Generating random graphs in biased Maker-Breaker games

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Abstract

We present a general approach connecting biased Maker-Breaker games and problems about local resilience in random graphs. We utilize this approach to prove new results and also to derive some known results about biased Maker-Breaker games. In particular, we show that for $b = o(\sqrt{n})$, Maker can build a pancyclic graph (that is, a graph that contains cycles of every possible length) while playing a $(1 : b)$ game on $E(K_n)$. As another application, we show that for $b = \Theta(n/\ln n)$, playing a $(1 : b)$ game on $E(K_n)$, Maker can build a graph which contains copies of all spanning trees having maximum degree $\Delta = O(1)$ with a bare path of linear length (a bare path in a tree T is a path with all interior vertices of degree exactly two in T).

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the $(a : b)$ Maker-Breaker game $\mathcal{F}$, two players, called Maker and Breaker, take turns in claiming previously unclaimed

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elements of X. The set X is called the *board* of the game and the members of $\mathcal{F}$ are referred to as the *winning sets*. Maker claims a board elements per turn, whereas Breaker claims b elements. The parameters a and b are called the *bias* of Maker and of Breaker, respectively. Maker wins the game as soon as he occupies all elements of some winning set. If Maker does not fully occupy any winning set by the time every board element is claimed by either of the players, then Breaker wins the game. We say that the $(a : b)$ game $\mathcal{F}$ is *Maker's win* if Maker has a strategy that ensures his victory against any strategy of Breaker, otherwise the game is *Breaker's win*. The most basic case is $a = b = 1$, the so-called *unbiased* game, while for all other choices of a and b the game is called a *biased* game. Note that being the first player is never a disadvantage in a Maker-Breaker game. Therefore, in order to prove that Maker can win some Maker-Breaker game as the first or the second player it is enough to prove that he can win this game as a second player. In this paper we are concerned with providing winning strategies for Maker and hence we will always assume that Maker is the second player to move.

It is natural to play Maker-Breaker games on the edge set of a graph $G = (V, E)$. In this case, $X = E$ and the winning sets are all the edge sets of the edge-minimal subgraphs of G which possess some given monotone increasing graph property $\mathcal{P}$. We refer to this game as the $(a : b)$ game $\mathcal{P}(G)$. In the *connectivity game* Maker wins if and only if his edges contain a spanning tree. In the *perfect matching* game the winning sets are all sets of $\lfloor |V(G)|/2 \rfloor$ independent edges of G. Note that if $|V(G)|$ is odd, then such a matching covers all vertices of G but one. In the *Hamiltonicity game* the winning sets are all edge sets of Hamilton cycles of G. Given a positive integer k, in the *k-connectivity game* the winning sets are all edge sets of k-vertex-connected spanning subgraphs of G. Given a graph H, in the *H-game* played on G, the winning sets are all the edge sets of copies of H in G.

Playing unbiased Maker-Breaker games on the edge set of K_n is frequently in favor of Maker. For example, it is easy to see (and also follows from [21]) that for every $n \geq 4$, Maker can win the unbiased connectivity game in $n - 1$ moves (which is clearly also the fastest possible strategy). Other unbiased games played on $E(K_n)$ like the perfect matching game, the Hamiltonicity game, the k-vertex-connectivity game and the T-game where T is a spanning tree with bounded maximum degree, are also known to be easy win for Maker (see e.g, [10], [11], [16]). It is thus natural to give Breaker more power by allowing him to claim $b > 1$ elements in each turn.

Note that Maker-Breaker games are known to be *bias monotone*. That means that none of the players can be harmed by claiming more elements. Therefore, it makes sense to study $(1 : b)$ games and the parameter b^* which is the *critical bias* of the game, that is,

b^* is the maximal bias b for which Maker wins the corresponding $(1:b)$ game $\mathcal{F}$.

There is a striking relation between the theory of biased Maker-Breaker games and the theory of random graphs, frequently referred to as the *Erdős paradigm*. Roughly speaking, it suggests that the critical bias for the game played by two "clever players" and the appropriately defined critical bias for the game played by two "random players" are asymptotically the same. In this "random players" version of the game, both players use the *random strategy*, i.e., Maker claims one random unclaimed element, while Breaker claims b random unclaimed elements from the board $E(K_n)$, per move. Note that the resulting graph occupied by Maker at the end of the game is the random graph $\mathbb{G}(n,m)$, chosen uniformly among all graphs with n vertices and $m = \lfloor \frac{1}{1+b}\binom{n}{2} \rfloor$ edges. Therefore, if the winning sets consist of all the edge sets of subgraphs of K_n which possess some monotone graph property $\mathcal{P}$, a natural guess for the critical bias is b^* for which $m^* = \frac{1}{1+b^*}\binom{n}{2}$ is the threshold for the property that $\mathbb{G}(n,m)$ typically possesses $\mathcal{P}$. For this reason, the Erdős paradigm is also known as the *random graph intuition*.

Chvátal and Erdős were the first to indicate this phenomenon in their seminal paper [9]. They showed that Breaker, playing with bias $b = \frac{(1+\varepsilon)n}{\ln n}$, can isolate a vertex in Maker's graph while playing on the board $E(K_n)$. It thus follows that Breaker wins every game for which the winning sets consist of subgraphs of K_n with positive minimum degree. What is most surprising about their result is that at the end of the game, Maker's graph consists of roughly $m = \frac{1}{2}n\ln n$ edges which is (asymptotically) the threshold for a random graph $\mathbb{G}(n,m)$ to stop "having isolated vertices" (for more details on properties' thresholds for random graphs, the reader is referred to [7] and [17]). In this spirit, the results of Chvátal and Erdős in [9] hint that $b^* = \frac{n}{\ln n}$ is actually the critical bias for many games whose target sets consist of graphs having some property $\mathcal{P}$, for which the threshold is $m^* = \frac{1}{2}n\ln n$ (such as the connectivity game, the perfect matching game and the Hamiltonicity game). Gebauer and Szabó showed in [15] that the critical bias for the connectivity game played on $E(K_n)$ is asymptotically equal to $n/\ln n$. In a relevant development, Krivelevich proved in [18] that the critical bias for the Hamiltonicity game is indeed $(1+o(1))n/\ln n$.

Another striking result exploring the relation between results in Maker-Breaker games played on graphs and threshold probabilities for properties of random graphs is due to Bednarska and Łuczak in [4]. Given a graph G on at least three vertices we define

$$m(G) = \max\left\{\frac{|E(H)|-1}{|V(H)|-2} : H \subseteq G \text{ and } |V(H)| \geq 3\right\}.$$

Bednarska and Łuczak proved that the critical bias for the H-game is of order $\Theta\left(n^{1/m(H)}\right)$. The most surprising part in their proof is the side of Maker, where they proved the following:

Theorem 1.1 (Theorem 2 in [4]). *For every graph H which contains a cycle there exists a constant c_0 such that for every sufficiently large integer n and $b \leq c_0 n^{1/m(H)}$ Maker has a random strategy for the $(1 : b)$ H-game played on $E(K_n)$ that succeeds with probability $1 - o(1)$ against any strategy of Breaker.*

Stating it intuitively, they proved that an "optimal" strategy for Maker is just to claim edges at random without caring about Breaker's moves! Note that since a Maker-Breaker game is a deterministic game, it follows that if Maker has a random strategy that works with non-zero probability against any given strategy of Breaker, then the game is Maker's win (otherwise Maker's strategy should work with probability zero against Breaker's winning strategy).

In the proof of Theorem 1.1, the graph obtained by Maker at the end of the game is not exactly a random graph, since some *failure* edges might exist (that is, it might happen that by choosing random edges, Maker attempts occasionally to pick an edge e which already belongs to Breaker). Thus, in order to prove their result, Bednarska and Łuczak not only proved that random graphs typically contain copies of the target graph H, but they also showed that with a positive probability, even after removing a small fraction of the total number of edges, these graphs still contain many copies of H. This particular statement relates to the *resilience* of random graphs with respect to the property "containing a copy of H".

Given a monotone increasing graph property $\mathcal{P}$ and a graph G which satisfies $\mathcal{P}$, the *resilience of G with respect to $\mathcal{P}$* measures how much one should change G in order to destroy $\mathcal{P}$ (here we assume that an edgeless graph does not satisfy $\mathcal{P}$). There are two natural ways to define it quantitatively. The first one is the following:

Definition 1.2. *For a monotone increasing graph property $\mathcal{P}$, the* global resilience *of G with respect to $\mathcal{P}$ is the minimum number $0 \leq r \leq 1$ such that by deleting $r \cdot e(G)$ edges from G one can obtain a graph G' not having $\mathcal{P}$.*

Since one can destroy many natural properties by small changes (for example, by isolating a vertex), it is natural to limit the number of edges touching any vertex that one is allowed to delete. This leads to the following definition of *local resilience.*

Definition 1.3. *For a monotone increasing graph property $\mathcal{P}$, the* local resilience *of G with respect to $\mathcal{P}$ is the minimum number $0 \leq r \leq 1$ such that by deleting at each vertex v at most $r \cdot d_G(v)$ edges one can obtain a graph not having $\mathcal{P}$.*

Sudakov and Vu initiated the systematic study of resilience of random and pseudorandom graphs in [23]. Since then, this field has attracted substantial research interest (see, e.g. [2, 5, 6, 8, 14, 19, 20]).

Going back to Theorem 1.1, Bednarska and Łuczak actually proved that playing according to the random strategy, Maker can typically build a graph $G \sim \mathbb{G}(n, m)$ minus some ε-fraction of its edges. They then showed that for a given graph H and an appropriate m, the global resilience of a typical $G \sim \mathbb{G}(n, m)$ with respect to the property "containing a copy of H" is at least ε. It is thus natural to seek an alternative theorem which provides the analogous local resilience argument.

The main result in this paper uses a sophisticated version of the argument in [4]. Let G be a graph and let $0 < p < 1$. The model $\mathbb{G}(G, p)$ is a random subgraph G' of G, obtained by retaining each edge of G in G' independently at random with probability p. For the special case where $G = K_n$, we denote $\mathbb{G}(n, p) := \mathbb{G}(K_n, p)$, which is the well-known Erdős-Rényi model of random graphs. Let $\mathcal{P}$ be a graph property, and consider sequences of graphs $\{G_n\}_{n=1}^{\infty}$ (indexed by the number of vertices) and probabilities $\{p(n)\}_{n=1}^{\infty}$. We say that $\mathbb{G}(G_n, p(n)) \in \mathcal{P}$ *asymptotically almost surely*, or a.a.s. for brevity, if the probability that $\mathbb{G}(G_n, p(n)) \in \mathcal{P}$ tends to 1 as n goes to infinity. In this paper, we often abuse notation and simply write $G = G_n$ and $p = p(n)$ to denote those sequences. Before stating our main result we need the following definition:

Definition 1.4. *Let $\mathcal{P}$ be a monotone increasing graph property, let $G = G_n$ denote a family of graphs (where G_n is a graph on n vertices), let $0 < p = p(n) < 1$ and let $0 \leq r \leq 1$. We say that $\mathcal{P}$ is* (G, p, r)-resilient *if the local resilience of a graph $G' \sim \mathbb{G}(G, p)$ with respect to $\mathcal{P}$ is a.a.s. at least r.*

Our main result is the following.

Theorem 1.5. *For every constant $0 < \varepsilon \leq 1/100$ and a sufficiently large integer n the following holds. Suppose that*

(i) $0 < p = p(n) < 1$,

(ii) *G is a graph with $|V(G)| = n$,*

(iii) $\delta(G) \geq \frac{10 \ln n}{\varepsilon p}$, *and*

(iv) *$\mathcal{P}$ is a monotone increasing graph property which is (G, p, ε)-resilient.*

Then Maker has a winning strategy in the $(1 : \lfloor \frac{\varepsilon}{20p} \rfloor)$ game $\mathcal{P}(G)$.

Theorem 1.5 connects between Maker's side in biased Maker-Breaker games on graphs and local resilience; it thus allows to use (known) results about local resilience to give a lower estimate for the critical bias in biased Maker-Breaker games. We now present our concrete results for biased games, all of them are applications of Theorem 1.5 and corresponding local resilience results for random graphs.

First, as a warm up we prove the following theorem which shows that the critical bias for the Hamiltonicity game played on $E(K_n)$ is $\Theta(\frac{n}{\ln n})$.

Theorem 1.6. *There exists a constant $\alpha > 0$ for which for every sufficiently large integer n the following holds. Suppose that $b \leq \alpha n/\ln n$, then Maker has a winning strategy in the $(1 : b)$ Hamiltonicity game played on $E(K_n)$.*

This result is presented here mainly for illustrative purposes, and also due to the historical importance of the biased Hamiltonicity game and the long road it took before having been resolved finally in [18].

As a second application we show that by playing a $(1 : b)$ game on $E(K_n)$, if $b = o(\sqrt{n})$, then Maker wins the *pancyclicity* game. That is, Maker can build a graph which consists of cycles of any given length $3 \leq \ell \leq n$.

Theorem 1.7. *Let $b = o(\sqrt{n})$. Then in the $(1 : b)$ game played on $E(K_n)$, Maker has a winning strategy in the pancyclicity game.*

Note that the result is asymptotical tight possible in the sense that for $b \geq 2\sqrt{n}$, Chvátal and Erdős showed in [9] that Maker cannot even build a triangle.

Ferber, Hefetz, and Krivelevich showed in [12] that if T is a tree on n vertices and $\Delta(T) \leq n^{0.05}$ then the following holds. In the $(1 : b)$ game, Maker has a strategy to win the T-game in $n + o(n)$ moves, for every $b \leq n^{0.005}$. They also asked for improvements of the parameter b (regardless of the number of moves required for Maker to win). In this paper, as a third application of our main result, we show how to obtain such an improvement for a large family of trees. Those are trees T with $\Delta(T) = O(1)$ containing a *bare path* of length $\Theta(n)$, where a bare path is a path for which all the interior vertices are of degree exactly two in T. In fact we prove the following much stronger result:

Theorem 1.8. *For every $\alpha > 0$ and $\Delta > 0$ there exists a $\delta := \delta(\alpha, \Delta) > 0$ such that for every sufficiently large integer n the following holds. For $b \leq \frac{\delta n}{\log n}$, in the $(1 : b)$ Maker-Breaker game played on $E(K_n)$, Maker has a strategy to build a graph which contains copies of all the spanning trees T such that:*

(i) $\Delta(T) \leq \Delta$, *and*

(ii) T *has a bare path of length at least* αn.

Remark 1.9. *Note that the bias* b *in Theorem 1.8 is tight (up to a constant factor), as Chvátal and Erdős showed [9] that for* $b = \lfloor \frac{(1+\varepsilon)n}{\ln n} \rfloor$ *Breaker can isolate a vertex in Maker's graph.*

The rest of the paper is organized as follows. In Section 2 we present some auxiliary results. In Section 3 we prove Theorem 1.5, and in Section 4 we show how to apply Theorem 1.5 combined with local resilience statements (introduced in Subsection 2.3) to various games.

1.1 Notation

A graph G is given by a pair of its (finite) vertex set $V(G)$ and edge set $E(G)$. For a subset X of vertices, we use $E(X)$ to denote the set of edges spanned by X, and for two disjoint sets X, Y, we use $E(X, Y)$ to denote the number of edges with one endpoint in X and the other in Y. Let $G[X]$ denote the subgraph of G induced by a subset of vertices X. We write $N(X)$ to denote the collection of vertices that have at least one neighbor in X. When X consists of a single vertex, we abbreviate $N(v)$ for $N(\{v\})$, and let $d(v)$ denote the cardinality of $N(v)$, i.e., the degree of v. Moreover, if X is a set of vertices, we let $G \setminus X$ to be the induced subgraph $G[V(G) \setminus X]$. When there are several graphs under consideration, we use subscripts such as $N_G(X)$ indicating the relevant graph of interest.

To simplify the presentation, we often omit floor and ceiling signs whenever these are not crucial and make no attempts to optimize the absolute constants involved. We also assume that the parameter n (which always denotes the number of vertices of the host graph) tends to infinity and therefore is sufficiently large whenever necessary. All our asymptotic notation symbols (O, o, Ω, ω, Θ) are relative to this variable n.

2 Auxiliary results

In this section we present some auxiliary results that will be used throughout the paper.

2.1 Binomial distribution bounds

We use extensively the following well-known bound on the lower and the upper tails of the Binomial distribution due to Chernoff (see, e.g., [1]).

Lemma 2.1. *If* $X \sim \text{Bin}(n, p)$*, then*

- $\mathbb{P}\left[X < (1-a)np\right] < \exp\left(-\frac{a^2 np}{2}\right)$ *for every* $a > 0$.
- $\mathbb{P}\left[X > (1+a)np\right] < \exp\left(-\frac{a^2 np}{3}\right)$ *for every* $0 < a \leq 1$.

The following is a trivial yet useful bound.

Lemma 2.2. *Let* $X \sim \text{Bin}(n, p)$ *and* $k \in \mathbb{N}$*. Then*

$$\mathbb{P}[X \geq k] \leq \left(\frac{enp}{k}\right)^k.$$

Proof. $\mathbb{P}[X \geq k] \leq \binom{n}{k} p^k \leq \left(\frac{enp}{k}\right)^k$. $\square$

2.2 The MinBox game

Consider the following variant of the classical *Box Game* introduced by Chvátal and Erdős in [9], which we refer to as the *MinBox game.* The game $MinBox(n, D, \alpha, b)$ is a $(1 : b)$ Maker-Breaker game played on a family of n disjoint sets (*boxes*), each having size at least D. Maker's goal is to claim at least $\alpha|F|$ elements from each box F. In the proof of our main result, we make use of a specific strategy S for Maker in the MinBox game. This strategy not only ensures his victory, but also allows Maker to maintain a reasonable proportion of elements in all boxes throughout the game.

Before describing the strategy, we need to introduce some notation. Assume that a MinBox game is in progress, let $w_M(F)$ and $w_B(F)$ denote the number of Maker's and Breaker's current elements in box F, respectively. Furthermore, let $\mathsf{dang}(F) := w_B(F) - b \cdot w_M(F)$ be the *danger value* of F. Finally, we say that a box F is *free* if it contains an element not yet claimed by either player, and it is *active* if $w_M(F) < \alpha|F|$. Maker's strategy is as follows:

Strategy S**:** In any move of the game, Maker identifies one free active box having maximal danger value (breaking ties arbitrarily), and claims one arbitrary free element from it.

We are ready to state the following theorem.

Theorem 2.3. *Let n, b, and D be positive integers, and $0 < \alpha < 1$. Assume that Maker plays the game $MinBox(n, D, \alpha, b)$ according to the strategy S described above. Then he ensures that, throughout the game, every active box F satisfies*

$$\mathsf{dang}(F) \leq b(\ln n + 1).$$

In particular, if $\alpha < \frac{1}{1+b}$ and $D \geq \frac{b(\ln n+1)}{1-\alpha(b+1)}$, then S is a winning strategy for Maker in this game.

The proof of this result can be found in the Appendix. We remark that it is very similar to the proof of Theorem 1.2 in [15].

2.3 Local resilience

In this subsection we describe several results related to local resilience of monotone graph properties. The main result of this paper (Theorem 1.5) shows a connection between local resilience of graphs and Maker-Breaker games, therefore, in order to be able to apply it, we first need to present some results related to local resilience of various properties of random graphs.

The first statement of this section is a theorem from [20] providing a good bound on the local resilience of a random graph with respect to the property "being Hamiltonian". This result will be used in the proof of Theorem 1.6 for the Hamiltonicity game. We remark that for our purposes, prior (and weaker) results on the local resilience of a random graph with respect to Hamiltonicity (for example those in [14]) would suffice.

Theorem 2.4 (Theorem 1.1, [20])**.** *For every positive $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that for $p \geq \frac{C \ln n}{n}$, a graph $G \sim \mathbb{G}(n, p)$ is a.a.s. such that the following holds. Suppose that H is a subgraph of G for which $G' = G \setminus H$ has minimum degree at least $(1/2 + \varepsilon)np$, then G' is Hamiltonian.*

The following result from [19] is related to the local resilience of a typical $G \sim \mathbb{G}(n, p)$ with respect to pancyclicity.

Theorem 2.5 (Thereom 1.1, [19])**.** *If $p = \omega(n^{-1/2})$, then the local resilience of $G \sim \mathbb{G}(n, p)$ with respect to the property "being pancyclic" is a.a.s. $1/2 + o(1)$.*

The following theorem shows that a sparse random graph $G \sim \mathbb{G}(n,p)$ is typically such that even if one deletes a small fixed fraction of edges from each vertex $v \in V(G)$, it still contains a copy of every tree T having a bare path of linear length and having bounded maximum degree. This result relates to the local resilience of the property of being *universal* for this particular class of trees, and it is an essential component in the proof of Theorem 1.8.

Theorem 2.6. *For every $\alpha > 0$ and $\Delta > 0$, there exist $\varepsilon > 0$ and C_0 such that for every $p \geq C_0 \ln n/n$, $G \sim \mathbb{G}(n,p)$ is a.a.s. such that the following holds. For every subgraph $H \subseteq G$ with $\Delta(H) \leq \varepsilon np$, the graph $G' = G \setminus H$ contains copies of all spanning trees T such that:*

(i) $\Delta(T) \leq \Delta$, *and*

(ii) T *contains a bare path of length at least* αn.

In order to prove Theorem 2.6 we need the following theorem due to Balogh, Csaba and Samotij [2] about the local resilience of random graphs with respect to the property "containing all the almost spanning trees with bounded degree".

Theorem 2.7 (Theorem 2, [2])**.** *Let β and γ be positive constants, and assume that $\Delta \geq 2$. There exists a constant $C_0 = C_0(\beta, \gamma, \Delta)$ such that for every $p \geq C_0/n$, a graph $G \sim \mathbb{G}(n,p)$ is a.a.s. such that the following holds. For every subgraph H of G for which $d_H(v) \leq (1/2 - \gamma)d_G(v)$ for every $v \in V(G)$, the graph $G' = G \setminus H$ contains all trees of order at most $(1-\beta)n$ and maximum degree at most Δ.*

Proof of Theorem 2.6 . Let $\alpha > 0$ and $\Delta > 0$ be two positive constants. Let $\varepsilon := \varepsilon(\alpha) > 0$ be a sufficiently small constant and let $C_0 = C_0(\varepsilon, \Delta) > 0$ be a sufficiently large constant. Let $G \sim \mathbb{G}(n,p)$ be a random graph, $H \subseteq G$ be any subgraph with $\Delta(H) \leq \varepsilon np$ and denote $G' = G \setminus H$. We wish to show that G' contains a copy of every spanning tree T which satisfies (i) and (ii). This can be done as follows. Assume that G has been generated by a two-round-exposure and is presented as $G = G_1 \cup G_2$, where $G_1 \sim \mathbb{G}(n, p/2)$, $G_2 \sim \mathbb{G}(n,q)$, and q is a positive constant for which $1 - p = (1-p/2)(1-q)$. Observe that $q > p/2$. Let V_0 be a random subset of $V(G)$ of size $|V_0| = 0.99\alpha n$ and denote $G'_1 = G_1 \setminus V_0$. Note that $G'_1 \sim \mathbb{G}((1-0.99\alpha)n, p/2)$ and that a.a.s. $d_{G'_1}(v) \geq (1 - 0.99\alpha - \varepsilon)np/2$ for every $v \in V(G'_1)$ (this can be easily shown using Chernoff's inequality, choosing C_0 appropriately, and applying the union bound). In addition, note that for every $v \in V(G)$, the degree of v into V_0 (in G_1) is at least $(0.99\alpha - \varepsilon)np/2$. Let T be a tree which satisfies (i) and (ii), and let $P = v_0 v_1 \dots v_t$ be a bare path of T with $t = \alpha n$. Let T' be the tree obtained from T by deleting $v_1, \dots, v_{t-1}$ and adding the edge $v_0 v_t$. Note that $|V(T')| = (1-\alpha)n + 1$.

Let β be such that $(1-\beta)|V(G_1')| = |V(T')|$. Applying Theorem 2.7 to G_1', with (say) $\gamma = 1/4$ and β, using the fact that ε is sufficiently small we conclude that there exists a copy T'' of T' in $G_1' \setminus H$. Let x and y denote the images (in T'') of v_0 and v_t (from T'), respectively.

Let $V' = (V(G) \setminus V(T'')) \cup \{x, y\}$. In order to complete the proof, we should be able to show that $(G \setminus H)[V']$ contains a Hamilton path with x and y as its endpoints. Note that $V_0 \subseteq V'$ and that $V' \setminus V_0$ and the two designated vertices x and y heavily depend on the tree T which we are trying to embed. Therefore, we wish to show that G is a.a.s. such that for every possible option for V' (with two designated vertices x and y), $(G \setminus H)[V']$ contains a Hamilton path with x and y as its endpoints. For this, note that a.a.s. $\delta\left(\left(G_1 \setminus H\right)[V']\right) \geq 0.491\alpha np$. Indeed, as we previously remarked, a.a.s. every $v \in V(G)$ has degree (in G_1) at least $(0.99\alpha - \varepsilon)np/2$ into V_0. Since ε is small enough, it follows that $\delta(G_1[V']) \geq (0.99\alpha - \varepsilon)np/2 \geq 0.494\alpha np$. Combining the last inequality with the fact that $\Delta(H) \leq \varepsilon np$, we obtain $\delta((G_1 \setminus H)[V']) \geq (0.494\alpha - \varepsilon)np \geq 0.491\alpha np$. Now, using the following claim (will be proven later) we deduce that a graph $G_1 \sim \mathbb{G}(n, p/2)$ is a.a.s. such that any subgraph $D \subseteq G_1$ on $\alpha n + 1$ vertices with $\delta(D) \geq 0.49\alpha np$ has "good" expansion properties (our candidate for D will be $\left(G_1 \setminus H\right)[V']$, and we assume that $\varepsilon < 0.001\alpha$).

Claim 2.8. *A graph $G_1 \sim \mathbb{G}(n, p/2)$ (where p is the same as in Theorem 2.6) is a.a.s. such that for any subgraph $D \subset G_1$ with $|V(D)| = \alpha n + 1$ and with $\delta(D) \geq 0.49\alpha np$, the following holds:*

$$|N_D(X) \setminus X| \geq 2|X| + 2$$

for every $X \subseteq V(D)$ with $|X| \leq |V(D)|/5$.

Next, we show how to use the edges of G_2 in order to turn the graph $(G_1 \setminus H)[V']$ into a graph which contains a Hamilton path connecting x and y. A routine way to turn a non-Hamiltonian graph D that satisfies some expansion properties (as in Claim 2.8) into a Hamiltonian graph is by using *boosters*. A booster is a non-edge e of D such that the addition of e to D creates a path which is longer than a longest path of D, or turns D into a Hamiltonian graph. In order to turn D into a Hamiltonian graph, we start by adding a booster e of D. If the new graph $D \cup \{e\}$ is not Hamiltonian then one can continue by adding a booster of the new graph. Note that after at most $|V(D)|$ successive steps the process must terminate and we end up with a Hamiltonian graph. The main point using this method is that it is well-known (for example, see [7]) that a non-Hamiltonian graph D with "good" expansion properties has many boosters. However, our goal is a bit different. We wish to turn D into a graph that contains a Hamilton path with x and y as its endpoints. In order to do so, we add one (possibly) fake edge xy to D and try to find a Hamilton cycle that contains the edge xy. Then, the path obtained by deleting this edge from the Hamilton cycle will be the desired path. For that we need to define the notion of *e-boosters*.

Given a graph D and a pair $e \in \binom{V(D)}{2}$, consider a path P of $D \cup \{e\}$ of maximal length which contains e as an edge. A non-edge e' of D is called an e-booster if $D \cup \{e, e'\}$ contains a path P' which passes through e and which is longer than P, or that $D \cup \{e, e'\}$ contains a Hamilton cycle that uses e. The following lemma shows that every connected and non-Hamiltonian graph D with "good" expansion properties has many e-boosters for every possible e.

Lemma 2.9. *Let D be a connected graph for which $|N_D(X) \setminus X| \geq 2|X|+2$ holds for every subset $X \subseteq V(D)$ of size $|X| \leq k$. Then, for every pair $e \in \binom{V(D)}{2}$ such that $D \cup \{e\}$ does not contain a Hamilton cycle which uses the edge e, the number of e-boosters for D is at least $(k+1)^2/2$.*

The proof of the previous lemma is very similar to the proof of the well-known Pósa's lemma using the ordinary boosters ([14], Lemma 4), and hence we postpone it to the appendix. The only difference is that in the proof of Lemma 2.9 we forbid rotations that destroy the edge e; and so the number of possible rotations with a given fixed endpoint drops by at most two.

Note that in order to turn $(G_1 \setminus H)[V']$ into a graph that contains a Hamiltonian cycle passes through e, one should repeatedly add e-boosters, one by one, at most $|V'| = \alpha n$ times. Therefore, in order to complete the proof, it is enough to show that a.a.s. a graph $G_2 \sim \mathbb{G}(n,q)$ is such that $G_2 \setminus H$ contains "many" e-boosters for any graph obtained from $(G_1 \setminus H)[V']$ by adding a set of edges E_0 of size at most αn. In the following lemma we formalize and prove this statement. This is the final ingredient in the proof of Theorem 2.6.

Lemma 2.10. *Assume that G_1 satisfies the conclusion of Claim 2.8. Then $G_2 \sim \mathbb{G}(n,q)$ is a.a.s. such that the following holds. Suppose that*

(i) *$V' \subseteq V(G_1)$ is a subset of size $|V'| = \alpha n + 1$,*

(ii) *$H \subset G_1$ is a subgraph such that $\delta((G_1 \setminus H)[V']) \geq 0.49\alpha np$,*

(iii) *$e = xy \in \binom{V'}{2}$, and*

(iv) E_0 *is a subset of at most* αn *pairs of* V'.

Then, $G_{E_0,e,V',H} = (G_1 \setminus H)[V'] \cup E_0 \cup \{e\}$ *contains a Hamilton cycle that passes through* e*, or* $E(G_2)$ *contains at least* $\alpha^2 n^2 p/200$ e*-boosters for* $G_{E_0,e,V',H}$.

Proof. Since G_1 satisfies the conclusion of Claim 2.8, by combining it with Lemma 2.9, it follows that for every $V' \subseteq V(G_1)$ of size $\alpha n + 1$ and $e \in \binom{V'}{2}$, for every subset E_0 of at most αn pairs of V', for every H with $\delta(G_1 \setminus H)[V'] \geq 0.49\alpha np$, the graph $G_{E_0,e,V',H}$ has at least $\alpha^2 n^2/50$ e-boosters. Fix such V', e, E_0 and H, and observe that the expected number of e-boosters in G_2 is at least $(\alpha^2 n^2/50) \cdot q \geq \alpha^2 n^2 p/100$ (recall that $q > p/2$). Therefore, by Chernoff's inequality (Lemma 2.1) it follows that the probability for $E(G_2)$ to have at most $\alpha^2 n^2 p/200$ e-boosters for $G_{E_0,e,V',H}$ is at most $\exp(-Cn^2 p)$, where C is a constant which depends only on α. Applying the union bound, running over all the options for choosing V', H, e and E_0, we obtain that the probability for having such V', e, H and E_0 for which G_2 contains at most $\alpha^2 n^2 p/200$ e-boosters for $G_{E_0,e,V',H}$ is at most

$$\sum_{t=1}^{\varepsilon n^2 p} 2^n \binom{e(G_1[V'])}{t} n^2 \binom{\alpha^2 n^2}{\alpha n} \exp(-Cn^2 p) \leq$$

$$2^n \varepsilon n^4 p \left(\frac{e\alpha^2 n^2 p}{\varepsilon n^2 p}\right)^{\varepsilon n^2 p} (e\alpha n)^{\alpha n} \exp(-Cn^2 p) = o(1).$$

where the last inequality holds for ε which is much smaller than α and for $p \geq C_0 \ln n/n$ where C_0 is sufficiently large. This completes the proof. $\square$

Before we complete the proof of Theorem 2.6, we prove Claim 2.8.

Proof of Claim 2.8. Let $S \subseteq V(G_1)$ be any subset of vertices of size $|S| \leq \frac{2\alpha n}{\sqrt{\ln n}}$, and note that $|E(G_1[S])| \sim \text{Bin}(\binom{|S|}{2}, p/2)$. Therefore, using Lemma 2.2 we obtain that

$$\begin{aligned} \mathbb{P}[|E(G_1[S])| \geq |S|np/\ln\ln n] &\leq \left(\frac{e|S|^2 p \ln\ln n}{4|S|np}\right)^{|S|np/\ln\ln n} \\ &= \left(\frac{e|S|\ln\ln n}{4n}\right)^{|S|np/\ln\ln n}. \end{aligned} \tag{1}$$

Let $\mathcal{E}$ denote the event "there exists a subset $S \subseteq V(G_1)$ of size $|S| \leq \frac{2\alpha n}{\sqrt{\ln n}}$ for which $|E(G_1[S])| \geq |S|np/\ln\ln n$". By applying the union bound and the estimate (1) we obtain that

$$\begin{aligned}\Pr\left[\mathcal{E}\right] &= \sum_{s=1}^{\frac{2\alpha n}{\sqrt{\ln n}}} \binom{n}{s} \left(\frac{es\ln\ln n}{4n}\right)^{snp/\ln\ln n} \\ &\leq \sum_{s=1}^{\frac{2\alpha n}{\sqrt{\ln n}}} \left(\frac{en}{s}\right)^{s} \left(\frac{es\ln\ln n}{4n}\right)^{snp/\ln\ln n} = o(1). \end{aligned} \tag{2}$$

Now, let $D \subseteq G_1$ be a subgraph on $\alpha n + 1$ vertices with $\delta(D) \geq 0.49\alpha np$, and we wish to show that for every $X \subseteq V(D)$, if $|X| \leq |V(D)|/5$, then $|N_D(X) \setminus X| \geq 2|X| + 2$. First, we consider the case $|X| \leq \frac{\alpha n}{3\sqrt{\ln n}}$. Assume that there exists a subset $X \subseteq V(G_1)$ in this range for which $|N_D(X) \setminus X| \leq 2|X| + 1$. Using the fact that $\delta(D) = \Theta(np)$ we obtain $|E(D[X \cup N_D(X)])| \geq |X| \cdot \Theta(np)$. Now, since $|E(D[X \cup N_D(X)])| \leq |E(G_1[X \cup N_D(X)])|$ and since $|X \cup N_D(X)| \leq \frac{\alpha n}{\sqrt{\ln n}} + 2 < \frac{2\alpha n}{\sqrt{\ln n}}$, by (2) it happens with probability $o(1)$. Therefore, we conclude that a.a.s. $|N_D(X) \setminus X| \geq 2|X| + 2$ holds for every subset $X \subseteq V(D)$ of size at most $\frac{\alpha n}{3\sqrt{\ln n}}$.

Second, assume $\frac{\alpha n}{3\sqrt{\ln n}} < |X| \leq |V(D)|/5$. In this range it is enough to show that a.a.s. for every two disjoint subsets of vertices $X, Y \subseteq V(D)$ of sizes $|X| = \frac{\alpha n}{3\sqrt{\ln n}}$ and $|Y| = \alpha n/10$ we have $|E_D(X, Y)| \neq 0$. Indeed, let $X \subseteq V(D)$ be a subset in this range and assume that $|N_D(X) \setminus X| \leq 2|X| + 2$. In particular, since $|X| \leq |V(D)|/5$ we conclude that $|X \cup N_D(X)| \leq |X| + 2|X| + 2 \leq 4|V(D)|/5$. Therefore, one can find $X' \subseteq X$ of size $|X'| = \frac{\alpha n}{3\sqrt{\ln n}}$ and $Y \subseteq V(D) \setminus (X \cup N_D(X))$ of size $|Y| = \alpha n/10$ for which $|E_D(X, Y)| = 0$, a contradiction.

In order to show that the above mentioned property a.a.s. holds, let $X, Y \subseteq V(G_1)$ be two disjoint subsets of sizes $|X| = \frac{\alpha n}{3\sqrt{\ln n}}$ and $|Y| = \alpha n/10$. For a vertex $x \in X$, let $d_{G_1}(x, Y)$ denote the number of neighbors of x in Y, and observe that $d_{G_1}(x, Y) \sim \mathrm{Bin}(|Y|, p/2)$. Therefore, the probability that $d_{G_1}(x, Y)$ is outside the interval $(|Y|p/3, 2|Y|p)$ is at most $e^{-\Theta(|Y|p|)} = e^{-\Theta(np)}$. Since all the events $d_{G_1}(x', Y) \notin (|Y|p/3, 2|Y|p)$ $(x' \in X)$ are mutually independent, we conclude that

$$\begin{aligned}&\mathbb{P}[\text{ for at least } \frac{\ln\ln n}{p} \text{ vertices } x \in X \text{ we have } d_{G_1}(x, Y) \notin (|Y|p/3, 2|Y|p)] \\ &\qquad \leq \binom{n}{\ln\ln n/p} e^{-\Theta(np \cdot \ln\ln n/p)} = e^{-\Theta(n \ln\ln n)}.\end{aligned}$$

Now, by applying Chernoff's inequality and the union bound, we obtain that the probability for having two such sets X and Y such that for at least $\frac{\ln \ln n}{p}$ vertices $x \in X$ we have $d_{G_1}(x, Y) \notin (|Y|p/3, 2|Y|p)$, is at most

$$\binom{n}{\alpha n/(3\sqrt{\ln n})}\binom{n}{\alpha n/10} e^{-\Theta(n \ln \ln n)} \leq 4^n e^{-\Theta(n \ln \ln n)} = o(1).$$

Next, we wish to show that a.a.s. in G_1, there are at most $\frac{\ln \ln n}{p}$ vertices $v \in V(D)$ with $d_{G_1}(v, V(D)) \notin (0.99|V(D)|p/2, 1.01|V(D)|p/2)$. This can be done in the following way: for each subset of vertices $D \subset V(G_1)$ of size $\alpha n + 1$, we fix an arbitrary orientation of the complete graph induced by $V(D)$ which is as regular as possible and consider $G_1[V(D)]$ as an *oriented* graph. Now, note that clearly the event " there are at most $\frac{\ln \ln n}{p}$ vertices $v \in V(D)$ with $d_{G_1}(v, V(D)) \notin (0.99|V(D)|p/2, 1.01|V(D)|p/2)$" is contained in the event $\mathcal{E}'$ = " there are at most $\frac{\ln \ln n}{p}$ vertices $v \in V(D)$ with at least one of $d^+_{G_1}(v, V(D))$ or $d^-_{G_1}(v, V(D))$ not in $(0.99|V(D)|p/4, 1.01|V(D)|p/4)$". Let $\sigma \in \{+, -\}$, D and $v \in V(D)$, and note that $d^\sigma_{G_1}(v, V(D)) \sim \text{Bin}(|V(D)|/2, p/2)$ and that the random variables $\{d^\sigma_{G_1}(v, V(D)) : v \in V(D)\}$ are mutually independent. Therefore, using a similar calculation as before and taking the union bound over $\sigma \in \{+, -\}$ we obtain the claim. Assuming this, let X and Y be two subsets of sizes $|X| = \frac{\alpha n}{3\sqrt{\ln n}}$ and $|Y| = \alpha n/10$. By the above mentioned arguments, there exists a vertex $x \in X$ with $d_{G_1}(x, Y) \in (|Y|p/3, 2|Y|p)$ and $d_{G_1}(x, V(D)) \in (0.99|V(D)|p/2, 1.01|V(D)|p/2)$. Now, since $\delta(D) \geq 0.49\alpha np = 0.98\alpha np/2$, it follows that there are at most $0.03\alpha np/2$ edges touching x in G_1 which do not appear in $V(D)$. Since $d_{G_1}(x, Y) \geq |Y|p/3 \geq \alpha np/30$, it follows D still contains edges between x and Y, and therefore $|E_D(X, Y)| \neq 0$. All in all, we conclude that $|N_D(X) \setminus X| \geq 2|X| + 2$ holds for every $|X| \leq |V(D)|/5$. This completes the proof. $\square$

This also completes the proof of Theorem 2.6. $\square$

3 Proof of the main result

Proof of Theorem 1.5 . In order to prove the theorem, we provide Maker with a random strategy that enables him to generate a random graph $G' \sim \mathbb{G}(G, p)$, and a.a.s. claim at least $1 - \varepsilon$ fraction of the edges of G' touching each vertex. We then use the fact that $\mathcal{P}$ is (G, p, ε)-resilient to conclude that G' a.a.s. satisfies $\mathcal{P}$. Note that since a Maker-Breaker game is deterministic, and since the strategy we describe a.a.s. ensures Maker's win against any strategy of Breaker, it follows that Maker also has a deterministic winning strategy.

We now present the random strategy for Maker. In this strategy, Maker will gradually generate a random graph $G' \sim \mathbb{G}(G,p)$, by tossing a biased coin on each edge of G, and declaring that it belongs to G' independently with probability p. Each edge which Maker has tossed a coin for is called *exposed*, and we say that Maker is *exposing* an edge $e \in E(G)$ whenever he tosses a coin to decide about the appearance of e in G'. To keep track of the unexposed edges, Maker maintains a set $U_v \subseteq N_G(v)$ of the *unexposed neighbors* of v, for each vertex v in G; i.e. $u \in U_v$ if and only if the edge vu remains to be exposed. Initially, $U_v = N_G(v)$ for all $v \in V(G)$. We remark that Maker will expose all edges of G, even those that belong to Breaker.

In every turn, Maker chooses an *exposure vertex* v (we will later discuss the choice of the exposure vertex) and starts to expose edges connecting v to vertices in U_v, one by one in an arbitrary order, until one edge in G' is found (that is, until he has a first success). If this exposure happens to reveal an edge $vu \in E(G')$ not yet claimed by Breaker, Maker claims it and completes his move. Otherwise, either the exposure failed to reveal a new edge in G' (failure of *type I*), or the newly found edge already belongs to Breaker (failure of *type II*). In either case, Maker skips his move. Let $f_I(v)$ and $f_{II}(v)$ denote the number of failures of type I and II, respectively, for the exposure vertex v. We remind the reader that Maker's goal is to make sure that at the end of the game $f_{II}(v)$ is relatively small, namely, $f_{II}(v) \leq \varepsilon d_{G'}(v)$ for all $v \in V(G')$. We do not know a priori what is the degree of v in G', since G' is random. However, it is true that a.a.s.

$$d_{G'}(v) \geq \frac{9}{10} d_G(v)p \tag{3}$$

holds for all $v \in V(G)$. To see this, recall that $d_G(v) \geq \delta(G) \geq \frac{10 \ln n}{\varepsilon p}$, so for any fixed $v \in V(G)$, Lemma 2.1 implies that $\mathbb{P}[\mathrm{Bin}(d_G(v),p) < \frac{9}{10} d_G(v)p] = o(\frac{1}{n})$. Hence, by the union bound, a.a.s. (3) holds for all vertices in G.

In view of (3), to complete the proof of Theorem 1.5 it suffices to show that a.a.s. Maker can ensure that $f_{II}(v) \leq \frac{9}{10}\varepsilon d_G(v)p$ for all vertices $v \in V(G)$ at the end of the game. Since Maker's goal here is to build a random graph, if a failure of type I occurs it does not harm Maker.

To keep the failures of type II under control, concurrently to the game played on G, we simulate a game $MinBox(n, 4\delta(G), p/2, 2b)$. In this simulated game, there is one box F_v for each $v \in V(G)$ which helps us to keep track of the exposure of edges touching v. Initially, we set the sizes of the boxes as $|F_v| = 4d_G(v)$. Now, we describe Maker's strategy.

Maker's strategy S_M: Maker's strategy is divided into the following two stages.

Stage 1: Before his move, Maker updates the status of the simulated game by pretending that Breaker claimed one free element from both F_v and F_u, for each edge vu occupied in Breaker's last move. Maker then identifies a free active box F_v having highest danger value in the simulated game (breaking ties arbitrarily). If there is no such box, Maker proceeds to the second stage of the strategy. Otherwise, let F_v be such a box. Maker claims one free element from F_v, and selects v as the exposure vertex. Let $\sigma : [m] \to U_v$ be an arbitrary permutation on U_v, where $m := |U_v|$. Maker starts tossing a biased coin for vertices in U_v, independently at random, according to the ordering of σ.

(a) If there were no successes, then Maker declares this turn as a *failure of type I*, thereby incrementing $f_I(v)$, and skips his move in the original game. Maker then claims $\lceil \frac{p}{2} \cdot |F_v| \rceil - 1$ additional free elements from F_v (or all the remaining free elements of F_v if there are not enough such elements) in the simulated game, and updates $U_v := \emptyset$, and $U_{\sigma(i)} := U_{\sigma(i)} \setminus \{v\}$ for each $i \leq m$.

Assume that Maker's first success has happened at the kth coin tossing.

(b) If the edge $v\sigma(k)$ is not free, then Maker declares $v\sigma(k)$ as a *failure of type II*, increments $f_{II}(v)$ by one, and skips his move in the original game. Maker then updates $U_v := U_v \setminus \{\sigma(i) : i \leq k\}$, and $U_{\sigma(i)} := U_{\sigma(i)} \setminus \{v\}$ for each $i \leq k$.

(c) Otherwise, Maker claims the edge $v\sigma(k)$. In this case Maker also claims a free element from box $F_{\sigma(k)}$ and then updates $U_v := U_v \setminus \{\sigma(i) : i \leq k\}$, and $U_{\sigma(i)} := U_{\sigma(i)} \setminus \{v\}$ for each $i \leq k$.

Stage 2: In this stage, there are no free active boxes. Let $U := \{vu : v \in V(G), u \in U_v\}$. For each $e = vu \in U$, Maker declares a failure of type II on both u and v (i.e., increments both $f_{II}(u)$ and $f_{II}(v)$ by one) with probability p, independently at random. After the end of this stage, Maker stops playing the game altogether, and skips all his subsequent moves.

We now prove that by following S_M, Maker typically achieves his goal. For the sake of notation, at any point during the game, we denote by $d_M(v)$ and $d_B(v)$ the degrees of v in the subgraphs currently occupied by Maker and Breaker, respectively. The proof will follow from the next four claims.

In the first claim, we prove that no box in the simulated game is ever exhausted of free elements. This implies that Maker is always able to effectively simulate all the moves in the original game, that is, the moves from Breaker (which in the simulated game causes two elements to be claimed), and also the moves from Maker. In particular, whenever a failure of type I occurs, Maker will claim exactly $\lceil \frac{p}{2} \cdot |F_v| \rceil - 1$ additional free elements from the relevant box, as Maker's strategy for Stage 1 (case (a)) dictates.

Claim 3.1. *At any point during the first stage, we have $w_M(F_v) < 1 + (1+2p)d_G(v)$ and $w_B(F_v) \leq d_G(v)$ for every box F_v in the simulated game. In particular, $w_M(F_v) + w_B(F_v) \leq 4d_G(v)$ thus no box is ever exhausted of free elements.*

Proof. Clearly $w_B(F_v) = d_B(v) \leq d_G(v)$, and $d_M(v) + f_{II}(v) \leq d_G(v)$. Moreover, $w_M(F_v) = d_M(v) + \lceil \frac{p}{2}|F_v| \rceil f_I(v) + f_{II}(v)$. We claim that $f_I(v) \leq 1$. This is true because otherwise F_v would still have free elements after the first failure of type I on v, and hence Maker would have claimed at least $\lceil \frac{p}{2} \cdot |F_v| \rceil$ elements from F_v. This is a contradiction, because F_v would then be inactive, and thus Maker will never play on v again, which implies $f_I(v) \leq 1$. Therefore $w_M(F_v) < 1 + d_G(v) + \frac{p}{2} \cdot |F_v| \leq 1 + (1+2p)d_G(v)$, as required. □

Claim 3.2. *For every $v \in V(G)$, F_v becomes inactive before $d_B(v) \geq \varepsilon d_G(v)/4$.*

Proof. Let $v \in V(G)$ be any vertex of $V(G)$. Note that in the simulated game, Maker always claims a free element from one free active box having highest danger value. This, however, does not imply that Maker exactly follows the strategy described in Theorem 2.3, as he might occasionally claim more than one free element when a failure of type I occurs. Nonetheless, we claim that the assertion of Theorem 2.3 still holds in this case because of the following reason. If Maker has a winning strategy in a $(1 : b)$ Maker-Breaker game, then he also has a winning strategy in a game in which he is occasionally allowed to claim more than one position per move. This is due to the monotonic nature of these types of games (recall that $MinBox$ is a Maker-Breaker game). Hence, by Theorem 2.3, we have

$$\mathsf{dang}(F_v) = w_B(F_v) - 2b \cdot w_M(F_v) \leq 2b(\ln n + 1) \tag{4}$$

for every active box F_v. Assume that there exists a vertex $v \in V(G)$ for which F_v is still active and $w_B(F_v) = d_B(v) \geq \varepsilon d_G(v)/4$. Recall that $b = \lfloor \frac{\varepsilon}{20p} \rfloor$, and by (4) it follows that

$$w_M(F_v) \geq \frac{w_B(F_v)}{2b} - (\ln n + 1) \geq \frac{5}{2} d_G(v)p - (\ln n + 1).$$

By the assumption that $\delta(G) \geq \frac{10 \ln n}{\varepsilon p}$, we conclude that $w_M(F_v) > 2d_G(v)p = \frac{p}{2}|F_v|$. However, since we assumed that F_v is active in $MinBox(n, 4\delta(G), p/2, 2b)$, we must have $w_M(F_v) \leq \frac{p}{2}|F_v|$, which is a contradiction. □

Claim 3.3. *Asymptotic almost surely all edges of G' are exposed before Stage 2.*

Proof. Suppose there exists a vertex v at the beginning of the second stage, such that $U_v \neq \emptyset$. Since $U_v \neq \emptyset$, we must have $f_I(v) = 0$. Moreover, because F_v is not active, we must also have $w_M(F_v) = d_M(v) + f_{II}(v) \geq \frac{p}{2}|F_v| = 2d_G(v)p$. This implies that $d_{G'}(v) \geq d_M(v) + f_{II}(v) \geq 2d_G(v)p$. Now, since $d_{G'}(v) \sim \mathrm{Bin}(d_G(v), p)$, using Chernoff's inequality, it follows that

$$\mathbb{P}[\mathrm{Bin}(d_G(v), p) \geq 2d_G(v)p] \leq e^{-d_G(v)p/3} = o\left(\frac{1}{n}\right).$$

Applying the union bound, it thus follows that with probability $1 - o(1)$, there exists no such vertex, proving the claim. □

Claim 3.4. *Asymptotically almost surely, for every $v \in V(G)$ we have $f_{II}(v) \leq \frac{9}{10}\varepsilon d_G(v)p$.*

Proof. Let $v \in V(G)$ be any vertex. By Claim 3.2, during Stage 1 Breaker can touch v at most $\varepsilon d_G(v)/4$ times before F_v becomes inactive. Moreover, by Claim 3.3, with probability $1 - o(1)$ all the edges of G' were exposed before the beginning of Stage 2. Since a failure of type II in Stage 1 is equivalent to Maker having a success on one of Breaker's edges, it follows that $f_{II}(v)$ is stochastically dominated by $\mathrm{Bin}(m, p)$, where $m = \varepsilon d_G(v)/4$. Applying Lemma 2.2 to $f_{II}(v)$ we conclude that the probability for having more than $\varepsilon d_G(v)p$ edges vu which are failures of type II is at most

$$\mathbb{P}\left[\mathrm{Bin}(\varepsilon d_G(v)/4, p) \geq \frac{9}{10}\varepsilon d_G(v)p\right] \leq \left(\frac{e\varepsilon d_G(v)p/4}{\frac{9}{10}\varepsilon d_G(v)p}\right)^{\frac{9}{10}\varepsilon d_G(v)p} = o\left(\frac{1}{n}\right).$$

Applying the union bound we obtain that the probability that there is such a vertex is $o(1)$. Therefore, a.a.s. $f_{II}(v) \leq \frac{9}{10}\varepsilon d_G(v)p$ for all $v \in V(G)$. □

This completes the proof of Theorem 1.5. □

4 Applications

In this section we show how to apply Theorem 1.5 in order to prove Theorems 1.6, 1.7 and 1.8. We also derive a directed graph analog of Theorem 1.5. We start with proving Theorem 1.6, which states that Maker can win the Hamiltonicity game played on $E(K_n)$ against an asymptotically optimal (up to a constant factor) bias of Breaker.

Proof of Theorem 1.6 . Let $C_1 = C(\frac{1}{6})$ be as in Theorem 2.4, and let $C_2 := \max\{C_1, 1000\}$. First, observe that for $p \geq \frac{C_2 \ln n}{n}$ a.a.s. we have that $G \sim \mathbb{G}(n,p)$ satisfies $\delta(G) \geq \frac{5}{6}np$ (this follows immediately from Chernoff's inequality and the union bound). Next, note that the property $\mathcal{P} :=$"being Hamiltonian" is $(K_n, p, 1/6)$ resilient for $p \geq \frac{C_2 \ln n}{n}$. Indeed, let $H \subseteq G$ be a subgraph for which $d_H(v) \leq \frac{1}{6}d_G(v)$. Observe that in $G' := G - H$ we have $d_{G'}(v) \geq \frac{5}{6}d_G(v)$. Now, since a.a.s. $\delta(G) \geq \frac{5}{6}np$, it follows that $\delta(G') \geq \frac{25}{36}np > \frac{2}{3}np$. Therefore, by our choice of C_2 and Theorem 2.4, it follows that G' is Hamiltonian.

Lastly, applying Theorem 1.5 with $\varepsilon = \frac{1}{100}$ (recall that we have an upper bound for ε), K_n (as the host graph G), $p = \frac{C_2 \ln n}{n}$ and $\mathcal{P}$, we obtain that Maker has a winning strategy in the $(1 : \lfloor \frac{1}{120p} \rfloor)$ game $\mathcal{P}(K_n)$. Note that $\frac{1}{120p} = \frac{n}{120 C_2 \ln n}$, and therefore, by setting $\alpha := \frac{1}{120C_2}$ we complete the proof. $\square$

Next, we prove Theorem 1.7

Proof of Theorem 1.7 . Let $p = \omega(n^{-1/2})$ and note that by Theorem 2.5, it follows that the property $\mathcal{P} :=$"being pancyclic" is $(K_n, p, 1/2 + o(1))$-resilient. Therefore, by applying Theorem 1.5 with $\varepsilon = 1/100$, K_n (as the host graph), p and $\mathcal{P}$, we obtain that Maker has a winning strategy in the $(1 : \lfloor \frac{1}{60p} \rfloor)$ game $\mathcal{P}(K_n)$. This completes the proof. $\square$

We turn to prove Theorem 1.8.

Proof of Theorem 1.8 . Let $\alpha > 0$ and $\Delta > 0$ be two positive constants. Let $\varepsilon > 0$ and C_0 be as in Theorem 2.6 (applied to α and Δ). Let $C_1 \geq \max\left\{C_0, \frac{20}{\varepsilon}\right\}$ be a large enough constant for which $G \sim \mathbb{G}(n,p)$ a.a.s. satisfies $\Delta(G) \leq (1+\varepsilon)np$, provided that $p = \frac{C_1 \ln n}{n}$. Let $\mathcal{T}$ be the set of all trees T on n vertices satisfying:

(i) $\Delta(T) \leq \Delta$, and

(ii) T contains a bare path of length at least αn,

and let $\mathcal{P}$ be the property "being $\mathcal{T}$-universal" (that is, contains copies of all trees in $\mathcal{T}$). Observe that $\mathcal{P}$ is $\left(K_n, p, \frac{\varepsilon}{1+\varepsilon}\right)$ resilient, and hence $\left(K_n, p, \frac{\varepsilon}{2}\right)$ resilient for $p = \frac{C_1 \ln n}{n}$. Indeed, let H be a subgraph of G for which $d_H(v) \leq \frac{\varepsilon}{1+\varepsilon} \cdot d_G(v)$, for all vertices $v \in V(G)$. Thus $d_H(v) \leq \frac{\varepsilon}{1+\varepsilon} \cdot \Delta(G) \leq \varepsilon np$. Therefore, by Theorem 2.6, $G' := G \setminus H$ satisfies $\mathcal{P}$.

Lastly, by applying Theorem 1.5 with $\min\{\varepsilon/2, 1/100\}$ (as ε), K_n (as the host graph G), p, and $\mathcal{P}$, we obtain that Maker has a winning strategy in the $(1 : \lfloor \frac{\varepsilon}{40p} \rfloor)$ game $\mathcal{P}(K_n)$. By setting $\delta = \frac{\varepsilon}{40C_1}$, we complete the proof. Note that we used the fact that $C_1 \geq \frac{20}{\varepsilon}$ in order to verify assumption (iii) in Theorem 1.5. $\square$

As a last application, we establish an analog of Theorem 1.5 to directed graphs. A *directed* graph D consists of a set of vertices $V(D)$, and a set of *arcs* (or *directed* edges) $E(D)$ composed of elements of the form $(u, v) \in V(D) \times V(D)$, where $u \neq v$. For a directed graph D and a vertex $v \in V(D)$ we let $d^+(v)$ and $d^-(v)$ denote the out- and in-degrees of v, respectively. Furthermore, we define $\delta^+(D)$ and $\delta^-(D)$ to be the minimum out- and in- degrees of D, respectively, and set $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$. Analogously to graphs, we define $\mathbb{D}(D, p)$ to be the model of random sub-directed graphs of D obtained by retaining each arc of D with probability p, independently at random. We write $\mathbb{D}(n, p)$ for $\mathbb{D}(D, p)$ in the special case where D is the complete directed graph on n vertices. That is, $V(D) = [n]$ and $E(D)$ consists of all the possible arcs. Similarly as in Definition 1.4, for a monotone increasing directed graph property $\mathcal{P}$, we say that $\mathcal{P}$ is (D, p, r)-*resilient* if the local resilience of $D' \sim \mathbb{D}(D, p)$ with respect to $\mathcal{P}$ is at least r, where here we mean that by deleting at each vertex v at most $r \cdot d^+_D(v)$ out- and $r \cdot d^-_D(v)$ in-edges one can obtain a directed graph not having $\mathcal{P}$.

Theorem 4.1. *For every constant* $0 < \varepsilon \leq 1/100$ *and a sufficiently large integer* n *the following holds. Suppose that*

(i) $0 < p = p(n) < 1$,

(ii) D *is a directed graph with* $|V(D)| = n$,

(iii) $\delta^0(D) \geq \frac{10 \ln n}{\varepsilon p}$, *and*

(iv) $\mathcal{P}$ *is a monotone increasing directed graph property which is* (D, p, ε)*-resilient.*

Then Maker has a winning strategy in the $(1 : \lfloor \frac{\varepsilon}{20p} \rfloor)$ *game* $\mathcal{P}(D)$.

Proof. For a directed graph D one can define the following bipartite graph G_D: the parts of G_D are two disjoint copies of $V(D)$, denoted by A and B. For any $a \in A$ and $b \in B$, the (undirected) edge ab belongs to $E(G_D)$ if and only if the directed edge ab belongs to $E(D)$. Note that the mapping $D \to G_D$ is an injection from the set of all directed graphs on n vertices to the set of bipartite graphs with two parts of size n each, and apply Theorem 1.5 to G_D in the obvious way. Note that the property $\mathcal{P}$ of digraphs naturally translates to a property $\mathcal{P}'$ of bipartite graphs which is (G_D, p, ε)-resilient. $\square$

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A Proofs of Theorem 2.3 and Lemma 2.9

We begin with the proof of the $MinBox$ game.

Proof of Theorem 2.3 . The proof of this theorem is very similar to the proof of Theorem 1.2 in [15]. Since claiming an extra element is never a disadvantage for any of the players, we can assume that Breaker is the first player to move. For a subset X of boxes, let $\overline{\mathsf{dang}}(X) = \frac{\sum_{F \in X} \mathsf{dang}(F)}{|X|}$ denote the average danger of the boxes in X. The game ends when there are no more free elements left.

First we prove the upper bound for the danger values of active boxes. Suppose, towards a contradiction, that there exists a strategy for Breaker that ensures the existence of an active box F satisfying $\mathsf{dang}(F) > b(\ln n + 1)$ at some point during the game. Denote the first time when this happens by g. Let $I = \{F_1, \ldots, F_g\}$ be the set which defines Maker's game, i.e, in his i^{th} move, Maker plays at F_i for $1 \le i \le g-1$ and F_g is the first active box satisfying $\mathsf{dang}(F_g) > b(\ln n + 1)$. For every $0 \le i \le g-1$, let $I_i = \{F_{g-i}, \ldots, F_g\}$. Following the notation of [15], let $\mathsf{dang}_{B_i}(F)$ and $\mathsf{dang}_{M_i}(F)$ denote the danger value of a box F, *directly before* Breaker's and Maker's i^{th} move, respectively. Notice that in his g^{th} move, Breaker increases the danger value of F_g to more than $b(\ln n + 1)$. This is only possible if $\mathsf{dang}_{B_g}(F_g) > b(\ln n + 1) - b = b \ln n$.

Analogously to the proof of Theorem 1.2 in [15], we state the following lemmas which estimate the change of the average danger after a particular move (by either player). In the first lemma we estimate the changes after Maker's moves.

Lemma A.1. *Let i, $1 \le i \le g-1$,*

(i) if $I_i \neq I_{i-1}$, then $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \ge 0$.

(ii) if $I_i = I_{i-1}$, then $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \ge \frac{b}{|I_i|}$.

Proof. For part (i) we have that $F_{g-i} \notin I_{i-1}$. Since danger values do not increase during Maker's move, we have $\overline{\mathsf{dang}}_{M_{g-i}}(I_{i-1}) \ge \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})$. Before M_{g-i}, Maker selected the box F_{g-i} because its danger was highest among the active boxes. Thus $\mathsf{dang}(F_{g-i}) \ge \max(\mathsf{dang}(F_{g-i+1}), \ldots, \mathsf{dang}(F_g))$, which implies $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) \ge \overline{\mathsf{dang}}_{M_{g-i}}(I_{i-1})$. Combining the two inequalities establishes part (i).

For part (ii) we have that $F_{g-i} \in I_{i-1}$. In M_{g-i}, $w_M(F_{g-i})$ increases by 1 and $w_M(F)$ does not change for any other box $F \in I_i$. Besides, the values of $w_B(\cdot)$ do not change during Maker's move. So $\mathsf{dang}(F_{g-i})$ decreases by b, whereas $\mathsf{dang}(F)$ do not increase for any other box $F \in I_i$. Hence $\overline{\mathsf{dang}}(I_i)$ decreases by at least $\frac{b}{|I_i|}$, which implies (ii). □

In the second lemma we estimate the changes after Breaker's moves.

Lemma A.2. *Let i be an integer, $1 \le i \le g-1$. Then,*

$$\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i) \le \frac{b}{|I_i|}.$$

Proof. The increase of $\sum_{F \in I_i} w_B(F)$ during B_{g-i} is at most b. Moreover, since the values of $w_M(F)$ for $F \in I_i$ do not change during Breaker's move, the increase of $\overline{\mathsf{dang}}(I_i)$ (during

B_{g-i}) is at most $\frac{b}{|I_i|}$, which establishes the lemma. $\square$

Combining Lemmas A.1 and A.2, we obtain the following corollary which estimates the change of the average danger after a full round.

Corollary A.3. *Let i be an integer, $1 \leq i \leq g-1$.*

(i) if $I_i = I_{i-1}$, then $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq 0$.

(ii) if $I_i \neq I_{i-1}$, then $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq -\frac{b}{|I_i|}$

Next, we prove that before Breaker's first move, $\overline{\mathsf{dang}}_{B_1}(I_{g-1}) > 0$, thus obtaining a contradiction. To that end, let $|I_{g-1}| = r$ and let $i_1 < \ldots < i_{r-1}$ be those indices for which $I_{i_j} \neq I_{i_j-1}$. Note that $|I_{i_j}| = j+1$. Recall that $\mathsf{dang}_{B_g}(F_g) > b \ln n$, therefore

$$\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &= \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1} \Big(\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})\Big) \\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1} \Big(\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1})\Big) \quad \text{[by Corollary A.3 } (i)\text{]} \\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) - \sum_{j=1}^{r-1} \frac{b}{j+1} \quad \text{[by Corollary A.3 } (ii)\text{]} \\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) - b \ln n > 0,
\end{aligned}$$

and this contradiction establishes the upper bound for the danger values of active boxes.

Lastly, consider a $MinBox(n, D, \alpha, b)$ game where $\alpha < \frac{1}{b+1}$ and $D \geq \frac{b(\ln n+1)}{1-\alpha(b+1)}$. We will prove that S is a winning strategy for Maker in this setting. With this in mind, it suffices to show that there are no active boxes left at the very end of the game. Suppose not, and let F be a box which remained active, i.e., $w_M(F) < \alpha|F|$. Clearly F is not free, since the game has ended. Thus we have $w_M(F) + w_B(F) = |F|$. Moreover, since Maker played according to S, we must have $\mathsf{dang}(F) \leq b(\ln n + 1)$. Hence

$$b(\ln n + 1) \geq \mathsf{dang}(F) = w_B(F) - b \cdot w_M(F) = |F| - (b+1)w_M(F) > (1 - \alpha(b+1))|F|.$$

This implies that $D \leq |F| < \frac{b(\ln n+1)}{1-\alpha(b+1)}$, which is a contradiction, thereby proving that Maker is the winner, and concluding the proof of the theorem.

$\square$

We turn to prove the variant of Pósa's lemma for e-boosters.

Proof of Lemma 2.9 . Let D be a connected graph for which $|N_D(X) \setminus X| \geq 2|X| + 2$ holds for every subset $X \subseteq V(D)$ of size $|X| \leq k$. Let $e \in \binom{V(D)}{2}$ be a pair such that the graph $D \cup \{e\}$ does not contain a Hamilton cycle which uses e. We will prove that the number of e-boosters for D is at least $(k+1)^2/2$.

The idea behind the proof is fairly natural and is based on Pósa's *rotation-extension* technique. Let $P = x_0 x_1 \dots x_h$ be a path in $D \cup \{e\}$, starting at a fixed endpoint x_0. Suppose P contains e, say $e = x_i x_{i+1}$ for some $0 \leq i < h$. If D contains an edge $x_j x_h$ for some $0 \leq j < h - 1$ such that $j \neq i$, then one can obtain a new path P' of the same length as P which contains e. The new path is $P' = x_0 x_1 \dots x_j x_h x_{h-1} \dots x_{j+1}$, obtained by adding the edge $x_j x_h$ and deleting $x_j x_{j+1}$. This operation is called an *elementary rotation* at x_j with a fixed endpoint x_0. We can apply other elementary rotations repeatedly, and if after a number of rotations, an endpoint x of the obtained path Q is connected by an edge to a vertex y outside Q, then Q can be extended by adding the edge xy.

The power of the rotation-extension technique of Pósa hinges on the following fact. Let $P = x_0 \dots x_h$ be a longest path in $D \cup \{e\}$ containing e. Let $\mathcal{P}$ be the set of all paths obtainable from P by a sequence of elementary rotations with fixed x_0. Denote by R the set of the other endpoints (not x_0) of paths in $\mathcal{P}$, and by R^- and R^+ the sets of vertices immediately preceding and following the vertices of R along P, respectively. We claim that:

Claim A.4. $N_D(R) \setminus R \subseteq R^- \cup R^+ \cup e$.

Proof of Claim A.4. Fix $u \in R$, let $v \in V(D) \setminus (R \cup R^- \cup R^+ \cup e)$, and consider a path $Q \in \mathcal{P}$ ending at u. If $v \in V(D) \setminus V(P)$, then $uv \notin E(D)$, as otherwise the path Q can be extended by adding v, thus contradicting our assumption that P is a longest path in $D \cup \{e\}$ containing e. Suppose now that $v \in V(P) \setminus (R \cup R^- \cup R^+ \cup e)$. Then v has the same two neighbors in every path in $\mathcal{P}$, because an elementary rotation that removed one of its neighbors along P would, at the same time, put either this neighbor or v itself in R (in the former case $v \in R^- \cup R^+$). Then if u and v are adjacent, an elementary rotation at v can be applied to Q (since $v \notin e$), and produces a path in $\mathcal{P}$ whose endpoint is a neighbor of v along P, a contradiction. Therefore in both cases u and v are non-adjacent, thereby proving Claim A.4. $\square$

Equipped with Claim A.4 we turn back to the proof of the lemma. Again, let $P = x_0 x_1 \dots x_h$ be a longest path in $D \cup \{e\}$ containing e, and let R, R^-, R^+ be as in Claim A.4.

Note that $|R^-| \leq |R|$ and $|R^+| \leq |R| - 1$, since $x_h \in R$ has no following vertex on P, and thus does not contribute an element to R^+. According to Claim A.4, we have

$$|N_D(R) \setminus R| \leq |R^- \cup R^+ \cup e| \leq 2|R| + 1,$$

and it follows that $|R| > k$. We claim that, for each $v \in R$, the pair x_0v is an e-booster for D. To prove this claim, fix $v \in R$, and let $Q \in \mathcal{P}$ be a path ending at v. Note that by adding x_0v to Q, we turn Q into a cycle C containing e. This cycle is either Hamiltonian or $V(Q) \neq V(D)$. The former case would immediately imply that x_0v is an e-booster for D. Thus we may assume that $V(C) = V(Q) \neq V(D)$. Since D is connected, there exists an edge $yz \in E(D)$ connecting $y \in V(C)$ to $z \notin V(C)$. We can use the edge yz to obtain a path P' that contains e of length $h+1$ in the following way. In C there are two edges incident to y, and at least one of them is not e. By removing that edge from C and adding the edge yz, we obtain such path P' of length $h+1$. On the other hand, because we assumed that P was the longest path in $D \cup \{e\}$ containing e, we must conclude that x_0v is an e-booster for D, thereby proving our claim.

To finish the proof of the lemma, fix a subset $\{y_1, \ldots, y_{k+1}\}$ of R. For every y_i, there exists a path P_i ending at y_i, that can be obtained from P by a sequence of elementary rotations. Now fix y_i as the starting point of P_i and let Y_i be the set of other endpoints of all paths obtained from P_i by a sequence of elementary rotations with fixed y_i. As before, $|Y_i| \geq k+1$, and all edges connecting y_i to a vertex in Y_i are e-boosters for D. Altogether we have found $(k+1)^2$ pairs y_iz_{ij} for $z_{ij} \in Y_i$. As every booster is counted at most twice, the conclusion of the lemma follows. $\square$