

# The game of JumbleG

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## Abstract

JumbleG is a Maker-Breaker game. Maker and Breaker take turns in choosing edges from the complete graph  $K_n$ . Maker's aim is to choose what we call an  $\epsilon$ -regular graph (that is, the minimum degree is at least  $(\frac{1}{2} - \epsilon)n$  and, for every pair of disjoint subsets  $S, T \subset V$  of cardinalities at least  $\epsilon n$ , the number of edges  $e(S, T)$  between  $S$  and  $T$  satisfies:  $\left| \frac{e(S, T)}{|S||T|} - \frac{1}{2} \right| \leq \epsilon$ .) In this paper we show that Maker can create an  $\epsilon$ -regular graph, for  $\epsilon \geq 2(\log n/n)^{1/3}$ . We consider also a similar game, JumbleG2, where Maker's aim is to create a graph with minimum degree at least  $(\frac{1}{2} - \epsilon)n$  and maximum co-degree at most  $(\frac{1}{4} + \epsilon)n$ , and show that Maker has a winning strategy for  $\epsilon > 3(\log n/n)^{1/2}$ . Thus, in both games Maker can create a pseudo-random graph of density  $\frac{1}{2}$ . This guarantees Maker's win in several other positional games, also discussed here.

## 1 Introduction

JumbleG is a Maker-Breaker game. Maker and Breaker take turns in choosing edges from the complete graph  $K_n$  on  $n$  vertices. Maker's aim is to choose a graph which is  $\epsilon$ -regular (the definition follows).

Let  $G = (V, E)$  be a graph of order  $n$ . We usually assume that the vertex set is  $[n] =$

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$\{1, \dots, n\}$ . We call a pair  $S, T$  of non-empty disjoint subsets of  $[n]$   $\epsilon$ -unbiased if

$$\left| \frac{e_G(S, T)}{|S||T|} - \frac{1}{2} \right| \leq \epsilon, \tag{1}$$

where  $e_G(S, T)$  is the number of  $S - T$  edges in  $G$ . The graph  $G$  is  $\epsilon$ -regular if

**P1:**  $\delta(G) \geq (\frac{1}{2} - \epsilon)n$ .

**P2:** Any pair  $S, T$  of disjoint subsets of  $[n]$  with  $|S|, |T| \geq \epsilon n$  is  $\epsilon$ -unbiased.

**Theorem 1** *Maker has a winning strategy in JumbleG provided  $\epsilon \geq 2(\log n/n)^{1/3}$  and  $n$  is sufficiently large.*

We consider also a similar game, which we denote by JumbleG2. In this game Maker's aim is to create a graph with Properties P1 and P3, where

**P3:** Maximum co-degree is at most  $(\frac{1}{4} + \epsilon)n$ .

(The co-degree of vertices  $u, v \in V(G)$  is the number of common neighbours of  $u$  and  $v$  in  $G$ .)

Here, too, Maker can win provided  $\epsilon$  is not too small:

**Theorem 2** *Maker has a winning strategy in JumbleG2 for all  $\epsilon \geq 3(\log n/n)^{1/2}$  if  $n$  is sufficiently large.*

Theorems 1 and 2 are proved in Section 2. As shown in Section 3, our restrictions on  $\epsilon$  are best possible, up to a logarithmic factor.

Although the goals of the above two games appear to be quite different, they are in fact very similar to each other — in both Maker tries to create a *pseudo-random graph* of density around  $\frac{1}{2}$ . Informally speaking, a pseudo-random graph  $G = (V, E)$  is a graph on  $n$  vertices whose edge distribution resembles that of a truly random graph  $G(n, p)$  of the same edge density  $p = e(G) \binom{n}{2}^{-1}$ . The reader can consult [12] for a recent survey on pseudo-random graphs. The fact that an  $\epsilon$ -regular graph is pseudo-random with density  $\frac{1}{2}$  is apparent from the definition. To see that degrees and co-degrees can guarantee pseudo-randomness we need to recall some notions and results due to Thomason. He introduced the notion of *jumbled* graphs [17]. A graph  $G$  with vertex set  $[n]$  is  $(\alpha, \beta)$ -jumbled if for every  $S \subseteq [n]$  we have

$$\left| e_G(S) - \alpha \binom{|S|}{2} \right| \leq \beta |S|$$

where  $e_G(S)$  is the number of edges of  $G$  contained in  $S$ .

Thomason showed that one can check for pseudo-randomness via jumbledness by checking degrees and co-degrees. Suppose that  $G = (V, E)$  has minimum degree at least  $\alpha n$  and no two vertices have more than  $\alpha^2 n + \mu$  common neighbours. Then, (see Theorem 1.1 of [17] and its proof) for every  $s \leq n$ , every set  $S \subseteq V$  of size  $|S| = s$  satisfies:

$$\left| e(S) - \alpha \binom{s}{2} \right| \leq \frac{((s-1)\mu + \alpha n)^{1/2} + \alpha}{2} s, \quad (2)$$

and therefore  $G$  is  $(\alpha, \beta)$ -jumbled with  $\beta = ((\alpha n + (n-1)\mu)^{1/2} + \alpha)/2$ .

Now suppose that for some  $\epsilon = \Omega(1/n)$  a graph  $G$  on  $n$  vertices has minimum degree at least  $\alpha n = (\frac{1}{2} - \epsilon)n$  and maximum co-degree at most  $(\frac{1}{4} + \epsilon)n = \alpha^2 n + (2\epsilon - \epsilon^2)n$ . Then a routine calculation, based on (2), shows that  $G$  is  $\epsilon'$ -regular for  $\epsilon' = \Omega(\epsilon^{1/4})$ . Thus Theorem 2 can be used to show that Maker can create an  $\epsilon$ -regular graph with  $\epsilon = n^{-1/8+o(1)}$  — a weaker result than the one provided by the direct application of Theorem 1. Indeed, let  $|S| = s$ ,  $|T| = t \geq \epsilon'n$ ,  $\mu = (2\epsilon - \epsilon^2)n$ , and  $\epsilon' \geq \Omega(\epsilon^{1/4}) \geq \Omega(n^{-1/4})$ . Then

$$\begin{aligned} \left| \frac{e_G(S, T)}{st} - \frac{1}{2} \right| &= \frac{1}{st} |e_G(S \cup T) - e_G(S) - e_G(T) - (\alpha + \epsilon)st| \\ &\leq \frac{1}{st} \left( \left| e_G(S \cup T) - \alpha \binom{s+t}{2} \right| + \left| e_G(S) - \alpha \binom{s}{2} \right| + \left| e_G(T) - \alpha \binom{t}{2} \right| \right) + \epsilon \\ &\leq \frac{((s+t)\mu + \alpha n)^{1/2} + \alpha}{2st} (s+t) + \frac{(s\mu + \alpha n)^{1/2} + \alpha}{2t} + \frac{(t\mu + \alpha n)^{1/2} + \alpha}{2s} + \epsilon \\ &\leq \frac{(s+t)^{3/2} \mu^{1/2}}{2st} + \frac{(s\mu)^{1/2}}{2t} + \frac{(t\mu)^{1/2}}{2s} + \frac{4((\alpha n)^{1/2} + \alpha)}{2 \min\{s, t\}} + \epsilon \\ &\leq \frac{((1+\epsilon')n)^{1/2} (2\epsilon n)^{1/2}}{\epsilon'n} + \frac{(2n\epsilon n)^{1/2}}{2\epsilon'n} + \frac{(2n\epsilon n)^{1/2}}{2\epsilon'n} + \frac{2n^{1/2}}{\epsilon'n} + \epsilon \\ &\leq c \frac{\epsilon^{1/2}}{\epsilon'} \\ &\leq \epsilon'. \end{aligned}$$

Pseudo-random graphs are known to have many nice properties. Hence, Maker's ability to create a pseudo-random graph guarantees his win in several other positional games. For example, using a result of [11], one can guarantee Maker's success in creating  $\frac{n}{4} - O(n^{5/6} \log^{1/6} n)$  pairwise edge-disjoint Hamiltonian cycles. This is trivially best possible up to the error-term and confirms a conjecture of Lu [13] in a strong form. We will discuss this and other games in Section 4.

## 2 Playing JumbleG

In this section we prove Theorems 1 and 2. The proofs are quite similar and are based on the approach of Erdős and Selfridge [9] via potential functions.

**Lemma 3** *If the edges of a hypergraph  $\mathcal{F}$  satisfy  $\sum_{X \in \mathcal{F}} 2^{-|X|} < 1/4$  then Maker can force a 2-colouring of  $\mathcal{F}$ .*

**Proof.** Let a round consist of a move of Maker followed by a move of Breaker. At the start of a round, let  $C_M, C_B$  denote the set of edges chosen so far by Maker and Breaker,  $R$  denote the unchosen edges and for  $X \in \mathcal{F}$  let  $\delta_{X,M}, \delta_{X,B}$  be the indicators of  $X \cap C_M \neq \emptyset, X \cap C_B \neq \emptyset$  respectively. Let  $\delta_X = \delta_{X,M} + \delta_{X,B}$ . We use the potential function

$$\Phi = \sum_{\substack{X \in \mathcal{F} \\ \delta_X \leq 1}} 2^{-|X \cap R| + 1 - \delta_X}.$$

This represents the expected number of monochromatic sets if the unchosen edges are coloured at random. Our assumption is that  $\Phi < \frac{1}{2}$  at the start and we will see that it can be kept this way until the end of the last complete round. In case  $n$  is odd, Maker with his last choice can at most double the value of  $\Phi$ . In any case at the end of the play  $\Phi < 1$ . Also, at the end  $R = \emptyset$ ; thus  $\delta_X \geq 2$  for all  $X \in \mathcal{F}$ , showing that Maker has achieved his objective.

It remains to show that Maker can ensure that the value of  $\Phi$  never increases after one complete round is played. Suppose that in some round, Maker chooses an edge  $a$  and Breaker chooses an edge  $b$ . Let  $\Phi'$  be the new value of  $\Phi$ . Then

$$\begin{aligned} \Phi' - \Phi &= - \sum_{\substack{a, b \in X \\ \delta_X = 0}} 2^{1 - |X \cap R|} - \sum_{\substack{a \in X \\ \delta_{X,B} = 1}} 2^{-|X \cap R|} - \sum_{\substack{b \in X \\ \delta_{X,M} = 1}} 2^{-|X \cap R|} + \sum_{\substack{a \in X, b \notin X \\ \delta_{X,M} = 1}} 2^{-|X \cap R|} + \sum_{\substack{a \notin X, b \in X \\ \delta_{X,B} = 1}} 2^{-|X \cap R|} \\ &\leq - \left( \sum_{\substack{a \in X \\ \delta_{X,B} = 1}} 2^{-|X \cap R|} - \sum_{\substack{a \in X \\ \delta_{X,M} = 1}} 2^{-|X \cap R|} \right) + \left( \sum_{\substack{b \in X \\ \delta_{X,B} = 1}} 2^{-|X \cap R|} - \sum_{\substack{b \in X \\ \delta_{X,M} = 1}} 2^{-|X \cap R|} \right) \end{aligned}$$

which is non-positive if Maker chooses  $a$  to maximise  $\sum_{\substack{a \in X \\ \delta_{X,B} = 1}} 2^{-|X \cap R|} - \sum_{\substack{a \in X \\ \delta_{X,M} = 1}} 2^{-|X \cap R|}$ .  $\square$

**Lemma 4** *Let  $\epsilon = \epsilon(n)$  tend to zero with  $n$ . Let  $\delta > 1$  be fixed. Let  $t = \lceil \delta \epsilon^{-2} \log n \rceil$ . Then for all sufficiently large  $n$  Maker can ensure that any pair of disjoint subsets of  $V$ , both of size at least  $t$ , is  $\epsilon$ -unbiased.*

**Proof.** Assume that  $t \leq n/2$  for otherwise there is nothing to prove. This means that  $\epsilon > \left(\frac{2 \log n}{n}\right)^{1/2}$ .

Let  $k = \lceil (\frac{1}{2} + \epsilon)t^2 \rceil$ . Let  $\mathcal{T}$  consist of pairs  $(S, T)$  of disjoint subsets of  $V$ , both of size *exactly*  $t$ . Recall that  $e_M(S, T)$  counts the number of Maker's edges connecting  $S$  to  $T$ . A simple averaging argument shows that it is enough to show that Maker can guarantee that

$$t^2 - k < e_M(S, T) < k, \quad \text{for all } (S, T) \in \mathcal{T}. \quad (3)$$

(Indeed, let  $S', T'$  have size at least  $t$  each. The expectation of  $\frac{e_M(S, T)}{t^2}$ , where  $S, T$  are random  $t$ -subsets of  $S', T'$ , is  $\frac{e_M(S', T')}{|S'| |T'|}$ . By (3) this cannot differ from  $\frac{1}{2}$  by more than  $\epsilon$ , as required.)

If Maker is able to ensure that all  $k$ -element subsets of the edge-set  $S : T = \{\{x, y\} \mid x \in S, y \in T\}$  are properly 2-colored (i.e. not monochromatic) for every  $(S, T) \in \mathcal{T}$ , then he has achieved his goal. A direct application of Lemma 3 is not possible however: there are simply too many of these  $k$ -sets and the criterion does not hold. We need to cut down on the number of sets.

Define  $\ell = \lceil 2t^2\epsilon \rceil$  and  $\lambda = \lceil 2^\ell n^{-2t} \rceil$ . For  $(S, T) \in \mathcal{T}$  we prove the existence of a collection  $\mathcal{X}_{S, T}$ , of  $\ell$ -subsets of the edge-set  $S : T = \{\{x, y\} \mid x \in S, y \in T\}$  such that (i)  $|\mathcal{X}_{S, T}| = \lambda$  and (ii) each  $k$ -set  $B \subseteq S : T$  contains at least one member of  $\mathcal{X}_{S, T}$ . Let us show that if the elements of  $\mathcal{X}_{S, T}$  are chosen at random, independently with replacement, then this property is almost surely satisfied. In estimating this probability we will use the following auxiliary inequalities:  $\ell = o(t^2)$  and

$$\begin{aligned} \frac{\binom{k}{\ell}}{\binom{t^2}{\ell}} &= \prod_{i=0}^{\ell-1} \frac{k-i}{t^2-i} = \left(\frac{k}{t^2}\right)^\ell \prod_{i=0}^{\ell-1} \left(1 - \frac{i(t^2-k)}{t^2 k - ki}\right) \\ &\geq \left(\frac{1}{2} + \epsilon\right)^\ell \exp\left\{-\frac{\ell^2}{2t^2} + O(\epsilon^2 \ell)\right\}. \end{aligned}$$

The probability that there is a  $k$ -subset of  $S : T$  which does not contain a member of  $\mathcal{X}_{S, T}$  is at most

$$\begin{aligned} \binom{t^2}{k} \left(1 - \frac{\binom{k}{\ell}}{\binom{t^2}{\ell}}\right)^\lambda &\leq 2^{t^2} \exp\left\{-\lambda \left(\frac{1}{2} + \epsilon\right)^\ell e^{-\frac{\ell^2}{2t^2} + O(\epsilon^2 \ell)}\right\} \\ &= 2^{t^2} \exp\left\{-n^{-2t} e^{2\epsilon \ell - \frac{\ell^2}{2t^2} + O(\epsilon^2 \ell)}\right\} = 2^{t^2} \exp\left\{-e^{-2t \log n + (2+o(1))\epsilon^2 t^2}\right\} = o(1), \end{aligned}$$

so a family  $\mathcal{X}_{S, T}$  with the required property does exist.

Let  $\mathcal{F} = (\binom{[n]}{2}, \mathcal{E})$  be the hypergraph with hyper-edges  $\mathcal{E} = \bigcup_{(S, T) \in \mathcal{T}} \mathcal{X}_{S, T}$ . (We will use the term *hyper-edges* to distinguish them from the edges of  $K_n$ ). To complete the proof it is

enough to show that Maker can ensure that the choices  $E_M, E_B \subset \binom{[n]}{2}$  of Maker, Breaker respectively are a 2-colouring of  $\mathcal{F}$ . This follows from Lemma 3 in view of the inequality

$$|\mathcal{E}| 2^{-\ell} \leq \binom{n}{t}^2 \lambda 2^{-\ell} = o(1). \quad (4)$$

□

**Proof of Theorem 1.** To ensure that all degrees of Maker's graph are appropriate we use a trick similar to the one in the proof of the previous lemma. Let  $k = \lceil (1/2 + \epsilon)n \rceil$ . Maker again would like to use Lemma 3 and ensure that all  $k$ -subsets of the edges incident with vertex  $i$  are properly 2-colored. These are again too many; we define  $\ell = \lceil 10\epsilon^{-1} \log n \rceil$ ,  $M = \lceil 2^\ell/n^2 \rceil$ , and  $\mu = nM$ . We want to find a collection  $A_1, A_2, \dots, A_\mu$  of  $\ell$ -sets such that, for  $1 \leq i \leq n$ , every  $k$ -subset of the edges incident with  $i$  contains at least one of  $A_{(i-1)M+j}$ ,  $1 \leq j \leq M$ . As before, we construct the sets  $A_i$  randomly. The probability that there is a bad  $k$ -subset (containing no chosen  $\ell$ -set) is at most

$$\binom{n-1}{k} \left(1 - \frac{\binom{k}{\ell}}{\binom{n-1}{\ell}}\right)^M \leq 2^n \exp \left\{ -\frac{2^\ell k^\ell}{n^2 n^\ell} e^{-\ell^2/n} \right\} \leq 2^n \exp \left\{ -((1+\epsilon)e^{-\ell/n})^\ell \right\} < n^{-2}$$

for large  $n$ , and so the desired sets exist.

For Property **P2** let  $t = \lceil 6\epsilon^{-2} \log n \rceil$ . By our assumption on  $\epsilon$ , we have  $t < \epsilon n$ . Define  $\mathcal{X}_{S,T}$  as in the proof of Lemma 4. Namely, let  $\ell' = \lceil 2t^2\epsilon \rceil$  and  $\lambda = \lceil 2^{\ell'} n^{-2t} \rceil$ . For  $(S, T) \in \mathcal{T}$  (that is,  $S, T$  are disjoint  $t$ -sets) let  $\mathcal{X}_{S,T}$  be a collection  $\ell'$ -subsets of  $S : T$  such that (i)  $|\mathcal{X}_{S,T}| = \lambda$  and (ii) every  $\lceil (\frac{1}{2} + \epsilon)t^2 \rceil$ -set contains at least one member of  $\mathcal{X}_{S,T}$ .

Let  $\mathcal{F}$  be the hypergraph with the edge set  $\mathcal{E}_1 \cup \mathcal{E}_2 = \{A_1, A_2, \dots, A_\mu\} \cup \bigcup_{(S,T) \in \mathcal{T}} \mathcal{X}_{S,T}$ .

Lemma 4 (or rather its proof) implies that it suffices for Maker to force a 2-colouring of  $\mathcal{F}$ . Indeed, the definition of the sets  $A_i$  will imply Property P1. To see that P2 will also hold, observe that for any  $S, T \in \mathcal{T}$ , we will have

$$\left| \frac{e_M(S, T)}{t^2} - \frac{1}{2} \right| \leq \epsilon,$$

while the claim for general  $|S|, |T| \geq t$  follows by averaging.

It remains to check that  $\mathcal{F}$  satisfies the conditions of Lemma 3 for large  $n$ . The initial value  $\Phi$  of the potential function satisfies

$$\Phi \leq Mn2^{-\ell} + \Phi(\mathcal{E}_2) = o(1). \quad (5)$$

(Here we have used (4).) This completes the proof of Theorem 1. □

**Proof of Theorem 2.** This time for Property P1 we define  $\ell = \lfloor \epsilon n \rfloor$ , as before  $M = \lceil 2^\ell/n^2 \rceil$ ,  $\mu = nM$ ,  $k = \lceil (1/2 + \epsilon)n \rceil$ . The family  $A_1, A_2, \dots, A_\mu$  should satisfy: For  $1 \leq i \leq$

$n$ , every  $k$ -subset of the edges incident with  $i$  contains at least one of  $A_{(i-1)M+j}$ ,  $1 \leq j \leq M$ . We construct the  $A_i$  randomly. Suppose that we randomly choose  $M$   $\ell$ -subsets of  $[n-1]$  independently with replacement. The probability that there is a  $k$ -subset of  $[n-1]$  which contains no chosen  $\ell$ -set is at most

$$\begin{aligned}
& \binom{n-1}{k} \left(1 - \frac{\binom{k}{\ell}}{\binom{n-1}{\ell}}\right)^M \\
& \leq 2^n \exp \left\{ -\frac{\binom{k}{\ell} M}{\binom{n}{\ell}} \right\} \\
& = 2^n \exp \left\{ -M \frac{k \cdots (k - \lfloor \ell/2 \rfloor + 1)}{n \cdots (n - \lfloor \ell/2 \rfloor + 1)} \cdot \frac{(k - \lfloor \ell/2 \rfloor) \cdots (k - \ell + 1)}{(n - \lfloor \ell/2 \rfloor) \cdots (n - \ell + 1)} \right\} \\
& \leq 2^n \exp \left\{ -M \left(\frac{k - \ell/2}{n}\right)^{\lfloor \ell/2 \rfloor} \cdot \left(\frac{k - \ell}{n}\right)^{\lceil \ell/2 \rceil} \right\} \\
& \leq \exp \left\{ n \log 2 - \frac{2^\ell}{n^2} \left(\frac{1}{2} + \frac{\epsilon}{2}\right)^{\lfloor \ell/2 \rfloor} \left(\frac{1}{2}\right)^{\lceil \ell/2 \rceil} \right\} \\
& = \exp \left\{ n \log 2 - \frac{(1 + \epsilon)^{\lfloor \ell/2 \rfloor}}{n^2} \right\} \\
& < n^{-2}
\end{aligned}$$

for  $\epsilon \geq 3(\log n/n)^{1/2}$ , so the required family exists.

For Property P3 we take a collection  $B_1, B_2, \dots, B_\rho$  of  $\ell$ -sets where  $\rho = \binom{n}{2} N$  and  $N = \lceil 4^\ell/n^3 \rceil$ . For each pair  $i, j \in [n]$  select  $N$  random  $\ell$ -subsets of  $[n] \setminus \{i, j\}$  so that each  $\lceil (1/4 + \epsilon)n \rceil$ -set contains at least one of them. The hyper-edges are  $\{(i, x) : x \in A\} \cup \{(j, x) : x \in A\}$  for each random  $A \subseteq [n] \setminus \{i, j\}$ .  $B_1, B_2, \dots, B_\rho$  are chosen randomly and now with  $k = \lceil (1/4 + \epsilon)n \rceil$  the probability that there is a  $k$ -subset of  $[n-2]$  which contains no chosen  $\ell$ -set is at most

$$\binom{n-2}{k} \left(1 - \frac{\binom{k}{\ell}}{\binom{n-2}{\ell}}\right)^N \leq \exp \left\{ n \log 2 - \frac{(1 + 2\epsilon)^{\lfloor \ell/2 \rfloor}}{n^3} \right\} < n^{-3}$$

for large  $n$ , and so the sets exist.

We will use Lemma 3 and so we need to check that the initial potential is less than  $1/4$ . Now the initial value of the potential function is at most

$$Mn2^{1-\ell} + Nn^22^{1-2\ell} = o(1)$$

and this completes the proof of Theorem 2.  $\square$

### 3 Breaker's Strategies

In this section we show that up to a small power of  $\log n$ , our restrictions on  $\epsilon$  are sharp in both Theorems 1 and 2 or, even more strongly, with respect to each of the Properties P1–P3.

#### Property P1

Theorem 2 gives immediately that Maker can guarantee a graph with minimum degree at least  $n/2 - 3\sqrt{n \log n}$ . A similar result has been previously obtained by Székely [16] by applying a lemma of Beck [2, Lemma 3] which in turn is based on the Erdős–Selfridge method. This comes quite close to a result of Beck [3] who proved that Breaker can force the minimum degree of Maker's graph to be  $n/2 - \Omega(\sqrt{n})$ .

#### Property P2

Let  $c > 0$  be any constant which is less than  $6^{-1/3}$ ,  $n$  be large, and  $\epsilon = cn^{-1/3} \log^{1/3} n$ .

Here we prove that *no* graph of order  $n$  can satisfy Property P2 for this  $\epsilon$ , which shows that the restriction on  $\epsilon$  in Theorem 1 is sharp up to a multiplicative constant. The proof is based on ideas of Erdős and Spencer [10].

Let  $G$  be an arbitrary graph of order  $n$ . Let  $m = \lceil \epsilon n \rceil$ . Let  $X$  be a random  $m$ -subset of  $V(G)$  chosen uniformly. For  $y \in V(G)$ , let  $\mathcal{E}_y$  be the event that  $y \notin X$  and  $||\Gamma(y) \cap X| - m/2| > \epsilon m$ , where  $\Gamma(y)$  denoted the set of neighbours of  $y$  in  $G$ .

Let us show that for every  $y$ ,

$$\Pr(\mathcal{E}_y) \geq \frac{2m}{n}. \tag{6}$$

Let  $d = d(y)$  be the degree of  $y$ . By symmetry, we can assume that  $d \leq \frac{n-1}{2}$ . For such  $d$  we bound from below the probability  $p$  that  $y \notin X$  and  $|\Gamma(y) \cap X| \leq m/2 - \epsilon m$ , which equals

$$p = \sum_{i < m/2 - \epsilon m} \binom{d}{i} \binom{n-1-d}{m-i} \binom{n}{m}^{-1}.$$

The combinatorial meaning of  $p$  implies that it decreases with  $d$ , so it is enough to bound  $p$  for  $d = \lfloor \frac{n-1}{2} \rfloor$  only. Let us consider the summands  $s_h$  corresponding to  $i = m/2 - h$  with, say,  $\epsilon m < h \leq \epsilon m + n^{1/3}$ . Let

$$f(x) = (1+x)^{\frac{1+x}{2}} (1-x)^{\frac{1-x}{2}}.$$



Its Taylor series at 0 is  $1 + \frac{x^2}{2} + O(x^4)$ . By Stirling's formula, we obtain that each summand

$$\begin{aligned} s_h &= \Omega \left( \frac{n^{-1/3} (\log n)^{1/6}}{f^m \left(\frac{2h}{m}\right) f^{2d-m} \left(\frac{2h}{2d-m}\right)} \right) \\ &= \exp \left( -\frac{1}{3} \log n - \frac{2h^2}{m} - \frac{2h^2}{2d-m} + O(1) \right) \\ &= n^{-1/3-2c^3-o(1)}. \end{aligned}$$

Thus

$$\sum_{h=\epsilon m}^{\epsilon m+n^{1/3}} s_h = n^{-2c^3-o(1)} \geq \frac{2m}{n}.$$

It follows that there is a choice of an  $m$ -set  $X$  such that  $|Y| \geq 2m$ , where  $Y$  consists of the vertices for which  $R_x$  holds. By definition  $Y \cap X = \emptyset$ .

Assume without loss of generality that we have  $d_X(y) < m - \epsilon m$  for at least half of the vertices of  $Y$ . Let  $Z \subset Y$  consist of any  $m$  of these vertices. This pair  $(X, Z)$ , both sets having at least  $\epsilon n$  elements, has the required bias.

### Property P3

Here we show that Breaker can force Maker to create a co-degree of at least  $\frac{n}{4} + c\sqrt{n}$ . Our argument is based on a theorem of Beck [5], which states that Breaker can force Maker's graph to have maximum degree at least  $n/2 + \sqrt{n}/20$ . Then the following lemma shows that Breaker also succeeds in forcing a high co-degree in Maker's graph.

**Lemma 5** *Assume that  $c_1 > 0$  is constant. Then for sufficiently large  $n$ , the following holds: Let  $G = (V, E)$  be a graph on  $n$  vertices with  $n(n-1)/4$  edges. If  $G$  has a vertex of degree at least  $n/2 + c_1\sqrt{n}$ , then  $G$  has a pair of vertices  $w_1, w_2$  whose co-degree is at least  $n/4 + c_1\sqrt{n}/10$ .*

**Proof** Let  $c_2 = c_1/10$ . Let  $v$  be a vertex of maximum degree in  $G$ . Denote  $N_1 = N(v)$ ,  $N_2 = V - N_1$ . Then  $|N_2| \leq n/2 - c_1\sqrt{n}$ . If there is  $u \in V$  such that  $d(v, N_1) \geq n/4 + c_2\sqrt{n}$ , we are done. Otherwise, for every  $u$ ,  $d(u, N_1) \leq n/4 + c_2\sqrt{n}$ , implying:

$$\begin{aligned} A &\stackrel{def}{=} \sum_{u \in V} d(u, N_2) \\ &\geq \sum_{u \in V} (d(u) - d(u, N_1) - 1) \\ &\geq 2|E| - n(n/4 + c_2\sqrt{n}) - n \\ &= n^2/4 - c_2n^{3/2} - 3n/2. \end{aligned}$$

Therefore by convexity,

$$B \stackrel{\text{def}}{=} \sum_{u \in V} \binom{d(u, N_2)}{2} \geq n \binom{A/n}{2} \geq n^3/32 - c_2 n^{5/2} - O(n^2).$$

On the other hand,

$$B = \sum_{w_1 \neq w_2 \in N_2} \text{co-degree}(w_1, w_2),$$

and thus there is a pair  $w_1, w_2 \in N_2$  such that:

$$\begin{aligned} \text{co-degree}(w_1, w_2) &\geq |B| / \binom{|N_2|}{2} \\ &\geq \frac{n^3/32 - c_2 n^{5/2} - O(n^2)}{\binom{n/2 - c_1 n^{1/2}}{2}} \\ &\geq n/4 + c_2 \sqrt{n}. \end{aligned}$$

□

## 4 Consequences

As we have already mentioned in the introduction, Maker's ability to create a pseudo-random graph of density about  $\frac{1}{2}$  allows him to win quite a few other combinatorial games. We will describe some of them below. All these games are played on the complete graph  $K_n$  unless stated otherwise, Maker and Breaker choose one edge alternately, Maker's aim is to create a graph that possesses a desired graph property.

**Edge-disjoint Hamilton cycles.** In this game Maker's aim is to create as many pairwise edge disjoint Hamilton cycles as possible. Lu proved [13] that Maker can always produce at least  $\frac{1}{16}n$  Hamilton cycles and conjectured that Maker should be able to make  $(\frac{1}{4} - \epsilon)n$  for any fixed  $\epsilon > 0$ . This conjecture follows immediately from our Theorem 1 and Theorem 2 of [11]. In [11], Frieze and Krivelevich show that a  $2\epsilon$ -regular graph contains at least  $(\frac{1}{2} - 6.5\epsilon)n$  edge disjoint Hamilton cycles, for all  $\epsilon > 10(\log n/n)^{1/6}$ . Our argument applies equally to the bipartite version of the problem where the game is played on the complete bipartite graph  $K_{n,n}$ . Thus Maker can always produce at least  $(\frac{1}{4} - \epsilon)n$  edge disjoint Hamilton cycles, verifying another conjecture of Lu [14], [15]. Finally, there is an analogous game that can be played on the complete digraph  $D_n$  and here Maker can always produce at least  $(\frac{1}{2} - \epsilon)n$  edge disjoint Hamilton cycles.

**Vertex-connectivity.** Theorem 2 can be used to show that Maker can always force an  $(n/2 - 3\sqrt{n \log n})$ -vertex-connected graph. Indeed, let Maker's graph  $M$  have minimum degree at least  $n/2 - 3\sqrt{n \log n}$  and maximum co-degree at most  $n/4 + 3\sqrt{n \log n}$ . Suppose

that the removal of some set  $R$  disconnects  $M$ , say  $V(M) \setminus R = A \cup B$  with  $|A| \leq |B|$ . If  $|A| = 1$ , then obviously all neighbours of  $a \in A$  are in  $R$ , implying  $|R| \geq \delta(M) \geq n/2 - 3\sqrt{n \log n}$ . If  $|A| \geq 2$ , let  $a_1, a_2$  be two distinct vertices in  $A$ . Then all neighbours of  $a_1, a_2$  lie in  $A \cup R$ , and therefore

$$|A| + |R| \geq \deg_M(a_1) + \deg_M(a_2) - \text{co-deg}_M(a_1, a_2) \geq \frac{3n}{4} - 9\sqrt{n \log n}.$$

If  $|A| \geq n/4 - 6\sqrt{n \log n}$ , then  $|B| \geq |A| \geq n/4 - 6\sqrt{n \log n}$  as well, and by the  $o(1)$ -regularity of  $M$  there is an edge between  $A$  and  $B$  – a contradiction. We conclude that  $|A| \leq n/4 - 6\sqrt{n \log n}$ , implying  $|R| \geq n/2 - 3\sqrt{n \log n}$ , as required.

The result of Beck [3] showing that Breaker can force a vertex which has degree at most  $n/2 - \Omega(\sqrt{n})$  in Maker's graph indicates that the error term in our result about the connectivity game is tight up to a logarithmic factor.

**$c \log n$ -Universality.** A graph  $G$  is called  $r$ -universal if it contains an induced copy of every graph  $H$  on  $r$  vertices. We can show the following result.

**Theorem 6** *Let  $r = r(n)$  be an integer, which satisfies*

$$\frac{n-r+1}{r} \left( \frac{1}{2} - \epsilon \right)^{r-1} \geq \frac{2 \log n}{\epsilon^2},$$

*for some  $\epsilon = \epsilon(n) \rightarrow 0$ . Then for all sufficiently large  $n$  Maker can ensure that his graph  $M$  is  $r$ -universal.*

**Proof.** Let  $t = \lfloor \frac{2 \log n}{\epsilon^2} \rfloor$ . Let  $n$  be sufficiently large so that the conclusion of Lemma 4 is valid. Let  $M$  be an arbitrary graph satisfying this property, that is, any pair of disjoint subsets of  $V(M)$ , both of size at least  $t$ , is  $\epsilon$ -unbiased. Let  $G$  be any graph on  $[r]$ . We will show that  $G$  is an induced subgraph of  $M$ .

Partition  $V(M) = \cup_{i=1}^r V_i$  into  $r$  parts, each having at least  $\frac{n-r+1}{r}$  vertices. Initially, let  $A_i = V_i$ ,  $i \in [r]$ . We define  $f : [r] \rightarrow V(M)$  with  $f(i) \in A_i$  inductively.

Suppose we have already defined  $f$  on  $[i-1]$ . It will be the case that  $|A_j| \geq \frac{n-r+1}{r} \eta^{i-1}$  for any  $j \geq i$ , where for brevity  $\eta = \frac{1}{2} - \epsilon$ . We will choose  $f(i) = v \in A_i$  so that for any  $j > i$  we have

$$|A_{ji}(v)| \geq \eta |A_j|, \tag{7}$$

where we define  $A_{ji}(v) = A_j \cap \Gamma_M(v)$  if  $\{i, j\} \in E(G)$  and  $A_{ji}(v) = A_j \setminus \Gamma_M(v)$  otherwise. (Here  $\Gamma_M(v)$  is the set of neighbours of  $v$  in  $M$ .)

Let  $B_{ji}$  be the set of vertices of  $A_i$  violating (7), i.e.  $\{v \in A_i : |A_{ji}(v)| < \eta |A_j|\}$ . Then  $|B_{ji}| < t$  as the pair  $(B_{ji}, A_j)$  is not  $\epsilon$ -unbiased. (Observe that  $|A_j| \geq \frac{n-r+1}{r} \eta^{r-1} \geq t$ .) Update  $A_i$  by deleting  $B_{ji}$  for all  $j \in [i+1, r]$ . Thus at least  $\frac{n-r+1}{r} \eta^{i-1} - (r-i)t \geq t$

vertices still remain in  $A_i$ . This inequality is true for  $i = r$  by our assumption and for any other  $i$ , because  $\eta \leq \frac{1}{2}$ . So a suitable  $f(i)$  can always be found. Now, replace  $A_j$  with  $A_{j_i}(f(i))$  for  $j > i$ . This completes the induction step. At the end of the process  $f([r])$  induces a copy of  $G$  in  $M$ .  $\square$

It follows from Theorem 6 that Maker can create an  $r$ -universal graph with  $r = (1 + o(1)) \log_2 n$ . On the other hand, Maker cannot achieve  $r = 2 \log_2 n - 2 \log_2 \log_2 n + C$  because, as was shown by Beck [4, Theorem 4], Breaker can prevent  $K_r$  in Maker's graph.

There is a remarkable parallel between random graphs and Maker-Breaker games, see e.g. Chvátal and Erdős [8], Beck [3, 4], Bednarska and Łuczak [6]. As shown by Bollobás and Thomason [7], the largest  $r$  such that a random graph of order  $n$  is almost surely  $r$ -universal is around  $2 \log_2 n$ . We conjecture that games have the same universality threshold (asymptotically).

**Conjecture 7** *Maker can claim an  $r$ -universal graph with  $r = (2 + o(1)) \log_2 n$ .*

The following related result improves the unbiased case of Theorem 4 in Beck [3]. (His assumption  $n \geq 100r^3v3^{r+1}$  is stronger than ours.)

**Theorem 8** *Let integers  $r, v$  and a real  $\epsilon > 0$  (all may depend on  $n$ ) satisfy  $\epsilon \rightarrow 0$  and*

$$\frac{n - r + 1}{r} \left( \frac{1}{2} - \epsilon \right)^{r-1} \geq v + \frac{2 \log n}{\epsilon^2}.$$

*Then for sufficiently large  $n$ , Maker can ensure that any graph  $G$  of order at most  $v$  and maximum degree less than  $r$  is a subgraph (not necessarily induced) of Maker's graph  $M$ .*

**Outline of Proof.** Use the method of Theorem 6 with the following changes. Take a proper colouring  $c : V(G) \rightarrow [r]$ . The desired  $f$  will map  $i \in V(G)$  into  $A_{c(i)}$ . The proof goes the same way except that when choosing  $f(i)$  we have to worry only about those  $j \geq i$  which are neighbours of  $i$  in  $G$  and make sure that there are at least  $v$  good choices for  $f(i) \in A_{c(i)}$  (so that we can ensure that  $f$  is injective). The details are left to the Reader.  $\square$

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## References

- [1] N. Alon and J. Spencer, The probabilistic method, Second Edition, John Wiley and Sons, 2000.

- [2] J. Beck, Van der Waerden and Ramsey type games, *Combinatorica* **1** (1981), 103–116.
- [3] J. Beck, Deterministic graph games and a probabilistic intuition, *Combinatorics, Probability and Computing* **3** (1994), 13–26.
- [4] J. Beck, Positional games and the second moment method, *Combinatorica* **22** (2002), 169–216.
- [5] J. Beck, Arithmetic progressions and tic-tac-toe like games, (2004) manuscript.
- [6] M. Bednarska and T. Luczak, Biased positional games for which random strategies are nearly optimal, *Combinatorica* **20** (2000), 477–488.
- [7] B. Bollobás and A. Thomason, Graphs which contain all small graphs, *European Journal of Combinatorics* **2** (1981), 13–15.
- [8] V. Chvátal and P. Erdős, Biased positional games, *Ann. Discrete Math.* **2** (1978), 221–229.
- [9] P. Erdős and J.L. Selfridge, On a combinatorial game, *Journal of Combinatorial Theory A* **14** (1973) 298–301.
- [10] P. Erdős and J.H. Spencer, Imbalances in  $k$ -colorations, *Networks* **1** (1971/2) 379–385.
- [11] A.M. Frieze and M. Krivelevich, On packing Hamilton cycles in  $\epsilon$ -regular graphs, to appear.
- [12] M. Krivelevich and B. Sudakov, Pseudo-random graphs, in: Proceedings of Conference on Finite and Infinite Sets, to appear.
- [13] X. Lu, Hamiltonian games, *Journal of Combinatorial Theory B* **55** (1992) 18–32.
- [14] X. Lu, A Hamiltonian game on  $K_{n,n}$ , *Discrete Mathematics* **142** (1995) 185–191.
- [15] X. Lu, Hamiltonian cycles in bipartite graphs, *Combinatorica* **15** (1995), 247–254.
- [16] L.A. Székely, On two concepts of discrepancy in a class of combinatorial games, Finite and Infinite Sets, *Colloq. Math. Soc. János. Bolyai*, Vol. **37**, North-Holland, 1984, 679–683.
- [17] A.G. Thomason, Pseudo-random graphs, *Annals of Discrete Mathematics* **33** (1987) 307–331.