On Fractional K-Factors of Random Graphs

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Abstract

Let K be a graph on r vertices and let $G = (V, E)$ be another graph on $|V| = n$ vertices. Denote the set of all copies of K in G by $\mathcal{K}$. A non-negative real-valued function $f : \mathcal{K} \to \mathbb{R}_+$ is called a fractional K-factor if $\sum_{K:v\in K\in\mathcal{K}} f(K) \leq 1$ for every $v \in V$ and $\sum_{K\in\mathcal{K}} f(K) = n/r$. For a non-empty graph K let $d(K) = e(K)/v(K)$ and $d^{(1)}(K) = e(K)/(v(K) - 1)$. We say that K is strictly K_1-balanced if for every proper subgraph $K' \subsetneq K$, $d^{(1)}(K') < d^{(1)}(K)$. We say that K is imbalanced if it has a subgraph K' such that $d(K') > d(K)$. Considering a random graph process $\widetilde{G}$ on n vertices, we show that if K is strictly K_1-balanced then with probability tending to 1 as $n \to \infty$, at the first moment τ_0 when every vertex is covered by a copy of K, the graph $\widetilde{G}_{\tau_0}$ has a fractional K-factor. This result is best possible. As a consequence, if K is K_1-balanced, we derive the threshold probability function for a random graph to have a fractional K-factor. On the other hand, we show that if K satisfies an imbalance condition, then for asymptotically almost every graph process there is a gap between τ_0 and the appearance of a fractional K-factor. We also introduce and apply a criteria for perfect fractional matchings in hypergraphs in terms of expansion properties.

1 Introduction

A *graph property* is a set of graphs closed under graph isomorphism. The statement "G has Q" means $G \in Q$. A property Q is called *monotone increasing* if whenever $G \in Q$ and $E[G] \subset E[G']$ then also $G' \in Q$. Q is *monotone decreasing* if whenever $G \in Q$ and $E[G] \supset E[G']$ then also $G' \in Q$. Let $G = G(n, p)$ denote, as usual, the random graph with n vertices and edge probability $p = p(n)$. For a graph property Q and for a function $p = p(n)$, we say that $G(n, p)$ satisfies Q *asymptotically almost surely* (abbreviated a.a.s.) if the probability that $G(n, p(n))$ satisfies Q tends to 1 as n tends to infinity. We say that a function $p^*(n)$ is a *threshold function* for the property Q if $p(n)/p^*(n) \xrightarrow{n\to\infty} 0$ implies that a.a.s. $G(n, p(n)) \notin Q$ and $p(n)/p^*(n) \xrightarrow{n\to\infty} \infty$ implies that a.a.s.

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$G(n,p(n)) \in Q$. We will also use the notation $G(n,M)$ for the random graph with n vertices and $M = M(n)$ edges. The notion of threshold function is also defined for this model in the natural way.

Let G be a graph of order n and let K be a graph of order r. We say that G has a *K-factor* if it contains n/r vertex disjoint copies of K. Thus, for example, a K_2-factor is simply a perfect matching.

One graph parameter that is related to K-factors is the fractional arboricity, defined inter alia in [1] and [10]. We give its definition here, as this parameter appears in the next results. For a simple graph $K = (V,E)$ we define its *K_1-density*, $d^{(1)}(K)$, as $|E|/(|V|-1)$. The *fractional arboricity* of K is $m^{(1)}(K) = \max\{d^{(1)}(K')|K' \subset K, v(K') > 1\}$. If $d^{(1)}(K) = m^{(1)}(K)$ then K is said to be *K_1-balanced*, and if additionally for no $K' \subsetneq K$, $d^{(1)}(K') = d^{(1)}(K)$ then K is said to be *strictly K_1-balanced*.

The problem of determining the threshold probability for a random graph to have a K-factor is still open. The case $K = K_2$ has been solved by Erdős & Rényi [3] in 1966, but even for $K = K_3$ the threshold function is yet unknown. Alon and Yuster in [1], and independently Ruciński in [11] have shown that for graphs[1] having $m^{(1)}(K) > \delta(K)$, the threshold function for a K-factor in $G(n,p)$ is $p = n^{-1/m^{(1)}(K)}$, where $\delta(K)$ is the minimal degree of K. For triangles they have bounded the threshold from above with $p \geq c\left(\frac{\ln n}{n}\right)^{1/2}$. This result was improved later by Krivelevich [9] who showed that $p \geq cn^{-3/5}$ is already enough. Using a joint distribution technique Kim [6] proved that if $p \gg n^{-11/18}$ then a.a.s. $G(n,p)$ has a K_3-factor. The conjecture however is still open (see e.g. [6]):

Conjecture. *Let $3|n$. Then*

$$\Pr[G(n,p) \textit{ has a } K_3\textit{-factor}] \longrightarrow \begin{cases} 0 & \binom{n-1}{2}p^3 - \ln n \to -\infty, \\ e^{-e^{-c}} & \binom{n-1}{2}p^3 - \ln n \to c, \\ 1 & \binom{n-1}{2}p^3 - \ln n \to \infty. \end{cases}$$

The conjecture is, in fact, that $p = (2n^{-2}\ln n)^{1/3}$ is the threshold function. The 0 statement is not a part of the conjecture, as it is known that if $\binom{n-1}{2}p^3 - \ln n \to -\infty$ then a.a.s. there is a vertex that is not in a triangle.

One way to make progress towards proving the conjecture is to consider its fractional relaxation, that is considering the problem of finding the threshold probability for a *fractional K-factor*. Let G be a graph on the vertex set V, let K be a graph of order r and let $\mathcal{K}$ be the family of all (not necessarily induced) copies of K in G. A non-negative real valued function $f : \mathcal{K} \to \mathbb{R}_+$ is called a *fractional K-factor* if $\sum_{K \in \mathcal{K}} f(K) = |V|/r$, and also for every $v \in V$ one has $\sum_{K : v \in K \in \mathcal{K}} f(K) \leq 1$. In this version it is not necessary for r to divide n. Clearly if f takes values only from $\{0,1\}$ then the set of all positive valued copies forms a K-factor, hence the existence of a K-factor implies the

[1] In fact in [1] it is shown that $p = n^{-1/m^{(1)}(K)}$ is the threshold function for a larger family of graphs.

existence of a fractional one, but not vice versa. in Section 2 we give some additional definitions and facts about fractional K-factors.

It turns out that this fractional relaxation is much more tractable, and quite precise results can be obtained about it. To formulate these results we introduce the notion of a random graph process. A *random graph process* on a vertex set V of size n is a Markov chain $\widetilde{G} = (\widetilde{G})_0^\infty$, whose states are graphs on V. The process starts with the empty graph ($E[\widetilde{G}_0] = \emptyset$) and for every $1 \le t \le \binom{n}{2}$, the graph $\widetilde{G}_t$ is obtained from $\widetilde{G}_{t-1}$ by adding an edge from $\binom{V}{2} \setminus E[\widetilde{G}_{t-1}]$, all new edges being equiprobable. Clearly, the graph at time t has t edges, and $\widetilde{G}_{\binom{n}{2}} = K_n$. For $t > \binom{n}{2}$ we define $\widetilde{G}_t = K_n$. We turn the set of all random processes to a probability space by giving the same probability to every process. If we stop a random graph process at time M, we get a random graph with M edges with a uniform distribution over these graphs, so in fact we get $G(n, M)$. For a (nonempty) monotone increasing graph property Q, the *hitting time* of Q is defined as $t(Q, \widetilde{G}) = \min\{t | \widetilde{G}_t \in Q\}$. The phrase "for *asymptotically almost every* (abbreviated a.a.e.) graph process X" means that when n tends to infinity, $\Pr[X]$ tends to 1.

Now we have the terminology needed in order to formulate our main result.

Theorem 1.1. *Let K be a strictly K_1-balanced graph. For asymptotically almost every random graph process $\widetilde{G}$,*

$$t(G \text{ has a fractional } K\text{-factor}, \widetilde{G}) = t(\text{every vertex of } G \text{ lies in a copy of } K, \widetilde{G}).$$

In words, the theorem states that for a strictly K_1-balanced graph K, and for asymptotically almost every random graph process, at the very moment when the last vertex that was not in a copy of $K$is covered with such copy, one has a fractional K-factor. This theorem yields the following result concerning fractional K-factors in $G(n, p)$.

Corollary 1.2. *Let K be a strictly K_1-balanced graph on r vertices and with e edges. Then*

$$\Pr[G(n,p) \text{ has a fractional } K\text{-factor}] \longrightarrow \begin{cases} 0 & \text{if } cn^{r-1}p^e - \ln n \to -\infty, \\ 1 & \text{if } cn^{r-1}p^e - \ln n \to \infty, \end{cases}$$

where $c = c(K)$ is some constant depending only on K.

We will prove the theorem and the corollary in Section 5. Many of the ideas are taken from the proofs in [8].

We also use some hypergraph terminology. A *hypergraph* H is an ordered pair $H = (V, E)$, where V is a set (the *vertex set*) and $E \subseteq 2^V$ is a family of distinct subsets of V (the *edge set*). A hypergraph is said to be *r-uniform* if all the edges are of size r. Let $H = (V, E)$ be a hypergraph. The *degree*, $d(v)$, of a vertex $v \in V$ is $d(v) = |\{e \in E | v \in e\}|$. The *neighborhood*, $N(v)$, of a vertex $v \in V$ is $N(v) = \bigcup_{v \in e} e \setminus \{v\}$. A *path of length l* in H is a sequence $v_0 e_1 v_1 e_2 v_2 \ldots e_l v_l$, where for every i, $v_i \in V$, $e_i \in E$ and $v_{i-1}, v_i \in e_i$. The *distance* between two vertices $u, v \in V$ is the length of the shortest path starting at u and ending at v, and we denote it by $d(u, v)$. For a set of vertices $U \in V$ we denote the hypergraph restrained to U by $H[U]$, that is $H[U] = (U, \{e \in E | e \subseteq U\})$.

Throughout the paper, the parameter n is assumed to tend to infinity, and we also assume it to be sufficiently large whenever necessary. The notations $o()$ and $O()$ have the usual meaning, that is, $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and $f(n) = O(g(n))$ if there is a constant $c > 0$ such that $f(n) \leq cg(n)$ for all (sufficiently large) n.

All logarithms are to the base $e = 2.71828\ldots$.

2 Fractional matchings and covers in hypergraphs and linear programming

Let $H = (V, E)$ be a hypergraph. A set of disjoint edges is called a *matching*. The size of a largest matching is called the *matching number* of H and is denoted by ν. A non-negative real valued function $f : E \to \mathbb{R}_+$ is called a *fractional matching* if $\sum_{v \in e} f(e) \leq 1$ for every $v \in V$. The *value* of the fractional matching is $|f| = \sum_{e \in E} f(e)$. The maximum value over all fractional matchings of H is the *fractional matching number* of H, denoted by $\nu^*(H)$. If f achieves $|f| = \nu^*$ then f is said to be an *optimal* fractional matching of H. Similarly, a *fractional cover* of H is a non-negative real valued function $g : V \to \mathbb{R}_+$ such that $\sum_{v \in E} g(v) \geq 1$ for every $e \in E$. Again, the *value* of the cover is $|g| = \sum_{v \in V} g(v)$, and the minimum of $|g|$ over all the covers of H is the *fractional covering number* of H, denoted by $\tau^*(H)$. If $|g| = \tau^*$ then g is said to be *optimal*. These two problems are dual in the linear programming sense. Therefore we can apply the Linear Programming Duality Theorem and get:

Proposition 2.1. *Let $H = (V, E)$ be a hypergraph. Then*

1. *for every fractional cover g and every fractional matching f, one has $|g| \geq |f|$;*
2. *$\tau^* = \nu^*$;*
3. *if g is an optimal fractional cover of H and f is an optimal fractional matching of H, then*

$$f(e) > 0 \text{ implies } \sum_{v \in e} g(v) = 1;$$
$$g(v) > 0 \text{ implies } \sum_{v \in e} f(e) = 1.$$

Part (3) of the theorem above is called the *complementary slackness conditions.*

We also cite here Proposition 2 of [8], which is a well known consequence of linear programming:

Proposition 2.2. *For every r-uniform hypergraph $H = (V, E)$ one has:*

1. $\nu^*(H) \geq \nu(H)$*;*

2. if $V_0 \subseteq V$ *is the set of all non-isolated vertices of* H*, then* $\nu^*(H) \leq |V_0|/r$*, therefore* $\nu^*(H) \leq |V|/r$*;*

3. let g *be a fractional cover of* H *and let* $U \subseteq V$*, then the restriction of* g *to* U *is a fractional cover of* $H[U]$*;*

4. let $g : V \to \mathbb{R}_+$ *be an optimal fractional cover of* H *and denote* $V_1 = \{v \in V : g(v) > 0\}$*, then* $\nu^*(H) \geq |V_1|/r$*.*

Part 2 of the above Proposition states that if $H = (V, E)$ is r-uniform then $\nu^*(H) \leq |V|/r$. It is also true that for every r-uniform hypergraph $H = (V, E)$, $\tau^*(H) \geq |V|/r$. Thus the next definition is natural:

Definition 2.3. Let $H = (V, E)$ be an r-uniform hypergraph. Let f be a fractional matching and let g be a fractional cover. If $|f| = |V|/r$ then f is called a *perfect* fractional matching and if $|g| = |V|/r$ then g is called a *perfect* fractional cover.

3 Perfect fractional matchings in hypergraphs

In this section we omit floor and ceiling notation to the benefit of readability.

Let $H = (V, E)$ be a hypergraph and let $U \subset V$ be a set of vertices. Let E_0 be a set of edges, such that:

1. $|e \cap U| = 1$ for every $e \in E_0$.

2. $e_1 \cap e_2 \subset U$ for every $e_1 \neq e_2 \in E_0$.

U is said to have a *proper expansion* of size $|E_0|$.

Again, let $H = (V, E)$ be an r-uniform hypergraph on n vertices, and let c_1, c_2, c_3 be some constants such that

$$0 < c_1 < \frac{1}{r-1}, \qquad c_3 > 1, \qquad \frac{c_1}{c_3+1} < c_2 < 1. \tag{1}$$

Next we define two properties of r-uniform hypergraphs. These properties depend on the choice of the constants c_1, c_2 and c_3:

Q1 $\forall U \subset V,\ |U| \geq c_1 n \Rightarrow e(H[U]) > 0$. That is, every large enough subset spans an edge.

Q2 For every pair of disjoint sets $U_1, U_2 \subset V$, $|U_1| \leq c_2 n, |U_2| \leq (r-1)|U_1|$, there exists a set of edges $E_0 \subset E, |E_0| \geq c_3|U_1|$ such that for every $e \in E_0$ we have $|e \cap U_1| = 1, |e \cap U_2| = 0$ and for every $e, e' \in E_0$, $e \cap e' \subset U_1$. That is, U_1 has a proper expansion of size at least $c_3|U_1|$ with the additional requirement that every one of the expanding edges misses U_2.

We say that H has **Q1** and **Q2** if we can find constants c_1, c_2, c_3 that satisfy Relations (1) and H has **Q1** and **Q2** with these constants.

Note that Property **Q1** is in fact the property of having independence number smaller than $c_1 n$.

Lemma 3.1. *Let $H = (V, E)$ be an r-uniform hypergraph on n vertices that has Properties* **Q1** *and* **Q2** *with constants c_1, c_2, c_3 that satisfy Relations (1). Let c_4, c_5 be constants that satisfy*

$$c_4 \geq c_1(r-1), \qquad c_4 c_5 < c_2(r-1), \qquad 1 > c_5 \geq \frac{1}{c_3+1}.$$

Define $a_i = c_4 c_5^i n$, $i \geq 0$ and let $k_0 = \min\{j : a_j \leq c_3(r-1)\}$. Then for every $0 \leq i \leq k_0$ and every $U \subset V$ of size $|U| = a_i$ we have

$$\nu^*(H[V \setminus U]) > \frac{n-a_i}{r} - \frac{a_i}{(r-1)r} = \frac{n}{r} - \frac{a_i}{r-1}.$$

Proof. By induction on i. For $i = 0$ we need to show that if $U \subset V$, $|U| = c_4 n$ then $\nu^*(H[V \setminus U]) > \frac{n}{r} - \frac{c_4 n}{r-1}$. Let M be a maximal integral matching of $H[V \setminus U]$. If $|M| \leq \frac{n}{r} - \frac{c_4 n}{r-1}$ then there are at most $n - c_4 n - \frac{c_4 n}{r-1}$ vertices taking part in the matching, and thus there are at least $\frac{c_4 n}{r-1}$ vertices in $V \setminus U$ that are not in the matching. Since $\frac{c_4 n}{r-1} \geq c_1 n$, we have an edge spanned by these vertices (**Q1**), contradicting the maximality of M.

Assume now that the lemma is true for $i-1$ and that $i \leq k_0$. Let $U \subset V$, $|U| = c_4 c_5^i n$. Denote $U_1 = V \setminus U$ and consider an optimal fractional cover g of $H[U_1]$. Define $U_0 = \{v \in U_1 | g(v) = 0\}$. If $|U_0| < \frac{a_i}{r-1}$ then by Part 4 of Proposition 2.2,

$$\nu^*(H[U_1]) \geq \frac{|U_1 \setminus U_0|}{r} > \frac{n - a_i - \frac{a_i}{r-1}}{r} = \frac{n}{r} - \frac{a_i}{r-1},$$

as required.

On the other hand, if $|U_0| \geq \frac{a_i}{r-1}$, then g has a large number of zeroes and we would like to take advantage of this imbalance. The requirements on the constants imply

$$\frac{a_i}{r-1} = \frac{c_4 c_5^i n}{r-1} < c_2 c_5^{i-1} n < c_2 n,$$

so we can pick a vertices subset $U_0' \subset U_0$ such that $\frac{a_i}{r-1} \leq |U_0'| \leq c_2 n$. By Property **Q2** with U_0' as U_1 and U as U_2 , there exists a set of edges $E_0 \subset E$ such that:

- $|E_0| \geq c_3 |U_0'|$;
- every edge in E_0 has exactly one vertex in U_0' and none in U;
- for every $e_1, e_2 \in E_0$, $e_1 \cap e_2 \subset U_0'$ (which implies that $|e_1 \cap e_2| \leq 1$).

Let $T = \left(\bigcup_{e\in E_0} e\right) \setminus U_0'$. Recall that $c_3 \geq \frac{1}{c_5} - 1$, then

$$|E_0| \geq c_3|U_0'| \geq \left(\frac{1}{c_5} - 1\right)\frac{a_i}{r-1} \geq \left(\frac{c_4 c_5^{i-1} n}{c_4 c_5^i n} - 1\right)\frac{a_i}{r-1} \geq$$
$$\geq \left(\frac{a_{i-1}}{a_i} - 1\right)\frac{a_i}{r-1} = \frac{a_{i-1} - a_i}{r-1}.$$

Every edge donates $r-1$ distinct vertices to T, therefore $|T| \geq a_{i-1} - a_i$. The vertices in T must have a large weight in g, since g assigns a total weight of at least one to every edge, and every edge in E_0 has a vertex of weight zero. In fact, the average weight of the vertices in T is at least $\frac{1}{r-1}$. Let $T_0 \subset T$ be the set of the $a_{i-1} - a_i$ vertices with the largest g-weight in T, then $\sum_{v\in T_0} g(v) \geq \frac{a_{i-1}-a_i}{r-1}$. Now we apply the induction hypothesis on $U \cup T_0$ ($|U \cup T_0| = a_{i-1}$) and get that

$$\sum_{v\notin U\cup T_0} g(v) \geq \nu^*(H[U_1 \setminus T_0]) > \frac{n}{r} - \frac{a_{i-1}}{r-1},$$

due to the fact that g restricted to $H[U_1 \setminus T_0]$ is also a fractional cover (Part 3 of Proposition 2.2) and the value of every fractional cover is at least as large as ν^* (linear programming duality).

Now we can estimate the value of g:

$$\nu^*(H[U_1]) = \sum_{v\in U_1\setminus T_0} g(v) + \sum_{v\in T_0} g(v) > \frac{n}{r} - \frac{a_{i-1}}{r-1} + \frac{a_{i-1} - a_i}{r-1} = \frac{n}{r} - \frac{a_i}{r-1},$$

as required. □

Theorem 3.2. *Every r-uniform hypergraph satisfying* **Q1, Q2** *with constants c_1, c_2, c_3 satisfying Relations (1) has a perfect fractional matching.*

Proof. Let $H = (V, E)$ be an r-uniform hypergraph that satisfies **Q1, Q2** and let g be an optimal fractional cover of H. By Part 4 of Proposition 2.2 it is enough to verify that $g(v) > 0$ for every $v \in V$. Assume that $g(v) = 0$ for some $v \in V$. ¿From **Q2** with $U_1 = \{v\}$ and $U_2 = \emptyset$ we get that $d(v) \geq c_3$. Moreover, **Q2** ensures that we can find a set of at least c_3 edges that touch v and are disjoint outside of v. Denote this set by E_0 and note that $|E_0| \geq c_3(r-1)$. Every edge is covered by g, therefore the vertices of the edges in E_0 (except v) have an average weight of $\frac{1}{r-1}$ in g. Let $c_4 = c_1(r-1)$, $c_5 = \frac{1}{c_3+1} < 1$ and define the series a_i and the number k_0 as in Lemma 3.1. Recall that $a_{k_0} \leq c_3(r-1)$. Let T be the set of the a_{k_0} vertices with the largest value in g. Clearly, the average weight of a vertex in T is at least as large as that of a vertex in the union of the edges of E_0, excluding v_0. By Parts 1 and 2 of Proposition 2.1, Part 3 of Proposition 2.2 and Lemma 3.1 we have

$$\nu^*(H) = \sum_{v\notin T} g(v) + \sum_{v\in T} g(v) \geq \nu^*(H[V \setminus T]) + \sum_{v\in T} g(v) > \frac{n}{r} - \frac{a_{k_0}}{r-1} + \frac{a_{k_0}}{r-1} = \frac{n}{r},$$

which contradicts Part 2 of Proposition 2.2. □

We define here another property of r-uniform hypergraphs, which is similar to property **Q2**. Let $H = (V, E)$ be an r-uniform hypergraph. Define:

Q2': For every $U_1 \subset V$, $|U_1| \leq c_2 n$, there exists a set of edges $E_0 \subset E, |E_0| \geq (r - 1 + c_3)|U_1|$ such that for every $e, e' \in E_0$ we have $|e \cap U_1| = 1$ and $e \cap e' \subset U_1$. In other words, **Q2'** states that every small enough set has a large proper expansion.

Generally, **Q2'** is more convenient to use in applications (while **Q2** was tailor made for the proof of Lemma 3.1).

Proposition 3.3. *Property* **Q2'** *implies Property* **Q2**.

Proof. Let $H = (V, E)$ be an r-uniform hypergraph that has **Q2'** and let U_1, U_2 be as in **Q2**. By **Q2'** there exists a set of edges with the desired properties, E_0, s.t. $|E_0| \geq (r - 1 + c_3)|U_1|$. Since the edges in E_0 are disjoint, U_2 meets at most $(r-1)|U_1|$ of the edges and at least $c_3|U_1|$ are left to satisfy **Q2**. □

4 Balanced graphs

We give some definitions of graph density measurements. These notions appear in [5, Chapter 3].

Let $K = (V, E)$ be a graph on r vertices and with e edges. We define the *density* of K, $d(K) = e/r$, and the *maximum density* of K, $m(K) = \max\{d(K')|K' \subseteq K, v_{K'} > 0\}$. A graph K is *balanced* if $d(K) = m(K)$, that is, if no subgraph of K is denser than K itself. If K does have a subgraph denser than itself, we say that K is *imbalanced*.

K is called *strictly balanced* if $d(K') < d(K)$ for every $K' \subsetneq K$, that is if K is denser than every one of its proper subgraphs.

Example 4.1. Every complete graph is strictly balanced. A union of a cycle and a path of length ≥ 1 with one common vertex is balanced but not in the strict meaning.

We also use another related notation, the fractional arboricity, mentioned in the introduction. We repeat the definition here. Let K be a graph on r vertices and with e edges. Define $d^{(1)}(K) = e/(r-1)$ if $r \geq 2$ and $d^{(1)}(K) = 0$ if $K = K_1$. Then define

$$m^{(1)}(K) = \max\{d^{(1)}(K')|\emptyset \neq K' \subseteq K\}.$$

Analogously to the above we say that a graph K is K_1-*balanced* if $m^{(1)}(K) = d^{(1)}(K)$, that is, if for every subgraph $K' \subseteq K$, $d^{(1)}(K') \leq d^{(1)}(K)$. Furthermore, if for every proper subgraph $K' \subsetneq K$, $d^{(1)}(K') < d^{(1)}(K)$ then we say that K is *strictly* K_1*-balanced*.

Example 4.2. Every complete graph is also strictly K_1-balanced. A union of C_4 and C_3 with one common edge is a K_1-balanced graph but not in the strict meaning.

We are interested in the property of covering every vertex of a random graph $G(n,p)$ with a copy of another graph, K. Every vertex in $G(n,p)$ can take the role of any vertex in K. In this analysis the notion of a *rooted graph* is very useful. Let K be a graph and let u be a vertex of K, then the pair (u, K) is called a rooted graph. For example, K_3 has only one rooted version (up to isomorphism), while a union of K_3 and K_2 with one common vertex has three.

Let K be a graph on e edges and with $r > 1$ vertices. For a rooted version (u, K) we define its density $d(u, K) = d^{(1)}(K) = e/(r-1)$ and its maximal density

$$m(u, K) = \max_{K' : u \in K' \subseteq K} d(u, K').$$

Note that $d(u, K)$ does not depend on the choice of the root, while $m(u, K)$ generally does.

The notions of balance are naturally defined for rooted graphs as well. A rooted graph (u, K) is said to be *balanced* if $d(u, K) = m(u, K)$. It is said to be *strictly balanced* if, in addition, for every proper subgraph K' containing the root u one has $d(u, K') < d(u, K)$.

We state some properties of balanced graphs.

Proposition 4.3. *1. Every strictly balanced graph or strictly K_1-balanced graph is connected.*

2. A graph K is strictly K_1-balanced if and only if every rooted version of K is strictly balanced.

3. A non-empty K_1-balanced graph is strictly balanced.

4. Let K_1 be a balanced graph and let K_2 be another graph (not necessarily balanced) such that $d(K_2) \geq d(K_1)$. Let $G = K_1 \cup K_2$ (where K_1 and K_2 are not necessarily edge disjoint). Then, $d(G) \geq d(K_1)$.

Proof. The first three statements of the proposition are quite easy to verify and can be found in [5]. We will prove the fourth part.

For a graph F define $f(F) = d(K_1)v(F) - e(F)$. Then we have $f(K_1) = 0$, $f(K_2) \leq 0$ (by assumption) and $f(K_1 \cap K_2) \geq 0$ because $K_1 \cap K_2$ is a subgraph of the balanced graph K_1. Consequently by the modularity of f we have

$$f(G) = f(K_1 \cup K_2) = f(K_1) + f(K_2) - f(K_1 \cap K_2) \leq 0,$$

which is what we wanted to prove. □

Remark 4.4. The opposite direction of Part 3 above is generally not true, for example, C_6 with a chord connecting two vertices of distance two is strictly balanced but it is not K_1-balanced.

Remark 4.5. Note that for non-empty graphs one has: Strictly K_1-balanced $\Rightarrow$ K_1-balanced $\Rightarrow$ strictly balanced $\Rightarrow$ balanced.

When we look at a specific vertex in $G(n, p)$, the probability that it can take the role of some vertex $u \in H$ is a function[2] of $m(u, K)$, and the lower the density the more probable is that this role will be taken. Therefore it is natural to define:

[2]For a specific vertex i in $G(n, p)$, $p = n^{-1/m(v,K)}$ is the threshold for the property "i is contained in a copy of K as the vertex v" [12].

Definition 4.6.

$$m_* = m_*(K) = \min_{u \in K} m(u, K), \qquad M(K) = \{u \in V(K) | m(u, K) = m_*\}.$$

When $p = p_* := Cn^{-1/m_*(K)}$, a fixed vertex u in $G(n,p)$ is expected to be extended to $o(1)$ rooted versions where its role is not in $M(K)$. Also, u is expected to take the role of vertices from $M(K)$ in a constant number of rooted extensions. Let $a = |M(K)|/|Aut(K)|$ where $Aut(K)$ is the automorphism group of K. We state Theorem 3.22 from [5], concerning the threshold probability function for the property "every vertex lies in a copy of K" denoted here **EPI**K. This theorem was proved by Ruciński and by Spencer (independent works).

Theorem 4.7. *Let K be a graph with minimum degree at least 1. If for every $v \in M(K)$ the rooted graph (v, K) is strictly balanced then*

$$\lim_{n\to\infty} \Pr[G(n,p) \in \mathbf{EPI}K] = \begin{cases} 0 & \text{if } an^{v_K - 1}p^{e_K} - \log n \to -\infty, \\ 1 & \text{if } an^{v_K - 1}p^{e_K} - \log n \to \infty. \end{cases}$$

Note that by Part 2 of Proposition 4.3 this theorem applies for every strictly K_1-balanced graph.

4.1 Packs of strictly K_1-balanced graphs

Let K be a graph on more than two vertices. We build another graph in several steps. At the first step we just take one copy of K. At every other step we add another copy of K to the existing graph, but we require that the new copy will have at least two vertices in common with the old graph (so in fact we add only a part of K, and of course, we also require that at least one vertex will be a new one). If G has n vertices, then we need $\Theta(n)$ steps in the process of building it from K. The exact number of copies depends on K, and how we "use" it, but every step adds at least one vertex and at most $v(K) - 2$, which is a constant.

Remark 4.8. Note that even if G is composed of several copies of K, and every copy meets at least 2 vertices of other copies, it still may be that G can not be built from K in the way defined above. For example take a cycle of copies of K where every copy has one vertex in common with its neighbor on one side and another common vertex with the other neighbor. Then (unless K is a cycle of the same length) this graph can't be built from copies of K.

Lemma 4.9. *Let K be a strictly K_1-balanced graph. There exists a number $C = C(K)$ such that every graph G that can be built from K in C steps has $d(G) > d^{(1)}(K)$.*

Note that the lemma says that the *standard* density of the resulting graph is higher than the K_1-density of K.

Proof. Let G_i be the graph after i steps, so $G_1 = K$ and $G_C = G$. At step $i + 1$ we add another copy of K s.t. there is a subgraph $K' \subsetneq G_i, K' \subsetneq K, v(K') \geq 2$ that is common to K and G_i. Let

$d = d^{(1)}(K)$. In a similar manner to the proof of Proposition 4.3 we define f, a function on graphs, by $f(G) = dv(G) - e(G)$. Then, $f(G_1) = f(K) = dv(K) - e(K) = d$. For every $K' \subsetneq K$ with $v(K') = r'$ and $e(K') = e'$ one has

$$f(K) - f(K') = d - (dr' - e') < 0.$$

Let $\epsilon = \min_{K'} d(r' - 1) - e' > 0$. Then by the modularity of f,

$$f(G_i) = f(G_{i-1}) + f(K) - f(K') \leq f(G_{i-1}) - \epsilon,$$

and so $f(G_i) \leq d - (i-1)\epsilon$. Therefore if $C > \frac{d}{\epsilon} + 1$ then $f(G) = f(G_C) < 0$ which implies $d(G) > d^{(1)}(K)$. □

5 A hitting time result for fractional factors of random graphs with strictly K_1-balanced graphs

In this section we apply Theorem 3.2 that deals with fractional matchings in hypergraphs. The connection between fractional K-factors of graphs and fractional matchings in hypergraphs is straightforward:

Definition 5.1. Let $G = (V, E)$ be a graph, and let K be another graph on r vertices. We define the *K-hypergraph of G*, denoted by $\mathscr{G}_K$, in the following way:

- The vertex set of $\mathscr{G}_K$ is V, the vertex set of G.
- $\{v_1, v_2, \ldots, v_r\}$ is an edge of $\mathscr{G}_K$ if and only if there is a (not necessarily induced) copy of K in G on these vertices.

Note that $\mathscr{G}_K$ is r-uniform. Clearly there is a fractional K-factor in G if and only if there is a perfect fractional matching in $\mathscr{G}_K$. In this definition two copies of K that lie on the same r vertices correspond to the same edge in $\mathscr{G}_K$, so $\mathscr{G}_K$ may have less edges than the number of K-copies in G. Another way to define $\mathscr{G}_K$ is to put multiple edges in this case. For our purpose there is no difference between the definitions, as the addition (or removal) of multiple edges does not affect the existence of a fractional matching.

For a vertex $v \in G$ we define the "K-degree of v", $d_K(v)$, to be the degree of v in $\mathscr{G}_K$. We also use the notion of "disjoint K-degree of v", $d'_K(v)$ - the maximal number of copies of K in G that contain v and are vertex-disjoint outside of v.

5.1 Main theorem

We state Theorem 1.1 and Corollary 1.2 again. Let K be a strictly K_1-balanced graph. For a graph G we define two properties:

EPIK "Every point in K" - G has this property if every vertex of G lies in a (not necessarily induced) copy of K.

FK**F** "Fractional K-factor" - G has this property if it has a fractional K-factor.

Theorem. *For a graph process* $\widetilde{G}$ *let* $\tau_{\mathbf{EPI}K}$ *be the hitting time of the property* **EPI**K *and* $\tau_{\mathbf{F}K\mathbf{F}}$ *be the hitting time of the property* **F**K**F**. *Then for asymptotically almost every graph process*

$$\tau_{\mathbf{F}K\mathbf{F}} = \tau_{\mathbf{EPI}K}.$$

That is, the same edge that puts the last vertex in a copy of K also enables a fractional K-factor.

Corollary. *Let* K *be a strictly* K_1*-balanced graph, and let* $a = a(K)$ *be as defined after Definition 4.6. Then*

$$\lim_{n\to\infty} \Pr[G(n,p) \in \mathbf{F}K\mathbf{F}] = \begin{cases} 0 & \text{if } an^{v_K-1}p^{e_K} - \log n \to -\infty, \\ 1 & \text{if } an^{v_K-1}p^{e_K} - \log n \to \infty. \end{cases}$$

Proof. If $an^{v_K-1}p^{e_K} - \log n \to -\infty$ then by Theorem 4.7 a.a.s. there is a vertex in $G = G(n,p)$ not covered by a copy of K. This vertex will be isolated in $\mathscr{G}_K$, and by Part 2 of Proposition 2.2 we have

$$\nu^*(\mathscr{G}_K) \le (n-1)/r < n/r.$$

Therefore $\mathscr{G}_K$ does not allow for a perfect fractional matching and thus $G(n,p) \notin \mathbf{F}K\mathbf{F}$.

On the other hand, if $an^{v_K-1}p^{e_K} - \log n \to \infty$ then Theorem 4.7 states that a.a.s. $G(n,p)$ does have **EPI**K. Let $m = \binom{n}{2}p$. **EPI**K is a monotone property, and thus[3] $G(n,m)$ and equivalently $\widetilde{G}_m$ a.a.s. have **EPI**K. By Theorem 1.1 when $\widetilde{G}_m$ has **EPI**K it a.a.s. has **F**K**F** also, and again by monotonicity of **F**K**F**, $G(n,p)$ a.a.s. has **F**K**F**.

□

5.2 Proof Overview

The proof structure resembles the proof of Theorem 1 in [4]. If K is strictly K_1-balanced then by Part 2 of Proposition 4.3 K satisfies the requirements of Theorem 4.7 and the threshold function for **EPI**K is

$$p = \left(\frac{1}{a} n^{-(v(K)-1)} \ln n\right)^{1/e(K)}.$$

We set two probabilities and two times along the graph process $\widetilde{G}_n$:

$$p_1 = \left(\frac{1}{a} n^{-(v(K)-1)} (\ln n - \ln\ln\ln n)\right)^{1/e(K)}, \qquad m_1 = \binom{n}{2} p_1\,;$$

$$p_2 = \left(\frac{1}{a} n^{-(v(K)-1)} (\ln n + \ln\ln\ln n)\right)^{1/e(K)}, \qquad m_2 = \binom{n}{2} p_2\,.$$

[3] See [5, remark 1.18]

By Theorem 4.7 a.a.s. $G(n,p_1) \notin \mathbf{EPI}K$ and $G(n,p_2) \in \mathbf{EPI}K$ and since $\mathbf{EPI}K$ is a monotone property the same holds for $G(n,m_1)$ and $G(n,m_2)$. Therefore, for asymptotically almost every graph process $m_1 < \tau_{\mathbf{EPI}K} < m_2$.

Next, we define some (more) properties of graphs, **A0 - A6**, referred to later as "the **A** properties". The motivation for the definition of the **A** properties is given by Lemma 5.3 – these properties imply the existence of a fractional K-factor (Lemma 5.3 relies heavily on Theorem 3.2). To be more precise, the **A** properties are defined for a pair (G,S) where G is a graph and S is a subset of vertices of G. In the proof of the main theorem the set of vertices with small disjoint K-degree at time m_1 will take the role of S.

Now, Lemmas 5.6 - 5.12 show that a.a.e. graph process has all the properties except for $\mathbf{EPI}K$ along the whole time between m_1 and m_2. Since for a.a.e. graph process the hitting time of $\mathbf{EPI}K$ is also between m_1 and m_2 we conclude that for a.a.e. graph process, when $\mathbf{EPI}K$ hits we have all the **A** properties and thus a fractional K-factor. In fact, all the **A** properties are monotone[4], and Lemmas 5.6 - 5.12 simply demonstrate that the properties hold at time m_1 or at time m_2. The monotonicity of the properties implies their validity along the whole period between m_1 and m_2.

5.3 The A properties

Definition 5.2. Let K and $G = (V,E)$ be some graphs where $|V| = n$ and let $S \subset V$ be a set of vertices. Also, let $0 < d < 1$ be a constant depending[5] only on K. We define the following properties:

A0 S contains all the vertices with a disjoint K-degree of at most $\ln\ln\ln n$ (again, S may contain some more vertices).

A1 - EPIK Every point lies in a copy of K.

A2 The distance in $\mathscr{G}_K$ between any two vertices from S is at least 5.

A3 Every vertex subset of size greater than $n/|K|^3$ spans a copy of K.

A4 Every subset $U \subset V$ of size $n^d < |U| < \frac{1}{|K|^3}n$ has a proper expansion of size at least $|U|(|K|+2)$ in $\mathscr{G}_K$. (Proper expansion is defined at the beginning of Section 3).

A5 For every vertex subset U of size at most n^d and with $U \cap S = \emptyset$, U has a proper expansion of size at least $C|U|\ln\ln\ln n$ in $\mathscr{G}_K$, where C is some constant depending only on K.

A6 $|S| \le n^{d/100}$.

When it is clear what S is, we will treat the **A** properties as graph properties and not as properties of a graph and a vertices subset.

[4]**A0**,**A1**,**A3**,**A4** and **A5** are monotone increasing, **A2** is monotone decreasing and **A6** is both.

[5]d is required to satisfy the condition that appears in the end of Lemma 5.8.

Lemma 5.3. *Let K be some graph, let $G = (V, E)$ be a graph on n vertices and let $S \subset V$ be a subset of vertices of G. Assume that G has the properties* **A0** - **A6**. *Then G has a fractional K-factor.*

Proof. First, look at the vertices in S. For every such vertex we can find a copy of K s.t. all of these copies are disjoint (**A1** & **A2**). Denote the set of these copies by $\mathcal{K}_1$. Remove S and the vertices of the copies from $\mathcal{K}_1$. For every vertex u not in S, at most one copy of K was ruined in this process. Otherwise u has two paths (in $\mathscr{G}_K$) of length at most 2 starting from it and ending in S, but property **A2** forbids such paths.

Let G_0 be the graph after the removal of S and the vertices of the copies from $\mathcal{K}_1$, and let $\mathscr{G}_{0K}$ be the K-hypergraph of G_0. By **A0** and the above, $\mathscr{G}_{0K}$ has minimal degree at least $\ln\ln\ln n - 1$. $\mathscr{G}_{0K}$ has property **Q1** with $c_1 = 1/|K|^3$ as this is just property **A3** for G_0.

Also, $\mathscr{G}_{0K}$ has **Q2'** with $c_2 = 1/|K|^3$ and $c_3 = 2$. In order to prove this, let U be a set of vertices, $U \subset V[G_0], |U| = k$. If $n^d < k < |K|^{-3}n$ then by property **A4**, $\mathscr{G}_{0K}$ has a proper expansion of size $k(|K|+2)$ before the first step. In the first step we removed $|S|\cdot|K|$ vertices and every vertex ruined at most one edge from the proper expansion (since the edges are disjoint). By **A6**, $|S| \leq n^{d/100}$ and therefore the number of edges that were destroyed is less than $|S| \cdot |K| \ll k$, so after the first step U still has a proper expansion of size at least $k(|K| + 1)$.

If $1 \leq k \leq n^d$ then by property **A5**, U has a proper expansion with size at least $Ck \ln\ln\ln n$. In the process of removing S and some of its neighborhood, at most $(|K| - 1)k$ edges were destroyed (for each of U's vertices u, the copies touching u can touch only one copy from $\mathcal{K}_1$, again because of **A2**).

Therefore $\mathscr{G}_{0K}$ satisfies both **Q1** and **Q2'** and by Proposition 3.3 and Theorem 3.2 $\mathscr{G}_{0K}$ has a perfect fractional matching. Denote this fractional matching by M_0. Now we would like to find a perfect fractional matching for the original hypergraph $\mathscr{G}_K$, but this is fairly easy. Define a new fractional matching $M : E[\mathscr{G}_K] \to \mathbb{R}_+$:

$$M(e) = \begin{cases} M_0(e) & e \in E[\mathscr{G}_{0K}], \\ 1 & e \text{ corresponds to a copy from } \mathcal{K}_1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$|M| = |M_0| + |S| = \frac{n - |K| \cdot |S|}{|K|} + |S| = \frac{n}{|K|}.$$

Therefore $\mathscr{G}_K$ also has a perfect fractional matching which implies that G has **FKF**. ☐

5.4 Small

Let $\widetilde{G}$ be a graph process on n vertices. We are interested in a special subset of vertices - those with a low disjoint K-degree at time m_1, so we define:

$$\text{SMALL} = \text{SMALL}(\widetilde{G}) = \{v | \text{at time } m_1,\ d'_K(v) \leq \ln\ln\ln n\}.$$

SMALL is a function of graph processes, and therefore it is constant along the process. After time m_1 SMALL still contains every vertex with a low disjoint K-degree, as the property of having low disjoint K-degree is monotonically decreasing, but it may contain some other vertices as well. As already mentioned, SMALL will take the role of S when we'll discuss the **A** properties in the context of random graph processes. By saying "$\widetilde{G}$ has property **A** at time t" for one of the **A** properties, we mean that the graph $\widetilde{G}_t$ with the set SMALL$(\widetilde{G})$ has this property.

5.5 Proof of validity of the A properties between m_1 and m_2

Here we give a series of lemmas that principally say that properties **A1** – **A6** a.a.s. hold at time m_1 or m_2 (m_1 for monotonically increasing properties and m_2 for monotonically decreasing properties). **A0** is satisfied at time m_1 because of the way in which SMALL is defined for random graph processes. As mentioned already, all the **A** properties are monotone, so we can switch freely between $G(n, m_1)$, $G(n, p_1)$ and $\widetilde{G}_{m_1}$ (see [5, Remark 1.18]). Throughout this section we set $r = |V[K]|$, $e = |E[K]|$ and we use c as a general unspecified constant.

Proposition 5.4. *For a.a.e. graph process at time m_2, every pair of vertices lies in at most C copies of K, where $C = C(K)$ is a constant depending only on K.*

Proof. Clearly, if there are $C' > C$ copies of K that have 2 vertices in common, then they form a graph that can be built from K in C' steps. By Lemma 4.9, the density of every such graph is greater than $d^{(1)}(K)$, and by first moment arguments for a.a.e. graph process there are no such graphs in $G(n, p_2)$. □

Remark 5.5. The property of not containing a subgraph is monotonically decreasing, and therefore for a.a.e. graph process there are no such graphs between times m_1 and m_2.

Lemma 5.6. *A.a.e. graph process at time m_1 has Property* **A5***.*

Proof. Let $\epsilon > 0$ be some constant. Consider a set of vertices, $U \subseteq V(G(n, p_1))$, $|U| = k \leq n^{1-\epsilon}$, $U \cap \text{SMALL} = \emptyset$. We would like to estimate the size of the maximal proper expansion of U in $\mathscr{G}_K$. We know that every vertex in U has degree at least $\ln \ln \ln n$ in $\mathscr{G}_K$. Consider the size of the set of all the edges that touch U. Every edge is counted at most r times so there are at least $\frac{k \ln \ln \ln n}{r}$ edges of $\mathscr{G}_K$ touching U. Denote them by $\mathcal{K}$. Now switch back to $\widetilde{G}_{m_1}$. We will also use $\mathcal{K}$ for the set of copies of K that touch U. Pick one copy and remove from $\mathcal{K}$ any other copy that has 2 vertices in common with it. Continue this process until $\mathcal{K}$ is empty. Since we removed a bounded number of copies in every step (by Remark 5.5 and Proposition 5.4), we have found a set of $\frac{k \ln \ln \ln n}{rC(K)}$ edge-disjoint copies of H that touch U.

These copies still do not form a proper expansion of U since some of them may have more than one vertex in U or may have a common vertex with another copy in $V \setminus U$.

To complete the proof, we show that for a.a.e. graph process at time m_1 every set U of size k has:

1. at most Ck edge-disjoint copies of K that have 2 vertices (or more) in U;

2. at most Ck pairs of edge-disjoint copies of K that have a common vertex in $V \setminus U$.

where C is some constant.

Fix k. For every set U of size k let X_U be the indicator random variable for the event "There are at least A edge-disjoint copies of K that have 2 vertices or more in U". Let $X = \sum_{U:|U|=k} X_U$. Then

$$EX \leq C_{aut}\binom{n}{k}\binom{\binom{k}{2}\binom{n}{r-2}}{A}p^{Ae} \leq C_{aut}\left(\frac{en}{k}\right)^k \binom{\frac{1}{2}k^2\left(\frac{en}{r-2}\right)^{r-2}}{A}p^{Ae} \leq$$

$$\leq C_{aut}\left(\frac{en}{k}\right)^k \left(\frac{\frac{e}{2}k^2\left(\frac{en}{r-2}\right)^{r-2}}{A}\right)^A \cdot p^{Ae} = C_{aut}\left(\frac{en}{k}\right)^k\left(\frac{ck^2n^{r-2}p^e}{A}\right)^A =$$

$$= C_{aut}\left(\frac{en}{k}\right)^k\left(\frac{ck^2(\ln n - \ln\ln\ln n)}{nA}\right)^A,$$

where C_{aut} is a constant depending only on the size of $aut(K)$.

We are interested in the expected number of sets that meet more than $2k$ such copies of H, so we set $A = 2k$.

$$EX = \left(\frac{en}{k}\right)^k\left(\frac{ck^2(\ln n - \ln\ln\ln n)}{2kn}\right)^{2k} \leq \left(\frac{ck\ln^2 n}{n}\right)^k \leq n^{-0.9\epsilon k},$$

since $k \leq n^{1-\epsilon}$.

Therefore the expected number of "bad" sets is less than

$$\sum_{k=1}^{n^{1-\epsilon}} n^{-0.9\epsilon k} = \frac{1-n^{-0.9\epsilon(n^{1-\epsilon}+1)}}{1-n^{-0.9\epsilon}} - 1 = o(1),$$

and by the first moment method a.a.s. there are no such sets in $G(n, p_1)$.

The second part is similar. Let X_U be the indicator random variable for the event "U has A edge-disjoint pairs of copies of K that have common vertices in $V \setminus U$". Also let $X = \sum X_U$. Then

$$EX = C_{aut}\binom{n}{k}\binom{\binom{k}{2}\binom{n}{2r-3}}{A}p^{2eA} \leq \left(\frac{en}{k}\right)^k\left(\frac{ck^2n^{2r-3}p^{2e}}{A}\right)^A$$

$$\leq \left(\frac{en}{k}\right)^k\left(\frac{ck^2\ln^2 n}{nA}\right)^A.$$

Again, set $A = 2k$,

$$EX \leq \left(\frac{en}{k}\right)^k \left(\frac{ck^2 \ln^2 n}{n2k}\right)^{2k} \leq \left(\frac{ck \ln^4 n}{n}\right)^k \leq n^{-0.9\epsilon k}.$$

And again, by Markov's inequality a.a.s. no such set exists in $G(n, p_1)$.

Therefore for every $\epsilon > 0$, asymptotically almost every graph process at time m_1 has the property that every one of the subsets of size $k \leq n^{1-\epsilon}$ satisfies the requirements below:

1. $ck \ln\ln\ln n$ copies of H touching it such that every 2 copies have at most one common vertex.

2. Less than $2k$ of the copies have more than one vertex in U

3. There are less than $2k$ pairs of copies with a common vertex in $V \setminus U$.

If we remove the "bad" copies (listed in items (2) and (3)), we are left with a proper expansion of size $ck \ln\ln\ln n$, and the proof is complete. □

Remark 5.7. Note that in the proof of item (3) above we didn't use the assumption that U has no vertex from SMALL. In particular, if we take U to be a singleton, we get that for almost every graph process, at time m_1, every vertex $u \in V$ lies in at most one pair of copies of K having a common vertex other than u.

Lemma 5.8. *A.a.e. graph process at time m_1 has property* **A4**.

Proof. Let U be a set of size k, $n^d < k < r^{-3}n$. Let $\mathcal{K}$ be an inclusion-maximal family of copies of K, that have exactly one vertex in U and that are pairwise disjoint outside U. Let

$$V \supset W = \bigcup_{K \in \mathcal{K}} K \setminus U.$$

Also let $V_0 = V \setminus (U \cup W)$.

Set $b = (r+2)(r-1)$. If U has a proper expansion of size less than $k(r+2)$, then $|W| < k(r+2)(r-1) = bk$. Also, there is no copy of K with one vertex in U and the rest of the vertices in V_0. We claim that this situation a.a.s. never happens. In other words, we would like to show that a.a.s. for every set U of size k, $n^d < k \leq \frac{1}{r^3}n$, and for every set V_0 of size at least $n-(b+1)k$ there is a copy of K with one vertex in U and the rest in V_0. Fix k, and for every set U of size k and a set $V_0 \subseteq V \setminus U$ of size $|V_0| = n-(b+1)k$, let X_{U,V_0} be the number of such copies of K.

Clearly,

$$EX_{U,V_0} = C_{aut} k \binom{n-(b+1)k}{r-1} p^e \geq C_{aut} k \left(\frac{n(1-(b+1)r^{-3})}{n(r-1)}\right)^{r-1} \frac{\ln n - \ln\ln\ln n}{a} \geq Ck \ln n,$$

for some constants C_{aut} and $C > 0$. Let $\mathcal{K}_0$ be the set of all potential copies of K with one vertex in U and the rest in V_0. We define Δ_{U,V_0} as defined in Janson's inequality (see [5, Theorem 2.16]) -

$$\Delta_{U,V_0} = \sum_{\substack{K_1 \neq K_2 \in \mathcal{K}_0 \\ E[K_1] \cap E[K_2] \neq \emptyset}} E[I_{K_1} I_{K_2}] \leq ck \binom{|V_0|}{r-1} C_{aut} \sum_{s,t} \binom{n}{r-s} p^{2e-t},$$

where the second sum is over copies with s common vertices and t common edges, and C_{aut} is a constant depending only on K. Since K is strictly K_1-balanced we have $\frac{t}{s-1} < \frac{e}{r-1}$ and thus

$$\Delta_{U,V_0} \le ckn^{r-1}p^e \sum_{\substack{s,t \\ t\ge 1}} n^{r-s}p^{e-t} \le ck \ln n n^{-\epsilon} = o(EX).$$

By applying Janson's inequality we get

$$\begin{aligned}
&\Pr[\text{There exist } U \text{ and } V_0 \text{ as above}] \le \\
&\le \binom{n}{k}\binom{n-k}{n-bk} \Pr[\text{There is no copy of } K \text{ between } U \text{ and } V_0] \le \\
&\le \left(\frac{en}{k}\right)^k \left(\frac{en}{bk}\right)^{bk} e^{-EX+\Delta} \le \left(\frac{en}{k}\right)^k \left(\frac{en}{bk}\right)^{bk} \frac{1}{n^{Ck}} = \\
&= \left(\frac{n^{b+1-C}e^{b+1}}{k^{b+1}b^b}\right)^k .
\end{aligned}$$

Therefore, if $k \ge n^d$ for some $d > (b+1-C)/(b+1)$, there is a.a.s. no such U and V_0, and the proof is complete. ☐

Lemma 5.9. *A.a.e. graph process has* **A6**.

Recall that SMALL is defined as the set of all the vertices with low disjoint K-degree at time m_1. In order to estimate the probability that a vertex u lies in a small number of disjoint copies of K we apply Spencer's *maxdisfam* technique ([14], [13]). We use the notation[6] from [2, section 8.4].

In our application Ω is the set of $\binom{n}{2}$ potential edges, $R = E[G(n,p_1)]$, $\forall r \in R\ p_r = p_1$, I is the set of all potential copies of K in $G(n,p_1)$ that contain u, $|I| = \binom{n-1}{r-1}\frac{(r-1)!}{|\mathrm{Aut}((u,K))|}$, A_i is the set of edges of the i'th copy and B_i is the event "this copy is in $G(n,p_1)$". X_i is the indicator random variable for B_i, $X = \sum_{i\in I} X_i$ and $\mu = E[X]$. We denote $i \sim j$ if $i \ne j$ and $A_i \cap A_j \ne \emptyset$ and define $\Delta = \sum_{i\sim j} \Pr[B_i \wedge B_j]$ (the sum over ordered pairs). $J \subseteq I$ is called a *disfam* if

- B_j for every $j \in J$.
- For no $j, j' \in J$ is $j \sim j'$.

If, in addition

- If $j' \notin J$ and $B_{j'}$ then $j \sim j'$ for some $j \in J$.

then J is a *maxdisfam*. Finally,

$$\nu = \max_{j\in I} \sum_{i\sim j} \Pr[B_i].$$

The next theorem is also from [2, section 8.4]

[6] And the idea from [7].

Theorem 5.10. *For any integer s,*

$$\Pr[\textit{there exists a maxdisfam } J, |J| = s] \le \frac{\mu^s}{s!} e^{-\mu} e^{s\nu} e^{\frac{\Delta}{2}}.$$

Lemma 5.9 Proof. We would like to show that at time m_1 for a fixed vertex u,

$$\Pr[u \in \textsc{Small}] \le \frac{c(\ln n)^{\ln\ln\ln n} \cdot \ln\ln n}{n},$$

for some constant c. In order to apply Theorem 5.10 we need to estimate μ, ν and Δ.

Fix $u \in V[\widetilde{G}]$. For every vertex $i \in V[K]$ not in $M(K)$ choose an inclusion-minimal subgraph $K_i \subseteq K$ containing i such that $d(i, K_i) = m(i, K_i) > m_*$. Let $X_i(u)$ be the number of copies of (i, K_i) in $\widetilde{G}_{m_1}$ where u is the root, and let $X^*(u) = \sum_{i \notin M(K)} X_i(u)$. Clearly,

$$EX^*(u) \le C_{aut} \sum_{i \in V[K] \setminus M(K)} \binom{n-1}{v(K_i)-1} p_1^{e(K_i)} = o(1),$$

and therefore the probability that u lies in a copy of K in the role of vertex out of $M(K)$ tends to zero as n tends to infinity. The major contribution to the number of copies that contain u is done by vertices from $M(K)$. Let $M'(K) = \{v_1, v_2, \ldots, v_l\}$ be a maximal collection of vertices from $M(K)$ such that the rooted graphs (v_i, K) are pairwise non-isomorphic. Let $X(u)$ be a random variable counting the number of rooted versions of K where the root u is in the role of a vertex from $M'(K)$, and let X_K be a random variable counting the number of copies of K in $\widetilde{G}_{m_1}$. Then,

$$\sum_{u \in V[\widetilde{G}_{m_1}]} X(u) = |M(K)| X_K,$$

and thus, by symmetry, $EX(u) = \frac{1}{n}|M(K)| EX_K = \frac{|M(K)|}{n} \frac{1}{|\mathrm{Aut}(K)|} n^{v(K)} p_1^{e(H)} = \ln n - \ln\ln\ln n$. Therefore, $\mu = EX^*(u) + EX(u) = \ln n - \ln\ln\ln n + o(1)$. In order to estimate Δ we shall count pairs of copies of K by the number of common vertices and edges. As before, we will use s for the number of common vertices and t for the number of common edges.

$$\Delta = \sum_{i \sim j} \Pr[B_i \wedge B_j] = \sum_{s,t} \binom{n}{r-1} \binom{n}{r-s} p_1^{2e-t} \le$$
$$\le \sum_{s,t} c n^{2r-1-s} n^{-\frac{r-1}{e}(2e-t)} = \sum_{s,t} c n^{-(s-1) + \frac{r-1}{e} t}.$$

K is strictly K_1-balanced and thus $\frac{t}{s-1} < \frac{e}{r-1}$ which implies

$$\Delta \le \sum_{s,t} c n^{-(s-1) + \frac{r-1}{e} t} = o(1).$$

The estimation of ν is similar,

$$\nu = \max_{j \in I} \sum_{i \sim j} \Pr[B_i] = \sum_{s,t} \binom{n}{r-s} p_1^{e-t} \le$$
$$\le \sum_{s,t} n^{r-1-(s-1)} n^{-\frac{r-1}{e}(e-t)} \le n^{-\epsilon}$$

for some $\epsilon > 0$.

Before we apply Theorem 5.10 let us note that the maxdisfam technique only enables us to give an upper bound for the probability of a vertex having a low *edge*-disjoint degree, while we need to estimate the probability of having a low *vertex*-disjoint degree. A vertex may have a low vertex-disjoint degree and a high edge-disjoint degree if it lies in many copies of K who share a vertex but not an edge. By remark 5.7 in almost every graph process at time m_1 every vertex lies in at most one pair of copies of K having another common vertex. Thus the vertex-disjoint degree of a vertex is bounded by its edge-disjoint degree plus one.

Now we can apply Theorem 5.10 and give a bound for the probability that a vertex has a low edge-disjoint degree in $\mathscr{G}_K$ -

$$\begin{aligned}\Pr[d'_H(v) \leq \ln\ln\ln n] \leq & \sum_{s=0}^{\ln\ln\ln n+1} \frac{\mu^s}{s!} e^{-\mu} e^{sv} e^{\Delta/2} = \\ = e^{o(1)} \cdot \Pr[\mathrm{Poiss}_\mu \leq \ln\ln\ln n + 1] \leq & \\ \leq 2 \cdot \frac{\mu^{\ln\ln\ln n+1}}{(\ln\ln\ln n+1)!} e^{-\mu} \leq & \frac{(\ln n)^{\ln\ln\ln n+1} \cdot \ln\ln n}{n}.\end{aligned}$$

Finally, we apply Markov's inequality and conclude that for a.a.e. graph process at time m_1,

$$|\textsc{Small}| \leq c(\ln n)^{\ln\ln\ln n+1} \cdot \ln\ln n \ll n^{d/100}.$$

$\square$

Lemma 5.11. *A.a.e. graph process at time m_2 has* **A2**.

Proof. We wish to estimate the probability that there is a pair of vertices from Small with distance in $\mathscr{G}_K$ that is less than or equal to four. For $u, v \in V$ let $X_{u,v}$ be the indicator random variable for the event "There is a path of length l in $\mathscr{G}_K$ between u and v" where l is some integral constant. The corresponding structure in G for such path in $\mathscr{G}_K$ is a sequence of l copies of K, $K_1, K_2, \ldots, K_l$, where $u \in K_1$, $v \in K_l$ and for every $1 \leq i < l, V[K_i] \cap V[K_{i+1}] \neq \emptyset$.

Fix u and v and let $l \geq 1$ be an integer. For fixed sequences $\mathbf{s} = (s_1, s_2, \ldots, s_{l-1})$ and $\mathbf{t} = (t_1, t_2, \ldots, t_{l-1})$ let $\mathcal{P}_{\mathbf{s},\mathbf{t}}$ be the set of all l-sequences of K-copies, such that for every $1 \leq i < l$, $s_i = |V[K_i] \cap V[K_{i+1}]|, t_i = |E[K_i] \cap E[K_{i+1}]|$ and the corresponding structure in $\mathscr{G}_K$ is a path of length l from u to v. For $P \in \mathcal{P}_{\mathbf{s},\mathbf{t}}$ let X_P be the indicator random variable for the event "P is in $G(n, p_2)$", that is, the edges of all the l copies appear in $G(n, p_2)$. Now,

$$\begin{aligned}&\Pr[u, v \in \textsc{Small} \wedge X_{u,v} = 1] = \\ =& \Pr[u, v \in \textsc{Small} | X_{u,v} = 1] \cdot \Pr[X_{u,v} = 1] \leq \\ \leq& \sum_{\substack{0<s_1\\t_1}} \sum_{\substack{0<s_2\\t_2}} \cdots \sum_{\substack{0<s_{l-1}\\t_{l-1}}} \sum_{P \in \mathcal{P}_{\mathbf{s},\mathbf{t}}} \Pr[u, v \in \textsc{Small} | X_P = 1] \cdot \Pr[X_P = 1].\end{aligned}$$

For every $\mathbf{s}, \mathbf{t}$ the number of edges in every path $P \in \mathcal{P}_{\mathbf{s},\mathbf{t}}$ is finite, and therefore as n tends to infinity the influence of these edges is becoming negligible. Note that we are looking at the path on time m_2 while whether u and v is in Small or not is determined in time m_1. However knowing that a finite number of edges is on in time m_2 gives us even less information than knowing the same in time m_1, and in both cases the effect is negligible. Thus $\Pr[u, v \in \textsc{Small} | X_P = 1] \longrightarrow \Pr[u, v \in \textsc{Small}]$. Therefore

$$\begin{aligned}
&\Pr[u, v \in \textsc{Small} \ \wedge \ X_{u,v} = 1] \leq \\
\leq &(1+\epsilon) \Pr[u, v \in \textsc{Small}] \cdot \sum_{\substack{0<s_1 \\ t_1}} \sum_{\substack{0<s_2 \\ t_2}} \cdots \sum_{\substack{0<s_{l-1} \\ t_{l-1}}} \sum_{P \in \mathcal{P}_{\mathbf{s},\mathbf{t}}} \cdot \Pr[X_P = 1] \leq \\
\leq &(1+\epsilon) \Pr[u, v \in \textsc{Small}] \cdot \\
&C_{aut} \binom{n}{r-1} p^e \sum_{\substack{0<s_1 \\ t_1}} C_{aut} \binom{n}{r-s_1} p^{e-t_1} \cdots \sum_{\substack{0<s_{l-1} \\ t_{l-1}}} C_{aut} \binom{n}{r-s_{l-1}-1} p^{e-t_{l-1}}.
\end{aligned}$$

Since K is strictly K_1-balanced, $\binom{n}{r-s_i} p^{e-t_i}$ is maximal when $s_i = 1$ and $t_i = 0$ and then $\binom{n}{r-1} p^e = \ln n + \ln \ln \ln n$. Therefore we have

$$\Pr[u, v \in \textsc{Small} \ \wedge \ X_{u,v} = 1] \leq (1+\epsilon) \Pr[u, v \in \textsc{Small}] \cdot c \frac{(\ln n + \ln \ln \ln n)^l}{n}.$$

Now we need to estimate $\Pr[u, v \in \textsc{Small}]$. Again, we will use Spencer's maxdisfam technique described in Lemma 5.9. We will use the same notation as in the proof of Lemma 5.9, but this time we will consider all the K copies touching u *or* v (or both). Here $|I| = 2\binom{n-1}{r-1} \frac{(r-1)!}{|\mathrm{Aut}((u,K))|} - \binom{n-2}{r-2} C_{Aut}$. Again, we will estimate μ, Δ and ν and then apply Theorem 5.10.

Let $X(u)$ and $X(v)$ be the random variables counting the number of copies of K in $\widetilde{G}_{m_1}$ that are touching u and v respectively, and let $X(u, v)$ be the number of copies touching both u and v. Clearly $\mu = EX(u) + EX(v) - EX(u, v)$. $EX(u)$ was estimated in the proof of lemma 5.9, and by Proposition 5.4 $EX(u, v)$ is bounded by a constant. Therefore

$$\mu = 2 \ln n - 2 \ln \ln \ln n + O(1).$$

To estimate Δ we use s and t for the number of common vertices and edges (respectively) between A_i and A_j.

$$\begin{aligned}
\Delta = \sum_{i \sim j} \Pr[B_i \wedge B_j] &\leq 2 \sum_{s,t} \binom{n}{r-1} \binom{n}{r-s} p_1^{2e-t} \leq \\
&\leq \sum_{s,t} c n^{-(s-1) + \frac{r-1}{e} t},
\end{aligned}$$

but K is strictly K_1-balanced so $\frac{t}{s-1} < \frac{e}{r-1}$ and thus $\Delta = o(1)$. We give a bound from above for ν by $n^{-\epsilon}$ for some $\epsilon > 0$ in the same way as in Lemma 5.9. Again, the maxdisfam technique is applicable for edge-disjoint copies while being in SMALL is determined by the vertex-disjoint degree, but by remark 5.7 this is almost the same. Applying Theorem 5.10 we get that

$$\begin{aligned}
&\Pr[\text{maximal number of disjoint copies through } u \text{ or } v \leq 2\ln\ln\ln n] \leq \\
\leq &\sum_{s=0}^{2\ln\ln\ln n+2} \frac{\mu^s}{s!} e^{-\mu} e^{s\nu} e^{\Delta/2} = e^{o(1)} \cdot \Pr[\text{Poiss}_\mu \leq 2\ln\ln\ln n + 2] \leq \\
\leq &2\frac{\mu^{2\ln\ln\ln n+2}}{(2\ln\ln\ln n+2)!} e^{-\mu} \leq \frac{(2\ln n)^{2\ln\ln\ln n+2} \cdot \ln\ln n}{n^2}.
\end{aligned}$$

If u and v are in SMALL then the number of disjoint copies of K touching u or v is at most $2\ln\ln\ln n$. Thus

$$\begin{aligned}
&\Pr[u, v \in \text{SMALL}] \leq \\
\leq &\Pr[\text{maximal number of disjoint copies through } u \text{ or } v \leq 2\ln\ln\ln n] \leq \\
\leq &\frac{(2\ln n)^{2\ln\ln\ln n+2} \cdot \ln\ln n}{n^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Pr[\text{there exist two vertices from SMALL with distance at most } 4] \leq \\
\leq &\binom{n}{2}(1+\epsilon)\frac{(2\ln n)^{2\ln\ln\ln n+2} \cdot \ln\ln n}{n^2} \cdot c\frac{(\ln n + \ln\ln\ln n)^l}{n} = o(1),
\end{aligned}$$

and asymptotically almost every graph process has **A2**. □

Lemma 5.12. *A.a.e. graph process at time m_1 has* **A3**.

Proof. Let U be a set of vertices of size k, $k \geq n/r^3$, and let X_U be the number of copies of K in U. By Janson's inequality (see [5, Theorem 2.16]) we have $\Pr[X_U = 0] \leq e^{-EX_U+\Delta}$, where Δ is defined as below. Now,

$$\begin{aligned}
EX_U &= C_{aut}\binom{k}{r}p^e \geq \left(\frac{n}{r}\right)^r p^e \geq Cn\ln n; \\
\Delta &:= \sum_{\substack{K_1,K_2 \subset U \\ K_1 \neq K_2 \\ E[K_1]\cap E[K_2] \neq \emptyset}} E[I_{K_1} I_{K_2}] \leq C_{aut}\binom{k}{r}\sum_{\substack{s,t \\ t\geq 1}}\binom{k}{r-s}p^{2e-t},
\end{aligned}$$

where $t \geq 1$ runs over the number of common edges between K_1 and K_2 and $s \geq 2$ does the same for the common vertices.

$$\Delta \leq C_{aut}\binom{k}{r}\sum_{\substack{s,t \\ t\geq 1}}\binom{k}{r-s}p^{2e-t} \leq C\sum_{\substack{s,t \\ t\geq 1}} n^{2r-s-\frac{r-1}{e}(2e-t)}(\ln n)^{(2e-t)}.$$

The common part is a proper subgraph of K, and since K is strictly K_1-balanced, $\frac{e}{r-1} > \frac{t}{s-1}$ for every s,t. This implies

$$2r - s - \frac{r-1}{e}(2e-t) < 1 - \epsilon$$

for some $\epsilon > 0$. Therefore

$$\Delta \leq Cn^{1-\epsilon} \sum_{s,t} (\ln n)^{2e-t} = o(EX_U).$$

Applying Janson's inequality we get

$$\Pr[X_U = 0] \leq e^{-n \ln n (1+o(1))},$$

and by the union bound

$$\Pr[\exists U \subset V, |U| \geq n/r^3, E[U] = \emptyset] \leq 2^n e^{-n \ln n(1+o(1))} = o(1).$$

Therefore a.a.s. every set of size at least n/r^3 spans at least one copy, and the proof is complete. □

6 Graphs having $\tau_{\mathbf{EPI}K} \ll \tau_{\mathbf{F}K\mathbf{F}}$

Recall that $m_* = m_*(K) = \min_{v \in K} m(v, K)$, as defined in Definition 4.6.

Lemma 6.1. *Let K be a graph having $m^{(1)}(K) > m_*(K)$. Then for a.a.e. graph process $\tau_{\mathbf{EPI}K} \ll \tau_{\mathbf{F}K\mathbf{F}}$. To be more precise, if $m^{(1)}(K) > m_*(K)$, then there exist two functions, $f_K(n), g_K(n)$ and a constant $\epsilon(H) > 0$, such that*

1. $\frac{g_K(n)}{f_K(n)} \geq \binom{n}{2} n^{\epsilon(K)}$*;*
2. *for a.a.e. graph process on n vertices $\tau_{\mathbf{EPI}K} < f_K(n)$;*
3. *for a.a.e. graph process on n vertices $\tau_{\mathbf{F}K\mathbf{F}} > g_K(n)$.*

Proof. Let p_* be the threshold probability from Theorem[7] 4.7, $p_* = \text{polylog}(n) n^{-1/m_*}$. By the definition of $m^{(1)}(K)$, there exists a subgraph $M \subset K$ s.t. $d^{(1)}(M) > m_*(K)$ which implies that $1 > v(M) - \frac{e(M)}{m_*}$. The expected number of copies of M in $G(n, p_*)$ is

$$C \binom{n}{v(M)} p_*^{e(M)} \leq C n^{v(M) - e(M)/m_*} = Cn^{1-\epsilon}, \quad \epsilon > 0.$$

Therefore, by the Markov inequality a.a.s. the number of copies of M is $O(n^{1-\epsilon})$. Let $\mathcal{K}$, $\mathcal{M}$ be the sets of copies of K,M (respectively) in $G(n,p)$. Take an optimal fractional K-factor $f : \mathcal{K} \to \mathbb{R}_+$. Let $V_M = \bigcup_{M \in \mathcal{M}} M$, then

$$|f| = \sum_{K \in \mathcal{K}} f(K) \leq \sum_{v \in V_M} \sum_{K : v \in K} f(K) \leq v(M) \cdot |\mathcal{M}| \cdot 1 = O(n^{1-\epsilon}).$$

[7] In [14] it is shown that for every graph K the threshold function behaves like $p_* = \text{polylog}(n) n^{-1/m_*(K)}$. The exact behavior depends on the structure of K.

Therefore, at time $\binom{n}{2}p_*$ asymptotically almost every graph process does not allow a fractional K-factor. Moreover, $\mathbf{F}K\mathbf{F}$ will hit only after the graph process will contain $\frac{n}{|K|}$ copies of K, which is expected to occur only at time $\binom{n}{2}n^{-1/d^{(1)}(M)}$. □

Remark 6.2. We have shown that for strictly K_1-balanced graphs, $\mathbf{EPI}K$ and $\mathbf{F}K\mathbf{F}$ occur simultaneously in almost every graph process, while for graphs having $m^{(1)}(K) > m_*(K)$ for almost every graph process there is a gap. We believe that for graphs having $m^{(1)}(K) = m_*(K)$, $\tau_{\mathbf{EPI}K} = \tau_{\mathbf{F}K\mathbf{F}}$ for almost every graph process. The figure below summarizes our view on this problem.

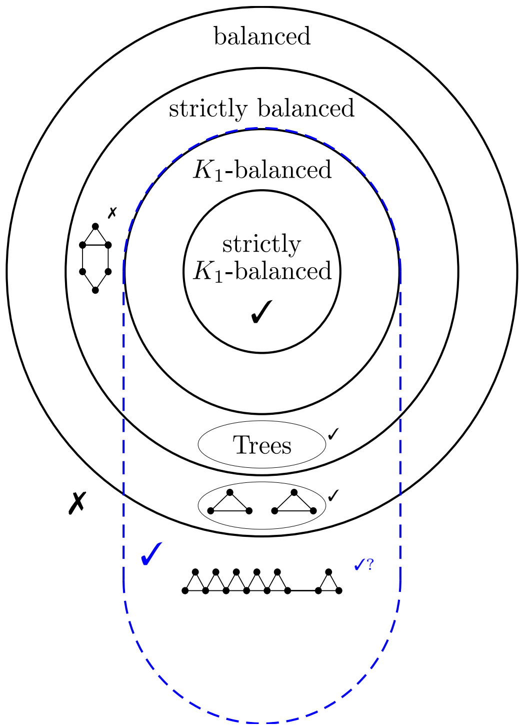

Figure 1: ✓ denotes a family of graphs for which $\mathbf{EPI}K$ and $\mathbf{F}K\mathbf{F}$ a.a.s occur simultaneously. ✗ denotes a family of graphs for which a.a.s there is a gap between $\tau_{\mathbf{EPI}K}$ and $\tau_{\mathbf{F}K\mathbf{F}}$. ✓ denotes a family of graphs for which we believe there should be a ✓. The dashed line marks the set of graphs with $m_* = m^{(1)}$.

References

[1] N. Alon and R. Yuster. Threshold functions for H-factors. *Combinatorics, Probability and Computing 2*, pages 137–144, 1993.

[2] Noga Alon and Joel H. Spencer. *The Probabilistic Method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley, New York, Second edition, 2000.

[3] P. Erdős and A. Rényi. On the existence of a factor of degree one in a connected random graph. *Acta Mathematica Academia Scientiarum Hungaricae*, (17):359–368, 1966.

[4] A. Frieze and M. Krivelevich. Hamilton cycles in random subgraphs of pseudo-random graphs. *Discrete Mathematics*, (256):137–150, 2002.

[5] Svante Janson, Tomasz Łuczak, and Andrzej Ruciński. *Random Graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley, New York, 2000.

[6] J. Kim. Perfect matchings in random uniform hypergraphs. *Random Structures and Algorithms*, (23):111–132, 2003.

[7] M. Krivelevich. Bounding Ramsey numbers through large deviation inequalities. *Random Structures and Algorithms*, (7):145–155, 1995.

[8] M. Krivelevich. Perfect fractional matchings in random hypergraphs. *Random Structures and Algorithms*, (9):317–334, 1996.

[9] M. Krivelevich. Triangle factors in random graphs. *Combinatorics, Probability and Computing*, (6):337–347, 1997.

[10] C. Payan. Graphes équilibrés et arboricité rationnelle (french) [balanced graphs and rational arboricity]. *European Journal of Combinatorics*, (7):263–270, 1986.

[11] A. Ruciński. Matching and covering the vertices of a random graph by copies of a given graph. *Discrete Mathematics*, (105):185–197, 1992.

[12] A. Ruciński and A. Vince. Strongly balanced graphs and random graphs. *Journal of Graph Theory*, (10):251–264, 1986.

[13] J. Spencer. Counting extensions. *Journal of Combinatorial Theory, Series A*, (55):247–255, 1990.

[14] J. Spencer. Threshold functions for extension statements. *Journal of Combinatorial Theory, Series A*, (53):286–305, 1990.