  and  $b \leq \frac{np}{20000 \ln n}$ . In a biased  $(1, b)$  Avoider-Enforcer game, Enforcer has a strategy to force Avoider to create a graph satisfying (R1) and (R2) with  $d = d(n) = \ln \ln n$  provided  $n$  is large enough.*

**Proof.** As we set  $d = d(n) = \ln \ln n$  we use the following notation:

$$k_1^* = k_1^*(n) = k_1(n, d) = \frac{n}{\ln n},$$

$$k_2^* = k_2^*(n) = k_2(n, d) = \frac{n \ln \ln n}{4130 \ln n}.$$



For every  $1 \leq k \leq k_1^*$  and for every  $S \subseteq V$ ,  $|S| = k$ , define the hypergraph  $\mathcal{F}(S)$  on  $N(S)$  in the following way: partition the vertices of  $N(S)$  into a family  $\mathcal{A}$  of  $2dk$  subsets, each of size  $|N(S)|/2dk$  (by (P5) and (P4) the size of  $N(S)$  is much greater than  $2dk$ , so this is well defined). An edge  $F \in \mathcal{F}(S)$  is the set of edges between  $S$  and any combination of  $dk$  subsets from  $\mathcal{A}$ .

For a given  $S$  of size  $k$ , if by the end of the game in Avoider's graph (let us denote it by  $Av$ ) the set  $S$  is connected by an edge to every hyperedge of  $\mathcal{F}(S)$ , then  $|N_{Av}(S)| > dk$ . Otherwise, there are  $dk$  subsets disconnected from  $S$  which form a hyperedge in  $\mathcal{F}(S)$ . So in order to force (R1) in Avoider's graph, it suffices to ensure that in his graph, for every  $1 \leq k \leq k_1^*$ , for every  $S \subseteq V$ ,  $|S| = k$ , and for every  $F \in \mathcal{F}(S)$ , there is an edge between  $S$  and  $F$ .

Notice that for every  $F \in \mathcal{F}(S)$ ,  $e(S, F) \geq \frac{1}{180}|S|np$ . Indeed, if  $1 \leq |S| \leq \frac{80}{p}$ , then by (P4) we obtain  $|N(S)| \geq \frac{1}{90}|S|np$ , and so the number of edges in  $G$  between  $S$  and half of its external neighborhood is at least the number of vertices there, which is at least  $\frac{1}{180}|S|np$ . If  $\frac{80}{p} \leq |S| \leq \frac{n}{\ln n}$ , then by Lemma 2.14 the number of edges in  $G$  between  $S$  and half of its external neighborhood is at least  $\frac{1}{50}|S|np > \frac{1}{180}|S|np$ .

In order to force (R2) in Avoider's graph, it is enough to ensure that he claims an edge between any two disjoint sets of size  $k_2^*$ . By (P7), for any such sets  $A, B$ ,  $e(A, B) \geq 0.5(k_2^*)^2 p$ .

Finally, in order to conclude that Enforcer can force Avoider to claim all these edges, we apply Lemma 2.6, where Enforcer in his head did a change of roles, pretending to be Avoider with the target sets  $E(S, F)$  for every  $F \in \mathcal{F}(S)$ , and  $E(A, B)$  for all pairs of disjoint sets  $A, B$  of size  $k_2^*$ . It is now sufficient to verify that:

$$\sum_{k=1}^{k_1^*} \sum_{|S|=k} |\mathcal{F}(S)| \left(1 + \frac{1}{b}\right)^{-\frac{1}{180}|S|np} + \sum_{|A|, |B|=k_2^*} \left(1 + \frac{1}{b}\right)^{-\frac{1}{2}(k_2^*)^2 p} < \left(1 + \frac{1}{b}\right)^{-b}.$$

By using the well known estimate  $1 + x = e^{x + \Theta(x^2)}$  for  $x \rightarrow 0$ , we can bound the term on the right hand side from below by (say)  $e^{-2}$ .

Observing that  $1 + \frac{1}{b} \geq \exp(0.9/b)$ , we can estimate the first summand on the left hand side from above by:

$$\begin{aligned} \sum_{k=1}^{k_1^*} \binom{n}{k} \binom{2dk}{dk} e^{-\frac{knp}{200b}} &\leq \sum_{k=1}^{k_1^*} \left[ n (2e)^d e^{-\frac{np}{20000 \ln n}} \right]^k \\ &\leq \sum_{k=1}^{k_1^*} \left[ n e^{2 \ln \ln n} e^{-100 \ln n} \right]^k = o(1). \end{aligned}$$

The second summand on the left hand side can be estimated from above by:

$$\begin{aligned} \binom{n}{k_2^*}^2 e^{-\frac{0.5(k_2^*)^2 p}{b}} &\leq \left[ \left( \frac{en}{k_2^*} \right)^2 e^{-\frac{10^4 k_2^* \ln n}{n}} \right]^{k_2^*} \\ &= \left[ \left( \frac{4130e \ln n}{\ln \ln n} \right)^2 e^{-\frac{10^4}{4130} \ln \ln n} \right]^{k_2^*} \end{aligned}$$

$$\leq \left[ (\ln n)^{2-2.4} \right]^{k_2^*} = o(1).$$

This completes the proof. □

## 5 Concluding remarks and open questions

In this paper we analyzed Maker-Breaker games and Avoider-Enforcer games played on the edge set of a random graph  $G \sim G(n, p)$ . We have shown that a.a.s. the following is true:

**Maker-Breaker games:** for  $p = \omega\left(\frac{\ln n}{n}\right)$ , the critical bias in the Hamiltonicity, perfect matching and  $k$ -connectivity games is  $b^* = (1 + o(1))\frac{np}{\ln n}$ . For  $p = \frac{K \ln n}{n}$  (where  $K > 1$ ), there exist  $b_1 = b_1(K)$  and  $b_2 = b_2(K)$  such that the critical bias in these games satisfies:  $b_1 \leq b^* \leq b_2$ .

**Avoider-Enforcer games:** for  $p \geq \frac{c \ln n}{n}$  (where  $c > 70000$ ), there exist  $c_1$  and  $c_2$  such that the critical bias in the Hamiltonicity, perfect matching and  $k$ -connectivity games satisfies:  $\frac{c_1 np}{\ln n} \leq b^* \leq \frac{c_2 np}{\ln n}$ .

Notice that while in the first case (Maker-Breaker with  $p = \omega\left(\frac{\ln n}{n}\right)$ ) we establish the exact threshold bias, in the latter two (Maker-Breaker with  $p = \Theta\left(\frac{\ln n}{n}\right)$ , and Avoider-Enforcer) we only establish the order of magnitude of the threshold bias. Although it is possible to achieve somewhat better constants than those appearing in this paper, we were not able to close the gap completely. It would be nice to get to the exact constant in these cases as well.

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