Efficient Hamilton covers and linear arboricity of random graphs

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Abstract

A Hamilton cover of a graph is a collection of Hamilton cycles whose union contains all edges. Since each Hamilton cycle covers two edges at every vertex, every Hamilton cover has size at least $\lceil \Delta(G)/2 \rceil$. We prove that this lower bound is algorithmically tight for binomial random graphs $G(n,p)$ throughout the widest possible range of edge probabilities: if $\omega(n) \to \infty$ and

$$\frac{\log n + \log\log n + \omega(n)}{n} \le p = p(n) \le 1 - \frac{\omega(n)}{n^2},$$

then there is a randomized polynomial-time algorithm which, given $G \sim G(n,p)$, with high probability outputs a Hamilton cover of size $\left\lceil \frac{\Delta(G)}{2} \right\rceil$. The main new contribution is the sparse regime near the Hamiltonicity threshold, where we prove a conjecture of Draganić, Glock, Munhá Correia and Sudakov. Our proof develops constructive tools for decomposing such graphs into controlled forest systems and extending them, using reserved pseudorandom structure, into Hamilton cycles.

We also prove the corresponding hitting-time result for the random graph process, answering a question of Hefetz, Kühn, Lapinskas and Osthus. Finally, we use our methods to show that $G \sim G(n,p)$ with high probability satisfies the celebrated Linear arboricity conjecture for every $p \le 1$, and that the appropriate linear forests can be found in polynomial time.

1 Introduction

A *Hamilton cover* of a graph G is a collection of Hamilton cycles whose union contains all edges of G. Every Hamilton cover has size at least $\left\lceil \frac{\Delta(G)}{2} \right\rceil$, since a Hamilton cycle uses exactly two edges at each vertex. We call a Hamilton cover *tight* if it has this minimum possible size. The question is therefore whether this local maximum-degree obstruction is the only obstruction.

Hamiltonicity is a central theme in graph theory and theoretical computer science, with wide-ranging applications. Sparse graph models have received particular attention, especially sparse random graphs. Pósa [36] proved that the binomial random graph $G(n,p)$ is with high probability (whp) Hamiltonian for $p \ge C \log n/n$, where C is a sufficiently large constant. The sharp threshold, due to Komlós–Szemerédi [28] and to Bollobás [7], is

$$p = \frac{\log n + \log\log n + \omega(n)}{n}$$

for every $\omega(n) \to \infty$; see, e.g., [8]. The corresponding hitting-time theorem, due independently to Ajtai, Komlós and Szemerédi [1] and to Bollobás [7], says that in the random graph process $\widetilde{G} = (G_i)_{i=0}^{\binom{n}{2}}$, whp the first Hamilton cycle appears exactly when the minimum degree $\delta(G_i)$ becomes two.

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Once Hamilton cycles appear, there are two natural robustness questions. The packing problem asks how many edge-disjoint Hamilton cycles one can find, and is governed by the minimum degree. The trivial upper bound $\lfloor \delta(G)/2 \rfloor$ is whp tight in $G(n,p)$, through a sequence of works beginning with Bollobás and Frieze [9] and including Frieze and Krivelevich [19], Krivelevich and Samotij [29], Knox, Kühn and Osthus [27], and Hamilton decomposition results for regular expanders [30]. The covering problem is the dual question: it asks how many Hamilton cycles are needed to cover all edges, and is governed by the maximum degree.

Hamilton covers in random graphs were first studied systematically by Glebov, Krivelevich and Szabó [20]. They proved that for every fixed $\alpha > 0$, $G(n,p)$ with $p \geq n^{-1+\alpha}$ has a Hamilton cover of size $(1+o(1))np/2$ whp, and conjectured that the optimal number $\lceil \Delta(G(n,p))/2 \rceil$ should be attainable essentially throughout the Hamiltonicity range. Hefetz, Kühn, Lapinskas and Osthus [24] proved the exact tight result for

$$\frac{\log^{117} n}{n} \leq p \leq 1 - n^{-1/8},$$

already pushing the cover theorem close to the complete graph in the upper bound for $p(n)$. Draganić, Glock, Munhá Correia and Sudakov [11] later lowered the sparse end to $C_0 \log n/n$, for a sufficiently large constant C_0.

In this paper we close the remaining gaps. In Section 4 we prove the tight bound for the range starting at the sharp threshold, confirming the conjecture by Draganić, Glock, Munhá Correia and Sudakov [11]. In Section 5 we complete the remaining dense range up to the natural endpoint. Consequently we obtain the following ultimate form of the random graph Hamilton cover theorem.

Theorem 1.1. *Let $\omega(n) \to \infty$, and let $p = p(n)$ satisfy*

$$\frac{\log n + \log\log n + \omega(n)}{n} \leq p \leq 1 - \frac{\omega(n)}{n^2}.$$

Then there is a randomized polynomial-time algorithm which, given $G \sim G(n,p)$, whp outputs a tight Hamilton cover of G.

Specifically, the range from the threshold up to $C_0 \log n/n$ is the new sparse result proved in Section 4; the range from $C_0 \log n/n$ up to $1 - n^{-1/8}$ follows by combining [11] with [24]; and the range $p = 1 - q$, where $q \leq n^{-1/8}$ and $qn^2 \to \infty$, is covered by our Theorem 5.2. Our proof is constructive, and yields a randomized polynomial time algorithm to find the required cover. In the appendix, we comment on the algorithmic details more explicitly, as our proof is presented existentially for clarity of presentation. Furthermore, we comment on how the proof in [11] and [24] for the range from $C_0 \log n/n$ up to $1 - n^{-1/8}$ can be made algorithmic as well.

The range of p in Theorem 1.1 is best possible in the following sense. The lower endpoint is forced by Hamiltonicity itself: below $p(n) = (\log n + \log\log n)/n$, the graph whp has a vertex of degree less than two, and hence has no Hamilton cover. The upper endpoint is also the natural endpoint for a whp statement. If $1 - p = c/n^2$ for a fixed $c > 0$, then, for odd n, with probability bounded away from zero the complement of $G(n,p)$ consists of exactly one edge. But K_n with one edge removed has no tight Hamilton cover, as observed in [24] and explained in Section 5. On the other hand, if $1 - p \ll n^{-2}$, then $G(n,p) = K_n$ whp, and Walecki's theorem (see, e.g., [6]) gives a tight Hamilton cover. Thus the only excluded part is the unavoidable critical window around $1 - p \asymp n^{-2}$.

The new sparse ingredient is also robust enough to give the corresponding hitting-time statement in the window where the Hamiltonicity threshold occurs. This answers a question of Hefetz, Kühn, Lapinskas and Osthus [24], who asked whether the tight-cover property appears at the hitting time of Hamiltonicity.

Theorem 1.2 (Hitting time for tight Hamilton covers)**.** *Let $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{\binom{n}{2}}$ be the random graph process on vertex set $[n]$, where G_i is obtained from G_{i-1} by adding one edge chosen uniformly at random from the missing edges. Let $\tau_2 = \min\{i : \delta(G_i) \geq 2\}$. Then whp G_{τ_2} has a tight Hamilton cover. Moreover, such a cover can be found in randomized polynomial time.*

Since the proof consists of the same algorithm and analysis as in the proof of Theorem 1.1, we briefly comment on the details in the concluding remarks in Section 7.

Hamilton covers are also natural from an algorithmic point of view. They form a covering analogue of Hamilton decompositions and Hamilton-cycle packings: rather than asking for one spanning cycle, or for many edge-disjoint spanning cycles, one asks for the minimum number of spanning tours needed to cover the whole edge set. Closely related Hamilton decomposition questions have appeared in the algorithmic study of TSP tour domination [22, 4], and Ferber and Mond [17] recently proved an algorithmic optimal packing theorem for edge-disjoint Hamilton cycles in random directed graphs with degree at least $\log^{15} n$. Our result gives an algorithmic covering counterpart for undirected random graphs.

Hamilton covers also fit into a broader line of work on pseudorandom graphs. The result of Glebov, Krivelevich and Szabó [20] applies to a broad class of expanders, and Hamilton decompositions of regular expanders were obtained by Kühn and Osthus [30]. More recently, Draganić, Kim, Lee, Munhá Correia, Pavez-Signé and Sudakov [12] proved asymptotically optimal packing and covering results for sparse pseudorandom graphs: for every $\varepsilon > 0$ there is $C > 0$ such that every (n, d, λ)-graph with $d/\lambda \geq C$ contains at least $(1/2 - \varepsilon)d$ edge-disjoint Hamilton cycles, and its entire edge set can be covered by at most $(1/2 + \varepsilon)d$ Hamilton cycles. In the directed setting, Ferber, Kronenberg and Long [16] proved asymptotic packing, counting and covering results for random directed graphs.

Linear arboricity

A closely related decomposition problem is the linear arboricity conjecture of Akiyama, Exoo and Harary [2]. A linear forest is a graph whose connected components are paths. The *linear arboricity* $\mathrm{la}(G)$ of a graph G is the smallest number of linear forests whose union covers $E(G)$, or equivalently, the smallest number of linear forests its edges can be partitioned into. The conjecture asserts that

$$\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil$$

for every graph G. This would be best possible, for example for regular graphs of odd degree, and can be viewed as a path-like strengthening of the basic degree barrier in edge decompositions. The conjecture remains open in general. Alon [3] proved the asymptotic form $\mathrm{la}(G) \leq (1/2 + o(1))\Delta(G)$; later work of Ferber, Fox and Jain [15] and Lang and Postle [31] improved the error term, which is now of order $\tilde{O}(\sqrt{\Delta(G)})$. Very recently, Christoph, Draganić, Girão, Hurley, Michel and Müyesser [10] proved that every n-vertex graph satisfies $\mathrm{la}(G) \leq \Delta(G)/2 + O(\log n)$, improving the bound for graphs with at least polylogarithmic degree. For random graphs, McDiarmid and Reed [32] proved the conjecture for random regular graphs, and Glock, Kühn and Osthus [21] proved it for a large range of binomial random graphs using the Hamilton-cover result of Hefetz, Kühn, Lapinskas and Osthus, and the range was later extended by the Hamilton cover result in [11]. Our Hamilton-cover theorem, together with the additional argument for sparse graphs in Section 6, yields the full binomial random graph result.

Theorem 1.3. *For every $p = p(n)$, whp $G \sim G(n, p)$ satisfies*

$$\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil .$$

1.1 Proof outline for the sparse range

We finish the introduction by outlining the new sparse-range argument in some more detail. The difficulty in this regime is that the graph is already Hamiltonian, but it is still far from regular in the sense relevant to a tight cover. Whp one has $\Delta(G) - \delta(G) = \Theta(\Delta(G)) = \Theta(\log n)$. A tight cover contains only $k := \lceil \Delta(G)/2 \rceil$ Hamilton cycles, so at a vertex of degree close to $\Delta(G)$ almost every available incident edge must be used without repetition, while at a vertex of very small degree the same few incident edges must be reused many times by the k cycles. Thus the cycles have to be routed through the exceptional vertices in a very controlled way.

The first step is to isolate the exceptional vertices. We fix a small constant $\alpha > 0$, depending only on the constant C in the upper bound $p \le C \log n/n$. Let B be the set of vertices of degree at least $(1-\alpha)\Delta(G)$, let S be the set of vertices of degree less than $(\log\log n)^4$, and put $W = B \cup S \cup N(S)$. Standard first-moment estimates show that these sets are typically small and well separated: there are no short paths between exceptional vertices, and no short cycles in their neighbourhoods. This separation is a useful feature of the random graph near the threshold. It allows us to deal with the high-degree and low-degree constraints locally, while leaving the typical part of the graph available for global routing.

We then cover the typical part $G - W$ by linear forests. The reason for using linear forests is that they are the right intermediate objects: every linear forest can potentially be completed to a Hamilton cycle, provided it has access to enough unused random edges and vertices. To keep this access available, we randomly split $G - W$ into a fixed constant number of subgraphs, where the number of parts depends only on C. Each part comes with its own linear-sized reservoir of vertices outside it. The split is chosen so that every part has maximum degree noticeably smaller than $\Delta(G)/k$ (recalling that $\Delta(G-W) \le (1-\alpha)\Delta(G)$), and every vertex still has many neighbours in each corresponding reservoir. Applying the approximate linear arboricity theorem separately to these parts gives a collection of linear forests covering $G - W$. Crucially, this collection has size smaller than k by a positive fraction of k. The unused fraction is the slack that will later absorb the exceptional vertices and the few edges lost when the typical and exceptional pieces are merged.

The vertices in B and S are handled by explicit local constructions. For each high-degree vertex $b \in B$, we distribute the edges incident to b among k cherries with centers at that vertex, so that edges are repeated only when necessary. For each low-degree vertex $s \in S$, the situation is different: every Hamilton cycle must pass through s, but there are very few edges available there. We build k small forest gadgets through $\{s\} \cup N(s)$, with endpoints pushed out to $N^2(s)$, so that all edges incident to the low-degree region are covered. Because the vertices in S are far apart, these gadgets do not interfere with one another. Combining the high-degree and low-degree gadgets gives k exceptional forests.

At this point we have two kinds of forests: typical forests covering $G-W$, and exceptional forests covering the edges incident to B, S, and $N(S)$. The next task is to combine them without increasing the total number of forests. We pair each typical forest with one exceptional forest. If the two forests conflict at a vertex, we delete the offending typical edge before taking their union. The deleted edges form a graph of bounded maximum degree, a consequence of the separation of B and S and the fact that vertices outside W have small neighbourhood in W. The exceptional forests that were not paired provide enough spare capacity to absorb this bounded-degree leftover graph by a simple merging lemma. Thus we obtain exactly k linear forests covering all edges of G, and each forest still has a private linear-sized reservoir in which all of its endpoints have many neighbours.

The final step is to extend each of these k forests to a Hamilton cycle. Conceptually, the task is to route the endpoints of the forests through the remaining random set. This is the point where the sparse regime creates a genuine technical difficulty: the available reservoir is large, but it comes with only a weak minimum-degree guarantee. Fortunately, one can show that for carefully chosen parameters, one can utilize the extendability method, introduced in Section 3.1, which allows us to connect the components of each forest using the reserved random edges and then close the resulting path into a Hamilton cycle. Applying this extension lemma to all k forests gives the desired tight Hamilton cover. The procedure is constructive throughout; on the high-probability event given by the random graph estimates, all steps can be implemented in randomized polynomial time.

2 Preliminaries

Here we state and show several basic graph theoretic results, as well as some probabilistic tools. We first introduce some standard notation.

2.1 Notation and definitions

We use standard graph theoretic notation. Given a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. Given $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $N_G(S)$ the external neighbourhood of S in G (omitting the subscript whenever it is clear from the context). We denote by $N_G^k(S)$ the set of vertices at distance k from S, in particular $N_G^1(S) = N_G(S)$. For subsets $S_1, S_2 \subseteq V(G)$ we denote by $e_G(S_1, S_2)$ the number of edges of G with one endpoint in S_1 and the other in S_2. For a vertex $v \in V(G)$, we denote by $\partial_G(v)$ the set of edges incident to v in G. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of G, respectively. A *linear forest* is a graph consisting of a disjoint union of paths. For a linear forest F, we let $End(F)$ denote the set of vertices of degree exactly 1 in F (that is, the path endpoints, with isolated vertices excluded), and for a collection $\mathcal{F}$ of linear forests we set $End(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} End(F)$. A Hamilton cycle in a graph G is a cycle containing every vertex in G. We let $G(n,p)$ denote the binomial random graph on n labeled vertices, where each edge is included independently with probability p. All logarithms are natural unless stated otherwise.

2.2 Auxiliary results about linear forests

We start with a simple lemma which allows us to extend a linear forest by edges of a low maximum degree graph.

Lemma 2.1. *Let H_1, H_2 be graphs, and let $\mathcal{F}$ be a collection of linear forests covering $E(H_1)$. Suppose that every vertex of $V(H_2)$ has degree at most d in both H_1 and H_2, and that every edge of H_1 incident to $V(H_2)$ lies in at most ℓ forests of $\mathcal{F}$. If*

$$|\mathcal{F}| \geq 4d\ell + 1,$$

then the edges of H_2 can be added to the forests in $\mathcal{F}$ so that the resulting collection is again a collection of $|\mathcal{F}|$ linear forests covering $E(H_1) \cup E(H_2)$. Moreover, each resulting forest is obtained from its original forest by adding edges of H_2 only, and the union of endpoint sets is contained in $End(\mathcal{F}) \cup V(H_2)$.

Proof. We add the edges of H_2 one by one to linear forests in $\mathcal{F}$. Suppose we have added $i < e(H_2)$ edges from H_2 to linear forests in $\mathcal{F}$, and consider an edge $e = (u,v)$ of H_2 which we have not added yet. The endpoints u, v lie in $V(H_2)$, so they each have degree at most d in H_1 and at most d in H_2, hence degree at most $2d$ in $H_1 \cup H_2$. Therefore the number of forests of $\mathcal{F}$ in which u is not isolated is at most $2d\ell$, and the same holds for v. Since $|\mathcal{F}| \geq 4d\ell + 1$, there is a forest of $\mathcal{F}$ in which both u and v are isolated, so we can add e to that forest. This completes the proof. □

We also need an approximate version of the linear arboricity conjecture, first shown by Alon [3].

Theorem 2.2. *For every $\varepsilon > 0$ there exists Δ_0 such that the following holds for all $\Delta \geq \Delta_0$. Every graph with maximum degree at most Δ can be decomposed into $\lceil (1+\varepsilon)\Delta/2 \rceil$ linear forests.*

2.3 Degree sequence in random graphs

In this section we will discuss various properties typically satisfied by vertices of high degree, that is, close to the maximum degree, in the random graph $G(n,p)$.

Lemma 2.3 ([8]). *Let $X \sim \mathrm{Bin}(n,p)$ with $pn \geq 1$ and $q := 1-p$. Then, if $hqn \geq 3$, we have*

$$\mathbb{P}(X \geq pn + h) < \sqrt{\frac{pqn}{2h^2\pi}} \cdot \exp\left(-\frac{h^2}{2pqn} + \frac{h}{pqn} + \frac{h^3}{p^2n^2}\right).$$

Lemma 2.4 ([8],[29]). *Let $\log n/n \leq p \leq n^{-1/2}$. With high probability, the minimum degree δ and the maximum degree Δ of $G(n,p)$ satisfy*

(i) $pn + (1-o(1))\sqrt{2pn\log n} \leq \Delta \leq pn + 2\sqrt{2pn\log n}$;

(ii) $\delta \le pn - \frac{1}{4}\sqrt{2pn\log n}$.

Proof. For the first part, the upper bound follows by a union bound over all vertices, since by Chernoff bounds the probability that a vertex has degree at least $pn + 2\sqrt{2pn\log n}$ is at most $o(1/n)$. The lower bound is a direct consequence of Theorem 3.12 in [8] with m being a function which tends to infinity arbitrarily slowly and noting that $\Delta = d_1 \ge d_m$. The second part is proven in ([29], Lemma 2.2). ☐

Proposition 2.5. *Let $C > 1$, consider $\log n/n \le p \le C\log n/n$, and let $\alpha = 1/10$. Define B to be the set of vertices with degree at least $pn + (1-\alpha)\sqrt{2pn\log n}$ in $G(n,p)$, and let S be the vertices with degree less than $(\log\log n)^4$. Then whp all of the following hold.*

1. *$|B|, |S| \le \sqrt{n}$.*
2. *There are no paths of length between 1 and 6 with both endpoints in $B \cup S$. There are also no cycles of length at most 6 that contain a vertex in $B \cup S \cup N(S)$.*
3. *If $np \le 1.01\log n$ then $\log n/10 \le d(v) \le 1.1\log n$ for all $v \in N(S)$.*
4. *If $np \ge 1.01\log n$ then $\Delta/20 \le d(v) \le (1-\varepsilon)\Delta$ for all $v \in N(S)$, where $\varepsilon = \varepsilon(C) > 0$ is small enough.*

Proof. We prove each item using first moment arguments.

Item 1. Each vertex v has degree $d(v) \sim \mathrm{Bin}(n-1,p)$ with mean $\mu := (n-1)p$. For B, write $\mu = (n-1)p$ and $h = (1-\alpha)\sqrt{2pn\log n}$. By the Chernoff bound (see, e.g., [25, Theorem 2.1]),

$$\mathbb{P}[d(v) \ge \mu + h] \le \exp\left(-\frac{h^2}{2(\mu + h/3)}\right).$$

Since $np \ge \log n$ and $\alpha = 1/10$, we have

$$\frac{h}{\mu} \le (0.9 + o(1))\sqrt{2},$$

and therefore

$$\frac{h^2}{2(\mu + h/3)} \ge \frac{0.9^2 \cdot 2pn\log n}{2pn\left(1 + 0.9\sqrt{2}/3 + o(1)\right)} = \frac{0.81}{1 + 0.3\sqrt{2} + o(1)}\log n > 0.56\log n.$$

Thus

$$\mathbb{P}[v \in B] \le n^{-0.56+o(1)} \quad \text{and hence} \quad \mathbb{E}[|B|] \le n^{0.44+o(1)} = o(\sqrt{n}).$$

For S, since $(\log\log n)^4 \ll \mu$, the standard Poisson-type bound gives

$$\mathbb{P}[v \in S] \le \left(\frac{e\mu}{(\log\log n)^4}\right)^{(\log\log n)^4} \cdot e^{-\mu} \le \exp\left(-\mu + O((\log\log n)^5\right) \le \exp\left(-\log n + O((\log\log n)^5)\right).$$

Hence $\mathbb{E}[|S|] \le \exp(O((\log\log n)^5)) = o(\sqrt{n})$. By Markov's inequality, $|B|, |S| \le \sqrt{n}$ whp.

Item 2. We use the first moment method. For a specific ordered tuple $(v_0, v_1, \dots, v_k)$ with $k \le 6$, the probability that $v_0v_1\cdots v_k$ is a path with $v_0, v_k \in B\cup S$ is at most $O(p^k \cdot \mathbb{P}[v_0 \in B\cup S] \cdot \mathbb{P}[v_k \in B\cup S])$, since conditioning on the path edges incident to each endpoint changes degrees by at most $O(1)$, negligibly affecting the tail probabilities. Summing over all tuples, the expected number of such paths is at most

$$\sum_{k=1}^{6} O(n(np)^k) \cdot \mathbb{P}[v \in B \cup S]^2 \le O(n(np)^6) \cdot n^{-1.18+o(1)} = O(n^{-0.18+o(1)}(\log n)^6) = o(1),$$

where we used $np = O(\log n)$ and $\mathbb{P}[v \in B] \le n^{-0.59+o(1)}$ (the contribution from S is dominated). Similarly, the expected number of cycles of length $k \le 6$ containing a vertex from $B \cup S$ is at most

$$\sum_{k=3}^{6} O((np)^k) \cdot \mathbb{P}[v \in B \cup S] \le O((\log n)^6 \cdot n^{-0.59+o(1)}) = o(1).$$

For cycles of length $k \le 6$ containing a vertex of $N(S)$: conditioned on a fixed cycle being present, the probability that a specified cycle vertex has an additional neighbour in S (and therefore lies in $N(S)$) is at most $np \cdot \mathbb{P}[v' \in S] = O(\log n \cdot n^{-1+o(1)})$, so the expected number of such cycles is at most

$$\sum_{k=3}^{6} O((np)^k) \cdot O(\log n \cdot n^{-1+o(1)}) = O((\log n)^7 \cdot n^{-1+o(1)}) = o(1).$$

By the first moment method, whp no such paths or cycles exist.

Item 3. Suppose $np \le 1.01 \log n$. For a pair of vertices u, v with $uv \in E(G)$, conditioned on the edge uv, the residual degrees $d(u) - 1$ and $d(v) - 1$ are independent $\mathrm{Bin}(n-2, p)$ random variables. The expected number of edges uv with $d(u) < (\log \log n)^4$ and $d(v) > 1.1 \log n$ is at most

$$n^2 p \cdot \mathbb{P}[\mathrm{Bin}(n-2,p) < (\log\log n)^4] \cdot \mathbb{P}[\mathrm{Bin}(n-2,p) > 1.1 \log n - 1].$$

The first probability is $\exp(-\log n + O((\log\log n)^5))$ as before. For the second one, since $(n-2)p \le 1.01 \log n$, the threshold exceeds the mean by a factor of $1.1/1.01 > 1.08$, and the Chernoff bound gives $\exp(-\Omega(\log n)) = n^{-\Omega(1)}$. The product is $n^2 p \cdot n^{-1+o(1)} \cdot n^{-\Omega(1)} = o(1)$, since $(\log \log n)^5 = o(\log n)$ absorbs the subpolynomial factor. The same argument applies for the lower bound: the expected number of edges uv with $d(u) < (\log\log n)^4$ and $d(v) < \log n/10$ is at most

$$n^2 p \cdot \exp(-\log n + O((\log\log n)^5)) \cdot \exp(-\Omega(np)) = o(1),$$

using $\mathbb{P}[\mathrm{Bin}(n-2,p) < \log n/10] \le \exp(-\Omega(np))$ by Chernoff (the threshold is a small fraction of the mean). Thus whp $\log n/10 \le d(v) \le 1.1 \log n$ for all $v \in N(S)$.

Item 4. By Item 2, every vertex $v \in N(S)$ satisfies $v \notin B$, so $d(v) < np + (1-\alpha)\sqrt{2np\log n}$. Writing $r = \sqrt{2\log n/(np)}$, by Lemma 2.4 we have $\Delta \ge np + (1-o(1))\sqrt{2np\log n} = np(1 + (1-o(1))r)$, and thus

$$\frac{d(v)}{\Delta} < \frac{1 + (1-\alpha)r}{1 + (1-o(1))r} \le 1 - \frac{(\alpha - o(1))r}{1+r} \le 1 - \varepsilon,$$

where $\varepsilon = \varepsilon(C) > 0$, since $np \le C \log n$ gives $r \ge \sqrt{2/C} \cdot (1 + o(1))$, and thus $\varepsilon \ge \alpha\sqrt{2/C}/(2(1 + \sqrt{2/C})) - o(1) > 0$. For the lower bound, condition on the high-probability event from Lemma 2.4 that $\Delta \le np + 2\sqrt{2np\log n} \le 4np$ (using $np \ge 1.01 \log n$), so $\Delta/20 \le np/5$. The same argument as in Item 3 then shows the expected number of edges uv from S to a vertex of degree less than $\Delta/20 \le np/5$ is at most

$$n^2 p \cdot \exp(-\log n + O((\log\log n)^5)) \cdot \mathbb{P}[\mathrm{Bin}(n-2,p) < np/5] = o(1),$$

since the Chernoff lower-tail bound gives $\mathbb{P}[\mathrm{Bin}(n-2,p) < np/5] \le \exp(-\Omega(np))$. Thus whp $\Delta/20 \le d(v) \le (1-\varepsilon)\Delta$ for all $v \in N(S)$. $\square$

Remark 2.6. The second property above implies that every vertex

$$x \in V(G) \setminus (B \cup S \cup N(S))$$

satisfies

$$d_G\big(x,\, B \cup N(B) \cup S \cup N(S) \cup N^2(S)\big) \le 1.$$

Indeed, if x had two neighbours in $B \cup N(B) \cup S \cup N(S) \cup N^2(S)$, then, since $x \notin B \cup S \cup N(S)$, each of these neighbours can be joined to a witness in $B \cup S$ by a distinct path of length at most 2. Together with x, these two witness paths produce either a path of length at most 6 with both endpoints in $B \cup S$, or a cycle of length at most 6 containing a vertex of $B \cup S \cup N(S)$, contradicting Item 2.

3 Extending a linear forest into a Hamilton cycle

In this section we show how an appropriate linear forest in a random graph can be extended into a Hamilton cycle. We start with some basic properties typically satisfied by random graphs.

Definition 3.1. For a real $C \geq 2$, a graph H is a *C-expander* if

(i) every set $X \subseteq V(H)$ with $|X| \leq |V(H)|/(2C)$ satisfies $|N_H(X)| \geq C|X|$; and

(ii) every two disjoint sets $A, B \subseteq V(H)$ with $|A|, |B| \geq |V(H)|/(2C)$ have at least one edge of H between them.

Lemma 3.2. *Let $G \sim G(n,p)$ and fix $\varepsilon > 0$ and a constant $c_0 \geq 2$. If $\log n/n \leq p \leq c_0 \log n/n$, then whp G has the following properties.*

(a) *For all integers $a, b \geq 1$ with $abp \geq 10n$, every pair of disjoint sets $A, B \subseteq V(G)$ with $|A| = a$ and $|B| = b$ have at least one edge between them.*

(b) *Let $S \subseteq V(G)$ with $|S| \geq \varepsilon n$ and suppose $d_S(v) \geq (\log\log n)^3$ for every $v \in S$. Then $G[S]$ is a $\log\log n$-expander.*

Proof. For (a), fix integers $a, b \geq 1$ and disjoint $A, B \subseteq V(G)$ of sizes a, b. Then $\mathbb{P}[e(A,B) = 0] = (1-p)^{ab} \leq e^{-pab} \leq e^{-10n}$. The number of choices of (a, b, A, B) is at most $n^2\binom{n}{a}\binom{n}{b} \leq n^2 \cdot 2^{2n}$, and $n^2 \cdot 2^{2n} \cdot e^{-10n} = o(1)$.

For (b), we show that every $X \subseteq S$ with $|X| \leq |S|/(2\log\log n)$ satisfies $|N_{G[S]}(X)| \geq |X|\log\log n$. We distinguish three cases.

Small sets: $|X| \leq (\log\log n)^2/2$. Pick any $v \in X$. By the minimum-degree assumption, v has at least $(\log\log n)^3$ neighbours in S, of which at most $|X|$ lie in X. Hence

$$|N_{G[S]}(X)| \geq (\log\log n)^3 - |X| \geq (\log\log n)^3/2 \geq |X|\log\log n.$$

Medium sets: $(\log\log n)^2/2 < |X| \leq \frac{100n}{\varepsilon \log n}$. We first show that whp every $Y \subseteq V(G)$ with $|Y| \leq y_0 := \frac{300\, n\log\log n}{\varepsilon \log n}$ spans fewer than $\alpha|Y|$ edges in G, where $\alpha := (\log\log n)^2/8$.

By a union bound, the probability that some such Y has at least $\alpha|Y|$ edges is at most

$$\sum_{2\leq y\leq y_0} \binom{n}{y}\binom{\binom{y}{2}}{\alpha y} p^{\alpha y} \;\leq\; \sum_{2\leq y\leq y_0} \left(\frac{en}{y}\right)^y \left(\frac{eyp}{2\alpha}\right)^{\alpha y} \;=\; \sum_{2\leq y\leq y_0} B(y)^y,$$

where $B(y) := \dfrac{en}{y}\left(\dfrac{eyp}{2\alpha}\right)^{\alpha}$. Factoring out the y-dependence,

$$B(y) = \left[en \cdot (ep/(2\alpha))^{\alpha}\right] \cdot y^{\alpha-1},$$

and since $\alpha > 1$ for n large, B is increasing in y, so $B(y) \leq B(y_0)$ for all $y \in [2, y_0]$. Plugging in $y = y_0$ and using $p \leq c_0 \log n/n$,

$$\frac{ey_0\, p}{2\alpha} \leq \frac{1200\, c_0\, e}{\varepsilon \log\log n},$$

so we have

$$B(y_0) \;\leq\; \frac{e\varepsilon \log n}{300\log\log n}\left(\frac{1200\, c_0\, e}{\varepsilon\log\log n}\right)^{(\log\log n)^2/8} = o(1).$$

In particular $B(y_0) < 1/2$ for n large, hence

$$\sum_{2\leq y\leq y_0} B(y)^y \;\leq\; \sum_{y\geq 2} B(y_0)^y \;\leq\; \frac{B(y_0)^2}{1 - B(y_0)} = o(1),$$

proving the claim.

Now suppose for contradiction that some $X \subseteq S$ in the medium range satisfies $|N_{G[S]}(X)| < |X| \log\log n$. Let $Y := X \cup N_{G[S]}(X)$, so

$$|Y| \le |X|(1 + \log\log n) \le 2|X| \log\log n \le \frac{200\, n \log\log n}{\varepsilon \log n} \le y_0.$$

By the minimum-degree assumption, $\sum_{v \in X} d_S(v) \ge |X|(\log\log n)^3$. On the other hand, every edge incident to a vertex of X with the other endpoint in S has that endpoint in $N_{G[S]}(X) \subseteq Y$, so

$$\sum_{v \in X} d_S(v) \;\le\; 2e_G(Y) \;<\; 2\alpha|Y| \;\le\; \frac{(\log\log n)^2}{4} \cdot 2|X| \log\log n \;=\; \frac{|X|(\log\log n)^3}{2},$$

a contradiction.

Large sets: $\frac{100n}{\varepsilon \log n} < |X| \le |S|/(2\log\log n)$. Suppose $|N_{G[S]}(X)| < |X| \log\log n$, and set $B := S \setminus (X \cup N_{G[S]}(X))$, so that $e_G(X, B) = 0$ and, for n large, $|B| \ge |S| - |X|(1+\log\log n) \ge |S| - \frac{|S|(1+\log\log n)}{2\log\log n} \ge |S|/3 \ge \varepsilon n/3$. Then

$$|X| \cdot |B| \cdot p \;\ge\; \frac{100n}{\varepsilon \log n} \cdot \frac{\varepsilon n}{3} \cdot \frac{\log n}{n} \;=\; \frac{100n}{3} \;\ge\; 10n,$$

contradicting (a).

Connectivity property. It remains to verify that every two disjoint $A, B \subseteq S$ with $|A|, |B| \ge |S|/(2\log\log n)$ have at least one edge in G between them. Since $|A|, |B| \ge \varepsilon n/(2\log\log n)$,

$$|A| \cdot |B| \cdot p \;\ge\; \left(\frac{\varepsilon n}{2\log\log n}\right)^2 \cdot \frac{\log n}{n} \;\ge\; 10n$$

for n large, and the claim follows from (a). □

In the next subsection we present the main tool of this section, the extendability technique.

3.1 The Friedman–Pippenger tree embedding technique with rollbacks

Building on the foundational tree-embedding results of Friedman and Pippenger [18] and Haxell [23], the Friedman–Pippenger technique (also known as the *extendability method*) enables the robust construction of linear-sized trees within expander graphs. The central result we use is Corollary 3.5, which builds such trees within an *m-joined* host graph.

The technique has played a key role in resolving several long-standing problems in graph theory; see, for instance, [13, 33, 14]. We will employ the framework developed in [33], with further details below.

We recall the notion of an (m, D)-extendable embedding from [33], and the auxiliary notion of an m-joined graph.

Definition 3.3. For $m \in \mathbb{N}$ with $m \ge 1$, a graph G is *m-joined* if every two disjoint sets $A, B \subseteq V(G)$ with $|A|, |B| \ge m$ have at least one edge of G between them.

Definition 3.4. Let $m, D \in \mathbb{N}$ with $D \ge 3$ and $m \ge 1$, let G be a graph, and let $S \subseteq G$ be a subgraph. We say that S is (m, D)-*extendable* if S has maximum degree at most D and

$$|(N_G(U) \cup U) \setminus V(S)| \;\ge\; (D-1)|U| - \sum_{x \in U \cap V(S)} (d_S(x) - 1) \tag{3.1}$$

for every set $U \subseteq V(G)$ with $|U| \le 2m$. (Here $N_G(U)$ denotes the external neighbourhood of U in G, so $(N_G(U) \cup U) \setminus V(S)$ counts the vertices of $U \setminus V(S)$ together with the neighbours of U lying outside $V(S)$.)

Corollary 3.5 (Corollary 3.12 in [33]). *Let $D, m, \ell \in \mathbb{N}$ satisfy $m \geq 1$ and $D \geq 3$, and set*

$$k = \left\lceil \frac{\log(2m)}{\log(D-1)} \right\rceil,$$

suppose $\ell \geq 2k+1$. Let G be an m-joined graph, and let S be an (m, D)-extendable subgraph of G with at most

$$|G| - 10\,Dm - (\ell - 2k - 1)$$

vertices. Suppose a, b are two distinct vertices of S, both with degree at most $D/2$ in S. Then there is an a,b-path P of length ℓ whose internal vertices lie outside $V(S)$, such that $S + P$ is (m, D)-extendable.

Before proving our main result, we recall the definition of Hamilton-connectivity and a recent Hamiltonicity theorem for C-expanders.

A graph is *Hamilton-connected* if for every two vertices of the graph there is a Hamilton path between them.

Theorem 3.6 ([14]). *For a sufficiently large constant C, every C-expander is Hamilton-connected.*

The following result is the key tool which allows us to extend a given linear forest F into a Hamilton cycle in a random graph, under the condition that every vertex of the random graph has many neighbours outside of $V(F)$.

Lemma 3.7. *Fix constants $C \geq 2$ and $K > 0$, let $\log n/n \leq p \leq C \log n/n$, and let $G \sim G(n, p)$. With high probability the following holds. Suppose $F \subseteq G$ is a nonempty linear forest with no isolated vertices, and such that $S := V(G) \setminus V(F)$ satisfies $|S| \geq Kn$ and $|N_S(v)| \geq 100(\log\log n)^3$ for every $v \in V(G) \setminus V(F)$ and every endpoint v of a path of F. Then there exists a Hamilton cycle in G which covers $E(F)$.*

Proof. We assume that G satisfies the properties of Lemma 3.2 (applied with the constant $c_0 := C$), together with the standard whp event $\Delta(G) = O(\log n)$ given by Lemma 2.4, and set $\varepsilon := K^2/100$. Throughout, the degree condition $|N_S(v)| \geq 100(\log\log n)^3$ will only be invoked for $v \in S$ and for endpoints of paths in F.

Split S uniformly at random into three parts S_1, S_2, S_3, each vertex assigned independently and uniformly. For each vertex v to which the hypothesis applies and each $i \in \{1, 2, 3\}$, let $E_{v,i}$ be the event that $|N_G(v) \cap S_i| < |N_S(v)|/10$. By Chernoff, $\mathbb{P}[E_{v,i}] \leq \exp(-c(\log\log n)^3) =: q$ for an absolute constant $c > 0$. The event $E_{v,i}$ depends only on the random choices for the neighbours of v, so $E_{v,i}$ is mutually independent of all events $E_{v',j}$ with v' at distance more than 2 from v in G; in particular, of all but at most $d := 3\Delta(G)^2 = O(\log^2 n)$ other events (using $\Delta(G) = O(\log n)$). Since $e \cdot q \cdot (d+1) = o(1) < 1$, the Lovász Local Lemma (see Ch. 5 in [5]) applies and, since $1 - eq \geq e^{-2eq}$, gives the quantitative bound

$$\mathbb{P}\Big[\bigcap_{v,i} \overline{E_{v,i}}\Big] \geq (1 - eq)^N \geq \exp(-2eqN) = \exp(-O(nq)),$$

where $N \leq 3n$ is the number of events. Separately, by Chernoff, the size event $\{|S_i| \geq |S|/4 \geq Kn/4 \text{ for every } i\}$ fails with probability at most $\exp(-\Omega(n))$. Since $nq = n\exp(-c(\log\log n)^3) \ll n$, we have $\exp(-O(nq)) \gg \exp(-\Omega(n))$, so a union bound gives that with positive probability both the LLL event and the size event occur simultaneously. Fix any such partition. In particular, $|S_i| \geq Kn/4$ for $i = 1, 2, 3$.

Step 1: reduce to at most εn components. If F has more than εn components, choose $T \subseteq V(F)$ containing exactly one endpoint from each path of F, so $|T| \geq \varepsilon n$. Partition T arbitrarily into two halves T_1, T_2 with $|T_1|, |T_2| \geq \varepsilon n/2$. Then $|T_1| \cdot |T_2| \cdot p \geq (\varepsilon n/2)^2 \cdot (\log n)/n \geq 10n$ for n large, so by (a) there is an edge between T_1 and T_2; adding this edge to F reduces the number of components by one. Repeat until F has at most εn components.

Step 2: reduce to at most $\frac{100n}{\varepsilon \log n}$ components, using S_1. At each step we update F to a linear forest with one fewer component and one additional vertex (drawn from S_1). Choose disjoint sets $R_1, R_2 \subseteq V(F)$ of endpoints, each of size $\frac{50n}{\varepsilon \log n}$, so that $R_1 \cup R_2$ contains at most one endpoint of any single path of F (this is possible whenever F has at least $\frac{100n}{\varepsilon \log n}$ paths, by picking one endpoint from each of $|R_1| + |R_2|$ distinct paths and partitioning). Note that $|S_1 \setminus V(F)| \geq |S_1| - \varepsilon n \geq \varepsilon n$, since at each previous step we add at most one vertex to $V(F)$ and Step 1 left F with at most εn components. For any $W \subseteq S_1 \setminus V(F)$ with $|W| \geq \varepsilon n/2$, we have

$$|R_i| \cdot |W| \cdot p \;\geq\; \frac{50n}{\varepsilon \log n} \cdot \frac{\varepsilon n}{2} \cdot \frac{\log n}{n} \;=\; 25n \;\geq\; 10n,$$

so by (a) each R_i has a neighbour in W. Hence each R_i is adjacent to more than half of $S_1 \setminus V(F)$, and by pigeonhole there is a vertex $w \in S_1 \setminus V(F)$ adjacent to both R_1 and R_2. Since the two endpoints picked up by w come from distinct paths (as $R_1 \cup R_2$ has at most one endpoint per path), joining w via these two edges merges two paths into one, reducing the component count by one.

Step 3: turn the linear forest into a single path, using S_2. After Steps 1–2, we have a linear forest F_E covering F with $V(F_E) \subseteq V(F) \cup S_1$ and at most $\frac{100n}{\varepsilon \log n}$ components. Order its paths as $P_1, \ldots, P_L$, and write a_i, b_i for the endpoints of P_i. Let I be the set of all such endpoints, regarded as a subgraph with no edges (so $d_I(x) = 0$ for $x \in I$). Set $G' := G[I \cup S_2]$. We connect b_i to a_{i+1} by vertex-disjoint paths in G' via repeated application of Corollary 3.5, with parameters $m = \lfloor n/(2 \log\log n) \rfloor$ and $D = \lfloor \log\log\log n \rfloor$.

We first check that G' is m-joined. For any two disjoint sets $A, B \subseteq V(G') \subseteq V(G)$ with $|A|, |B| \geq m$, we have $|A| \cdot |B| \cdot p \geq m^2 p \geq \frac{n^2}{4(\log\log n)^2} \cdot \frac{\log n}{n} = \frac{n \log n}{4(\log\log n)^2} \geq 10n$ for n large, so by Lemma 3.2(a) there is an edge between A and B in G (hence in G', since $A, B \subseteq V(G')$).

I is (m, D)-extendable in G'. We need to verify, for every $X \subseteq V(G')$ with $|X| \leq 2m$, that

$$|(N_{G'}(X) \cup X) \setminus V(I)| \;\geq\; (D-1)|X| - \sum_{x \in X \cap V(I)} (d_I(x) - 1) \;=\; (D-1)|X| + |X \cap V(I)|.$$

It suffices to show $|N_{S_2}(X)| \geq |X|D$, since $N_{S_2}(X) \cap V(I) = \emptyset$ implies $N_{S_2}(X) \subseteq (N_{G'}(X) \cup X) \setminus V(I)$, and $|X|D \geq (D-1)|X| + |X| \geq (D-1)|X| + |X \cap V(I)|$.

Fix such an X and set $G^* := G'[X \cup S_2]$. Note that $V(G^*) \setminus X = S_2 \setminus X$, so $N_{G^*}(X) \subseteq S_2$ and $|N_{S_2}(X)| = |N_{G^*}(X)|$. Every vertex of $V(G^*)$ has at least $10(\log\log n)^3 \geq (\log\log n)^3$ neighbours in $S_2 \subseteq V(G^*)$; since $|V(G^*)| \geq |S_2| \geq Kn/4 \geq \varepsilon n$ (using $\varepsilon \leq K/4$), Lemma 3.2(b) applied to $V(G^*)$ shows that G^* is a $\log\log n$-expander.

- If $|X| \leq |V(G^*)|/(2\log\log n)$, then the expansion property of G^* gives $|N_{S_2}(X)| = |N_{G^*}(X)| \geq |X| \log\log n \geq |X|D$.
- If $|V(G^*)|/(2\log\log n) < |X| \leq 2m$, then by the joining property of G^*, any subset of $V(G^*) \setminus (X \cup N_{G^*}(X))$ of size at least $|V(G^*)|/(2\log\log n)$ has an edge towards X . Hence

$$|N_{S_2}(X)| = |N_{G^*}(X)| \;\geq\; |V(G^*)| - |X| - \frac{|V(G^*)|}{2\log\log n} \;\geq\; \frac{Kn}{4} - \frac{2n}{\log\log n} \;\geq\; |X|D$$

for n sufficiently large (using $D = o(\log\log n)$ and $K > 0$ fixed).

Iterating Corollary 3.5. The number of (b_i, a_{i+1})-connections to find is $L - 1 \leq \frac{100n}{\varepsilon \log n}$. At each step we invoke Corollary 3.5 with path-length parameter $\ell = 2k+1$ where $k = \lceil \log(2m)/\log(D-1) \rceil$, so $\ell - 2k - 1 = 0$ and each path produced has length $\ell = 2k+1 = o(\log n)$. Thus the total number of internal vertices used over all steps is at most $L \cdot \ell = o(n)$, so at each step the current extendable subgraph has at most $|I| + o(n) = o(n) \leq |G'| - 10\,Dm$ vertices (and hence at most $|G'| - 10\,Dm - (\ell - 2k - 1)$ vertices), so Corollary 3.5 applies.

Step 4: extend the spanning path to a Hamilton cycle, using S_3. Steps 1–3 produce a path $P \subseteq V(G) \setminus S_3$ that covers $E(F)$; let x, y be its endpoints, and write $W := V(P) \setminus \{x, y\}$. The graph

$G'' := G - W$ has vertex set $V(G) \setminus W \supseteq S_3 \cup \{x, y\}$ (and may additionally contain vertices of $S_1 \cup S_2$ that were not used in Steps 2–3). The endpoints x, y remain endpoints of paths of the original forest F, since each of Steps 1–3 either adds edges between original endpoints or extends paths with internal vertices, never producing a new path endpoint; in particular the lemma's degree hypothesis applies to x and y. Every other vertex $v \in V(G'')$ lies in $S = V(G) \setminus V(F)$, so the hypothesis also applies to v. Combined with the splitting at the start of the proof, this gives $|N_G(v) \cap S_3| \geq 10(\log\log n)^3$ for every $v \in V(G'')$, hence v has at least $10(\log\log n)^3 \geq (\log\log n)^3$ neighbours inside $V(G'') \supseteq S_3$. By Lemma 3.2(b), G'' is a $\log\log n$-expander, hence is Hamilton-connected by Theorem 3.6, and so contains a Hamilton path from x to y. Uniting this path with P gives a Hamilton cycle of G covering F. $\square$

4 The sparse range

In this section we prove our main result.

Theorem 4.1. *Let $C > 0$ and let $\omega(n)$ be any function tending to infinity arbitrarily slowly. Let p be such that*

$$\frac{\log n + \log\log n + \omega(n)}{n} \leq p \leq \frac{C \log n}{n}.$$

Then $G(n, p)$ has a tight Hamilton cover whp.

Proof. Consider $G \sim G(n, p)$ with $(\log n + \log\log n + \omega(n))/n \leq p \leq C \log n/n$. We will show that whp G can be covered by $\lceil \Delta(G)/2 \rceil$ Hamilton cycles. We choose $t := \lceil 24/\alpha \rceil$ below (with α defined in the next paragraph) and set $K := 1/(3t)$, both positive constants depending only on C. Henceforth we fix G satisfying all the properties of Lemma 2.4 and Proposition 2.5. We additionally condition on two further whp events: $\delta(G) \geq 2$ (which holds since $p \geq (\log n + \log\log n + \omega(n))/n$, see e.g. [8]); and the event that, whenever $np \geq 1.01 \log n$, no vertex of G has degree less than $(\log\log n)^4$. For this last event, a Poisson-type Chernoff bound gives $\mathbb{P}[d(v) < (\log\log n)^4] \leq (e \cdot np/(\log\log n)^4)^{(\log\log n)^4} e^{-np} \leq n^{-1.01+o(1)}$, so a union bound yields probability $o(1)$.

Fix $\alpha := \alpha(C) > 0$ small enough so that $(\frac{1}{10} - \alpha)\sqrt{2/C} \geq 2\alpha$ holds (e.g. $\alpha := 1/(10^4\sqrt{C})$), and set $t := \lceil 24/\alpha \rceil$ (so $t \geq 24/\alpha$ and the bound $t(1 - 2\alpha/3)/(t - 2) \leq 1 - \alpha/2$ required below holds). Let ε be a constant with $\varepsilon \ll \min(\alpha, 1/C)$, and set $k := \lceil \Delta(G)/2 \rceil$. Let B denote the set of vertices v with $d(v) \geq (1 - \alpha)\Delta(G)$, and let S denote the set of vertices with $d(v) < (\log\log n)^4$. Note that for n large, by Lemma 2.4 we have $\Delta(G) \geq pn + (1 - o(1))\sqrt{2pn \log n}$, so

$$\begin{aligned}(1 - \alpha)\Delta(G) &\geq (1 - \alpha)pn + (1 - \alpha - o(1))\sqrt{2pn \log n} \\ &\geq pn + (1 - 1/10)\sqrt{2pn \log n}\end{aligned}$$

holds for n large. Hence B is contained in the set $B_{2.5}$ from Proposition 2.5; similarly $S = S_{2.5}$. All conclusions of Proposition 2.5 for $B_{2.5}, S_{2.5}$ therefore apply to B, S. Write $W := S \cup B \cup N(S)$.

4.1 Decomposing $G - W$ into linear forests

In the rest of the subsection, we find a collection $\mathcal{F}_M$ of linear forests which cover $G - W$ and have properties which are convenient for extending them later into Hamilton cycles. The following lemma allows us to partition the edges of $G - W$ into a constant number of subgraphs G_i, ensuring that each subgraph has approximately the same maximum degree. Additionally, each vertex $v \in V(G_i)$ retains a lot of neighbors in the set $V(G) \setminus V(G_i)$.

Lemma 4.2. *The edges of the graph $G - W$ can be partitioned into subgraphs $G_1, \ldots, G_t$ with the following properties*

- $\Delta(G_i) \leq (1 - 3\alpha/4)\frac{\Delta(G)}{t-2}$ *for each $i \in [t]$.*
- *Denote $R_i = V(G) \setminus (W \cup V(G_i))$. Then $|R_i| \geq n/2t$.*

- *Every vertex $v \in V(G) \setminus W$ satisfies $e_G(v, R_i) \geq d(v)/200t$*

Proof. We first record that $d_W(v) \leq 1$ for every $v \in V(G) \setminus W$. Indeed, v has at most one B-neighbour (else a length-2 path connecting vertices in B contradicts Proposition 2.5 item 2), at most one S-neighbour (similarly), and at most one $N(S)$-neighbour (two such w_1, w_2 with distinct S-witnesses give a length-4 path within S; with the same witness s, the 4-cycle w_1-v-w_2-s-w_1 contains $s \in S$, contradicting the cycle clause). Moreover, v cannot have neighbours in two of the sets $B, S, N(S)$: an S- and a B-neighbour give a length-2 path between B and S; an $N(S)$- and a B-neighbour give a length-3 path between B and S; and an S-neighbour together with an $N(S)$-neighbour would force $v \in N(S) \subseteq W$. So $d_W(v) \leq 1$.

Partition $V(G)$ randomly into sets $R'_1, \dots, R'_t$, where each vertex is assigned a part independently and uniformly at random. Define $R_i := R'_i \setminus W$. Colour every edge of $G[R'_i]$ uniformly at random with one of the colours in $[t] \setminus \{i\}$. For every pair $1 \leq i < j \leq t$, colour each edge of $G[R'_i, R'_j]$ uniformly at random with an element of $[t] \setminus \{i, j\}$. Denote by G_i the graph on $V(G) \setminus (W \cup R_i)$ consisting of all edges of colour i (with both endpoints in this vertex set).

Recalling that $|W| = O(\sqrt{n} \log n)$ by Proposition 2.5, by Chernoff we have that the size event $\mathcal{A} := \{|R_i| \geq n/(2t) \text{ for every } i\}$ holds with probability

$$1 - O(t) \cdot e^{-\Theta(n)} = 1 - e^{-\Theta(n)}.$$

For each $v \in V(G) \setminus W$ and each $i \in [t]$, let $A(v, i)$ be the event that $e_G(v, R_i) < d_G(v)/(200t)$ and $B(v, i)$ the event that $d_{G_i}(v) > (1 - 3\alpha/4)\Delta(G)/(t-2)$. The conditional distribution of $e_G(v, R_i)$ given $v \notin R_i$ is $\mathrm{Bin}(d_G(v) - d_W(v), 1/t)$ (since neighbours in W are removed from R_i), with mean at least $(d_G(v) - 1)/t \geq d_G(v)/(2t)$ for n large (using $d_G(v) \geq (\log \log n)^4 \to \infty$). By Chernoff, $\Pr[A(v, i)] \leq e^{-\Omega(d_G(v)/t)} \leq e^{-\Omega((\log \log n)^4/t)} = o(1/\Delta(G)^2)$ and similarly $\Pr[B(v, i)] \leq e^{-\Omega(\Delta(G)/t)} = o(1/\Delta(G)^2)$ (using that the expected degree of every vertex is at most $(1 - \alpha)\Delta(G)/t$. Each $A(v, i)$ and $B(v, i)$ depends only on the random choices at vertices of $N_G(v) \cup \{v\}$ and on the edges incident to v, hence is mutually independent of all but $O(t^2 \Delta(G)^2) = O(\log^2 n)$ other such events. With $2nt$ such events in total and each occurring with probability $o(1/\Delta(G)^2) = o(1/\log^2 n)$, the symmetric Lovász Local Lemma applies and gives

$$\Pr[\mathcal{B}] \geq \exp\bigl(-2 \cdot 2nt \cdot o(1/\log^2 n)\bigr) = \exp(-o(n)),$$

where $\mathcal{B} := \{\text{no } A(v, i) \text{ or } B(v, i) \text{ holds}\}$. Since $\Pr[\mathcal{A}^c] \leq e^{-\Theta(n)}$, we have $\Pr[\mathcal{A} \cap \mathcal{B}] \geq \Pr[\mathcal{B}] - \Pr[\mathcal{A}^c] \geq \exp(-o(n)) - e^{-\Theta(n)} > 0$, so there is a partition meeting all three properties simultaneously. $\square$

For each $i \in [t]$, applying Theorem 2.2 to G_i gives a collection of at most $(1 + \varepsilon)\frac{\Delta(G_i)}{2}$ linear forests covering G_i, for any fixed constant $\varepsilon > 0$ (we choose $\varepsilon \ll \alpha$ so that the bound below holds).

Definition 4.3. For each $i \in [t]$, let $K := \left\lceil \left(1 - \frac{2\alpha}{3}\right) \frac{\Delta(G)}{2(t-2)} \right\rceil$ and apply Theorem 2.2 to G_i (with $\varepsilon \ll \alpha$) to obtain a collection $\mathcal{F}_{G_i}$ of at most K linear forests covering G_i; we pad with empty forests (containing no edges) if needed so that $|\mathcal{F}_{G_i}| = K$ exactly. We denote the union of these collections by $\mathcal{F}_M$ and call it the *Middle collection.* By our choice of t, we have $t(1 - 2\alpha/3)/(t-2) \leq 1 - \alpha/2$, and thus

$$|\mathcal{F}_M| = t \cdot K \leq t \cdot \left(\left(1 - \frac{2\alpha}{3}\right) \frac{\Delta(G)}{2(t-2)} + 1 \right) = \frac{t(1 - 2\alpha/3)}{t - 2} \cdot \frac{\Delta(G)}{2} + t \leq \left(1 - \frac{\alpha}{2}\right) \lceil \Delta(G)/2 \rceil$$

for n large, since $\Delta(G) = \omega(t)$ absorbs the additive t. We may therefore assume that $\mathcal{F}_M$ has precisely $M := \left\lceil \left(1 - \frac{\alpha}{2}\right) \lceil \Delta(G)/2 \rceil \right\rceil$ linear forests by adding (if needed) padding forests to $\mathcal{F}_M$; each padding forest consists of a single edge of $G - W$, chosen to lie in some specific G_i (so each padding forest is associated with a well-defined unique index $i \in [t]$). We choose these padding edges to be distinct so that every edge of $G - W$ appears in at most two forests of $\mathcal{F}_M$.

Such padding edges exist whp, since we need only $O(\Delta(G)) = O(\log n)$ of them while $G - W$ has $\Omega(n \log n)$ edges. Each padding forest is the two-vertex graph whose edge set consists of one chosen padding edge.

4.2 Handling S and B

We start by covering the edges touching B with linear forests.

Lemma 4.4. *Let $k := \lceil \Delta(G)/2 \rceil$. There is a collection $\mathcal{F}_B = \{F_1^B, \dots, F_k^B\}$ of linear forests such that:*

1. *every edge incident to B is contained in at least one and at most two forests F_i^B;*
2. *for every $b \in B$ and every $i \in [k]$, we have $d_{F_i^B}(b) = 2$;*
3. *all endpoints of the forests F_i^B lie outside B.*

Proof. By Proposition 2.5 Item 2, the edges incident to B form disjoint stars. Let $k = \lceil \Delta(G)/2 \rceil$. For each $b \in B$, distribute the edges of its star among $F_1^B, \dots, F_k^B$ so that each forest gets two edges at b allowing an edge to be used twice if needed, but not more than twice; this is possible since $k \le d_G(b) \le 2k$. Then each F_i^B is a disjoint union of cherries centred in B, and the required properties are immediate. □

Now we turn to edges touching S. Recall that those are small degree vertices, and the edges touching them necessarily must be contained in many Hamilton cycles on average in the final cover. Our goal here is to cover the edges touching $S \cup N(S)$ with linear forests, but so that edges between $N(S)$ and $N^2(S)$ are not used many times.

Lemma 4.5. *Whp, for every $v \in S$, there is a collection $\mathcal{F}_v = \{F_{v,1}, \dots, F_{v,k}\}$, where $k = \lceil \Delta(G)/2 \rceil$, of linear forests contained in G such that:*

1. *every edge in G incident to a vertex of $\{v\} \cup N(v)$ is contained in at least one forest $F_{v,j}$;*
2. *every edge in G not incident to v is contained in at most 100 forests $F_{v,j}$;*
3. *for every j, every vertex of $\{v\} \cup N(v)$ has degree 2 in $F_{v,j}$;*
4. *for every j every vertex of $V(F_{v,j}) \setminus (\{v\} \cup N(v))$ has degree 1 in $F_{v,j}$ and lies in $N^2(v)$.*

Proof. We consider two regimes.

Case I: $np \le 1.01 \log n$. By Proposition 2.5 item 3, every $x \in N(v)$ satisfies

$$\log n/10 \le d(x) \le 1.1 \log n.$$

Also, by Lemma 2.4, whp

$$1.2 \log n \le k := \lceil \Delta(G)/2 \rceil \le 2 \log n$$

for n large. Since $\delta(G) \ge 2$, write

$$N(v) = \{x_1, \dots, x_r\}, \qquad 2 \le r = d(v) < (\log \log n)^4.$$

For each $i \in [r]$, put $L_i := N(x_i) \setminus \{v\}$. By Proposition 2.5 item 2, the sets L_i are pairwise disjoint subsets of $N^2(v)$: an edge inside $N(v)$ would give a triangle through v, and two vertices of $N(v)$ with a common neighbour other than v would give a 4-cycle through v.

For each $j \in [k]$, we first choose the two neighbours of v which will lie on the path through v. Let

$$A_j := \begin{cases} \{x_1, x_2\}, & j \notin \{3, \dots, r\}, \\ \{x_1, x_j\}, & 3 \le j \le r. \end{cases}$$

Thus the forests with $j = 3, \dots, r$ are responsible for covering the edges $vx_3, \dots, vx_r$, while all other forests use the edge-pair vx_1, vx_2.

Now form F_j as follows. If $A_j = \{x_a, x_b\}$, put into F_j a path

$$\ell_{a,j}\, x_a\, v\, x_b\, \ell_{b,j},$$

where $\ell_{a,j} \in L_a$ and $\ell_{b,j} \in L_b$. For every $x_i \in N(v) \setminus A_j$, put into F_j a cherry

$$\ell_{i,j}\, x_i\, \ell'_{i,j},$$

with distinct $\ell_{i,j}, \ell'_{i,j} \in L_i$.

Notice that we still did not exactly specify the vertices $\ell_{*,*}$, but merely the set L_* they belong to. It remains to choose these leaf-edges. For a fixed vertex x_i, count the number s_i of slots across F_j's requiring an edge from x_i to L_i. From the definition of the sets A_j,

$$s_1 = k, \qquad s_2 = k + r - 2, \qquad s_i = 2(k-1) + 1 \quad \text{for } i \geq 3.$$

Let $q_i := |L_i| = d(x_i) - 1$. Then

$$\log n/10 - 1 \leq q_i \leq 1.1 \log n,$$

while, for n large,

$$q_i \leq s_i \leq 5 \log n \leq 100 q_i.$$

Enumerate the edges from x_i to L_i cyclically and assign them to the s_i slots in the cyclic order, making the two slots of any cherry consecutive. Then every edge from x_i to L_i is used at least once and at most 100 times, and every cherry has two distinct leaves.

Since the sets L_i are pairwise disjoint and lie in $N^2(v)$, each F_j is a vertex-disjoint union of one path through v and cherries centred at the remaining vertices of $N(v)$. Hence each F_j is a linear forest whose endpoints lie in $N^2(v)$, and it uses no vertices outside $\{v\} \cup N(v) \cup N^2(v)$. Moreover, every vertex of $\{v\} \cup N(v)$ has degree 2 in every F_j, and every vertex outside $\{v\} \cup N(v)$ used by F_j has degree 1 and lies in $N^2(v)$.

Finally, every edge incident to v is covered: vx_1 appears in every forest, vx_2 appears in all forests with $j \notin \{3, \ldots, r\}$, and each vx_i with $i \geq 3$ appears in F_i. All edges incident to $N(v)$ but not to v are the edges from some x_i to L_i, and these are covered by the cyclic assignment above, each with multiplicity at most 100. This completes Case I.

Case II: $np \geq 1.01 \log n$. In this regime we have conditioned (at the start of the proof of Theorem 4.1) on the whp event that no vertex of G has degree less than $(\log \log n)^4$; hence $S = \emptyset$, and the conclusion is vacuous. This completes the proof. $\square$

Since we know that the vertices in S are pairwise at distance at least 5, the above lemma implies the following.

Corollary 4.6. *There exists a collection $\mathcal{F}_S = \{F_1^S, \ldots, F_k^S\}$, where $k = \lceil \Delta(G)/2 \rceil$, of linear forests such that:*

1. *every edge incident to $S \cup N(S)$ is contained in at least one forest F_i^S;*
2. *every edge between $N(S)$ and $N^2(S)$ is contained in at most 100 forests F_i^S;*
3. *every vertex of $S \cup N(S)$ has degree 2 in every F_i^S;*
4. *every vertex of $V(F_i^S) \setminus (S \cup N(S))$ has degree 1 and lies in $N^2(S)$.*

Remark 4.7. By Proposition 2.5, B and S have no path of length at most 3 between them, which implies $(B \cup N(B)) \cap (S \cup N(S) \cup N^2(S)) = \emptyset$, so $\mathcal{F}_B$ and $\mathcal{F}_S$ are vertex disjoint. Both contain precisely $\lceil \Delta/2 \rceil$ linear forests, so we can arbitrarily pair up the forests in those collections to obtain a collection $\mathcal{F}_E$ of $\lceil \Delta/2 \rceil$ linear forests (the E in the index stands for *extremal*, as we cover the vertices of smallest and highest degrees).

4.3 Joining $\mathcal{F}_M$ and $\mathcal{F}_E$

We will now merge $\mathcal{F}_M$ with $\mathcal{F}_E$ in order to create one collection $\mathcal{F}$ of $\lceil \Delta(G)/2 \rceil$ linear forests covering the edges of G, such that each endpoint in each forest has a large neighbourhood outside of that forest in G. In the end, we will transform each of these linear forests into a Hamilton cycle using Lemma 3.7, thus completing the proof. The construction of $\mathcal{F}$ proceeds as follows. Recall the notation $W = B \cup S \cup N(S)$.

First, recall that by Definition 4.3, we have $|\mathcal{F}_M| = M := \lceil (1 - \alpha/2) \lceil \Delta(G)/2 \rceil \rceil$. Take a subcollection $\mathcal{F}'_E \subseteq \mathcal{F}_E$ of size M, and fix a bijection $\phi : \mathcal{F}_M \to \mathcal{F}'_E$. For each pair (F_M, F_E) where $F_E = \phi(F_M)$, we can define the linear forest $\hat{F} := (F_M \cup F_E) \setminus Q$, where Q is the set of edges in F_M which are incident to vertices also incident to some edge of F_E. We now add all these new linear forests $\hat{F}$ to $\mathcal{F}$, so that currently, $|\mathcal{F}| = M$. Let $G' \subseteq G$ be the subgraph of G consisting of the edges in $G - W$ that are not part of any linear forest $\hat{F} \in \mathcal{F}$; specifically, these are the edges belonging to some set Q as defined earlier. Then, the following properties hold.

Claim 1. $\Delta(G') \leq 500$.

Proof. First, $d_{G'}(v) = 0$ for all $v \in W$, as the edges in Q come from forests in $\mathcal{F}_M$ with vertices in $G - W$. So fix $x \in V(G) \setminus W$. If an edge xy belongs to G', then there is a pair (F_M, F_E) with $xy \in Q \cap F_M$. There are two reasons xy can be in Q: either y is incident to an edge of F_E, or x is.

The former type appears at most once for a fixed $x \in V(G) \setminus W$. To see this, note that y lies in $N(W) \setminus W = N(B) \cup N^2(S)$ (since edges of F_E incident to $V(G) \setminus W$ have their $V(G) \setminus W$ endpoint in $N(W) \setminus W$). We have $d_G(x, N(B) \cup N^2(S)) \leq 1$ by Remark 2.6. This graph edge may occur in two forests of $\mathcal{F}_M$, because of the padding forests (see Definition 4.3), and so we count it with multiplicity at most two.

For the latter type, x being incident to an edge of some $F_E \in \mathcal{F}_E$ means $x \in N(W) \setminus W = N(B) \cup N^2(S)$, and x's neighbour along that edge lies in W; again $d_G(x, W) \leq 1$ gives at most one such edge of G, and since each edge appears in at most 100 forests of $\mathcal{F}_E$, x lies in at most 100 such forests. For each such forest, at most 2 edges incident to x in F_M can land in Q. Together this gives $d_{G'}(x) \leq 1 + 100 \cdot 2 = 201$, so $\Delta(G') \leq 500$, with room to spare. □

Let $G'_i := G' \cap G_i$ for each $1 \leq i \leq t$ with G_i's from Lemma 4.2. We will now add linear forests to $\mathcal{F}$ which cover the edges in G', as well as the edges used in $\mathcal{F}''_E := \mathcal{F}_E \setminus \mathcal{F}'_E$. We do this so that at the end we have $|\mathcal{F}| \leq \lceil \Delta(G)/2 \rceil$ as desired. Let us partition $\mathcal{F}''_E$ arbitrarily into t collections $\mathcal{F}''_E(1), \ldots, \mathcal{F}''_E(t)$ each of size at least $\lfloor |\mathcal{F}''_E|/t \rfloor \geq \lfloor (\lceil \Delta(G)/2 \rceil - M)/t \rfloor$, which exceeds $4 \cdot 100 \cdot \max\{1, \Delta(G')\} + 1$ (since $\lceil \Delta/2 \rceil - M \geq (\alpha/2) \lceil \Delta/2 \rceil - O(1)$, $t = t(C)$ is constant, and $\Delta(G') \leq 500$). For each $i \leq t$, we apply Lemma 2.1 with $H_2 := G'_i$ and the collection $\mathcal{F} := \mathcal{F}''_E(i)$, and with H_1 defined as the subgraph of G consisting of the edges of $\mathcal{F}''_E(i)$.

Every vertex in $v \in V(H_2) \subseteq V(G) \setminus W$ has at most one neighbour in $H_1 \subset W \cup N_G(W)$ by Remark 2.6. Furthermore, trivially each $v \in V(H_2)$ has degree at most $\Delta(G') \leq 500$ in H_2 by Claim 1.

By the construction of $\mathcal{F}_S$ and $\mathcal{F}_B$, every edge of H_1 incident to $V(H_2)$ appears in at most $\ell := 100$ forests of $\mathcal{F}''_E(i)$: such an edge is either a B-$N(B)$ edge, used at most twice, or an $N(S)$-$N^2(S)$ edge, used at most 100 times. Setting $d := \max\{1, \Delta(G')\}$, the hypothesis $|\mathcal{F}''_E(i)| \geq 4d\ell + 1$ holds for n large. So Lemma 2.1 finds a collection $\mathcal{F}(i)$ of $|\mathcal{F}''_E(i)|$ linear forests covering the edges of G'_i and those used in $\mathcal{F}''_E(i)$. We add all the collections $\mathcal{F}(i)$ to $\mathcal{F}$; then $|\mathcal{F}| = M + |\mathcal{F}''_E| \leq \lceil \Delta(G)/2 \rceil$ and the linear forests in $\mathcal{F}$ cover all edges of G: the $\hat{F}$'s cover $\mathcal{F}'_E$ together with all $\mathcal{F}_M$-edges except those in G', while the $\mathcal{F}(i)$'s cover G' together with $\mathcal{F}''_E$.

Lemma 4.8. *With $K := 1/(3t)$, every $F \in \mathcal{F}$ satisfies the conditions of Lemma 3.7: $|V(G) \setminus V(F)| \geq Kn$, and $|N_G(v) \setminus V(F)| \geq 100(\log \log n)^3$ for every $v \in V(G) \setminus W$ (which includes every endpoint of a path in F).*

Proof. Set $K := 1/(3t)$, a positive constant depending on C (since $t = t(C)$). First, recall that each $F \in \mathcal{F}$ is such that $F[V(G) \setminus W] \subseteq G_i$ for some $i \in [t]$ (the padding forests in $\mathcal{F}_M$ were chosen to be single edges of $G - W$, each of which lies in a unique G_i that we record; the $\hat{F}$'s inherit the index from their F_M component; the $\mathcal{F}(i)$'s use index i). So $R_i \setminus (N(B) \cup N^2(S)) \subseteq V(G) \setminus V(F)$. Further, by Lemma 4.2 we have $|R_i| \geq n/(2t)$, by Proposition 2.5 we have $|B|, |S| \leq \sqrt{n}$, and by Lemma 2.4 we have $\Delta(G) = O(\log n)$. Combining these,

$$|V(G) \setminus V(F)| \;\geq\; |R_i \setminus (N(B) \cup N^2(S))| \;\geq\; n/(2t) - O(\log^2 n)(|B| + |S|) \;\geq\; n/(2t) - o(n) \;\geq\; Kn,$$

using $K = 1/(3t) < 1/(2t)$. The hypothesis $|V(G) \setminus V(F)| \geq Kn$ of Lemma 3.7 therefore holds. For the second part, fix $v \in V(G) \setminus W$ (every endpoint of a path of F lies in $V(G) \setminus W$: by construction the endpoints of $\hat{F}$ and of $\mathcal{F}(i)$-forests lie there, since vertices of W are internal in every $\mathcal{F}_E$-forest and $\mathcal{F}_M$-forests are contained in $G - W$). By Remark 2.6, for every $v \notin W$:

$$e_G\big(v, R_i \setminus (N(B) \cup N^2(S))\big) \;\geq\; e_G(v, R_i) - 1 \;\geq\; d(v)/(200t) - 1.$$

Since $v \notin S$ we have $d(v) \geq (\log\log n)^4$, and $t = t(C)$ is constant, so for n sufficiently large $d(v)/(200t) - 1 \geq 100(\log\log n)^3$. Therefore $|N_G(v) \setminus V(F)| \geq e_G(v, R_i \setminus (N(B) \cup N^2(S))) \geq 100(\log\log n)^3$. □

4.4 Extending forests in $\mathcal{F}$ into Hamilton cycles

To finish the proof, we want to extend the collection $\mathcal{F}$ of at most $\lceil \Delta(G)/2 \rceil$ linear forests covering the edges of G into a Hamilton cover. Each $F \in \mathcal{F}$ is nonempty: the padding forests are single edges of $G - W$, each $\hat{F}$ contains all of its associated F_E-edges (which are nonempty as F_E pairs an $\mathcal{F}_B$-cherry with an $\mathcal{F}_S$-component), and each $\mathcal{F}(i)$-forest contains the edges of some $F \in \mathcal{F}''_E(i)$. We may further assume each $F \in \mathcal{F}$ has no isolated vertices by removing such vertices from $V(F)$; this changes $S := V(G) \setminus V(F)$ only by enlarging it, so the hypotheses of Lemma 4.8 are preserved. Then Lemma 4.8 shows that each $F \in \mathcal{F}$ satisfies the conditions of Lemma 3.7, so we can apply this lemma to extend each linear forest to a Hamilton cycle in G. This gives a covering of G with $\lceil \Delta(G)/2 \rceil$ Hamilton cycles, as required. □

5 The very dense range

In this section we close the remaining gap near $p = 1$. We write $q = 1 - p$. Recall that $q \leq n^{-1/8}$, and $qn^2 \to \infty$;

We shall use the following Hamilton decomposition theorem for dense regular graphs due to Kühn and Osthus [30].

Theorem 5.1. *For every $\gamma > 0$ there exists n_0 such that every even-regular*[1] *graph J on $n \geq n_0$ vertices with $\delta(J) \geq (1/2 + \gamma)n$ has a Hamilton decomposition.*

It is enough to prove the following.

Theorem 5.2. *Let $\varepsilon > 0$, and let $q = q(n)$ satisfy $q \leq n^{-\varepsilon}$ and $qn^2 \to \infty$. Then whp $G(n, 1-q)$ has a tight Hamilton cover.*

We first regularise G by adding extra copies of edges already present in G.

Lemma 5.3. *Let $G \sim G(n, 1-q)$, where $q \leq n^{-\varepsilon}$ and $qn^2 \to \infty$. Set $k := \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Then whp there exists a multigraph R, with every edge of R lying in $E(G)$, such that*

$$d_R(v) = 2k - d_G(v) \qquad \text{for every } v \in V(G).$$

Moreover, $\Delta(R) = o(n)$.

[1] A graph is even-regular if it is d-regular for an even integer d.

Proof. Let $H := \overline{G}$, and write $\delta := \delta(H)$ and $\Delta := \Delta(H)$. Since $\Delta(G) = n - 1 - \delta$, we have

$$2k = n - 1 - \delta + \xi$$

for some $\xi \in \{0, 1\}$. Thus the required degree in R is

$$b(v) := 2k - d_G(v) = d_H(v) - \delta + \xi.$$

Let $B := \sum_{v \in V(G)} b(v)$. Notice that B is even, since $B = 2kn - 2e(G)$. We shall use the following standard estimates for $H \sim G(n, q)$. Since $q = o(1)$ and $qn^2 \to \infty$, whp $\Delta = o(n)$ and

$$(\Delta + 1)(\Delta - \delta + 1) = o(B). \tag{1}$$

Indeed, if $nq \le 2 \log n$, then whp $B = \Omega(e(H))$ and $\Delta^2 = o(e(H))$. If $nq > 2 \log n$, then the usual estimates for the extreme degrees of $G(n, q)$ whp give $\Delta - \delta = O(\sqrt{nq \log n})$ and $B \ge cn\sqrt{nq \log n}$ for some absolute constant $c > 0$, while $\Delta = O(nq)$. Hence

$$(\Delta + 1)(\Delta - \delta + 1) = O(nq\sqrt{nq \log n}) = o(n\sqrt{nq \log n}) = o(B),$$

as $q = o(1)$.

Now create $b(v)$ labelled stubs at each vertex v. Define an auxiliary graph Γ on these B stubs by joining two stubs if they belong to distinct vertices u, v with $uv \in E(G)$. A perfect matching in Γ gives precisely the desired multigraph R.

A stub at v is non-adjacent only to the other stubs at v, and to stubs at vertices $u \in N_H(v)$. Therefore its degree in Γ is at least

$$B - b(v) - \sum_{u \in N_H(v)} b(u) \ge B - (\Delta + 1)(\Delta - \delta + 1) > B/2$$

by (1). Hence Γ has a perfect matching, for instance by Dirac's theorem. This matching defines R. Finally, whp we have

$$\Delta(R) = \max_v b(v) \le \Delta - \delta + 1 = o(n).$$

□

In particular, for R given in the previous lemma, we know that $G + R$ is a $2k$-regular multigraph. We also need the following fact which allows us to complete matchings to Hamilton cycles in very dense graphs, and which follows for example from Theorem 1 in [35]

Lemma 5.4. *Let J be a graph on n vertices with $\delta(J) \ge n - s$, where $s = o(n)$. Let M be a matching on $V(J)$. Then, for n large, there is a Hamilton cycle in $J \cup M$ containing every edge of M.*

Now we are ready for the proof of the main result of this section.

Proof. [Proof of Theorem 5.2] Let $G \sim G(n, 1 - q)$, and let R be the multigraph supplied by Lemma 5.3. By Shannon's theorem, the edges of R can be properly coloured with $r \le \frac{3}{2}\Delta(R) = o(n)$ colours. Thus $E(R) = M_1 \cup \cdots \cup M_r$, where each M_i is a matching in G.

We cover the edges of the regular multigraph $G + R$, one Hamilton cycle at a time: at step i, the Hamilton cycle contains the matching M_i. Start with $J_0 := G$. Suppose J_{i-1} has been defined. Since

$$\delta(J_{i-1}) \ge \delta(G) - 2r \ge n - 1 - \Delta(\overline{G}) - 2r = n - o(n),$$

Lemma 5.4 gives a Hamilton cycle C_i containing all edges of M_i, with all other edges lying in J_{i-1}. We regard the edges of M_i as extra edges coming from R, and delete from J_{i-1} only the remaining edges of C_i. Let the resulting graph be J_i.

After all r steps, for every vertex v,

$$d_{J_r}(v) = d_G(v) - 2r + d_R(v) = d_G(v) - 2r + 2k - d_G(v) = 2k - 2r.$$

Thus J_r is an even-regular simple graph. Moreover, $2k - 2r = n - o(n)$, so for n large we have $\delta(J_r) \ge 2n/3$. By Theorem 5.1, J_r decomposes into $k - r$ Hamilton cycles.

Together with $C_1, \ldots, C_r$, these cycles form a collection of exactly $k = \lceil \Delta(G)/2 \rceil$ Hamilton cycles. They cover every edge of G: the decomposition of J_r covers all base edges not used during the peeling process, while the cycles C_i cover the remaining base edges and use the edges of R only as repeated copies. Hence G has a tight Hamilton cover. □

Remark 5.5. The condition $qn^2 \to \infty$ is the natural endpoint for a whp statement. If $q = c/n^2$ and n is odd, then with probability bounded away from zero the complement of $G(n, 1-q)$ consists of exactly one edge, say xy. Then $G = K_n - xy$ has no tight Hamilton cover. Indeed, $k = \lceil \Delta(G)/2 \rceil = (n-1)/2$, so k Hamilton cycles altogether have only one more edge-occurrence than $|E(G)|$. But the two vertices x, y both have odd degree in G, and each would need a repeated incident edge; since $xy \notin E(G)$, these repeated edges must be distinct, a contradiction.

6 Linear arboricity

Recall that $\mathrm{la}(G)$ denotes the minimum number of linear forests whose union covers $E(G)$.

Proof. [Proof of Theorem 1.3] We split into ranges of p.

First suppose $p \geq 2\log n/n$ and $1 - p \gg n^{-2}$. In this range the Hamilton-cover result gives the desired conclusion by the standard deduction from Hamilton covers to linear arboricity; see, for instance, the proof of the linear-arboricity corollary in [21].

Next suppose $1 - p = O(n^{-2})$. Then whp $\overline{G}$ has $o(n)$ edges, and in particular G has a vertex of degree $n-1$. Hence $\Delta(G) = n - 1$. Furthermore, by the standard Walecki construction [6] the linear arboricity of K_n is $\lceil n/2 \rceil$, meaning that K_n decomposes into $\lceil n/2 \rceil$ linear forests. Intersecting these forests with $E(G)$ gives $\lceil n/2 \rceil = \lceil (\Delta(G) + 1)/2 \rceil$ linear forests covering G.

Now we consider $c/n \leq p \leq 2\log n/n$ for any fixed $c > 0$. Fix a sufficiently small constant $\alpha > 0$, put $k := \lceil \Delta(G)/2 \rceil$, and let B be the set of vertices of degree at least $(1 - \alpha)\Delta(G)$. By a high-degree separation result, similar to Proposition 2.5, whp B is independent, the stars from B are vertex-disjoint, $N(B)$ is independent and each $v \notin B$ has at most one neighbour in $N(B)$ (indeed, such a result follows from a simple first moment argument). We condition on these properties.

We first cover the edges incident to B. For each $b \in B$, enumerate $N(b) = \{x_1^{(b)}, \ldots, x_{d(b)}^{(b)}\}$. For $j \in [k]$, take the cherry at b using the two edges $bx_{2j-1}^{(b)}$ and $bx_{2j}^{(b)}$, with indices read cyclically modulo $d(b)$. Taking the union over all $b \in B$ gives linear forests $F_1^B, \ldots, F_k^B$. These forests cover every edge incident to B, and every such edge is used at most twice.

Let G_0 be obtained from G by deleting all edges incident to B. Since every vertex outside B has degree less than $(1 - \alpha)\Delta(G)$, we have $\Delta(G_0) \leq (1 - \alpha)\Delta(G)$. Applying the approximate linear arboricity theorem (Theorem 2.2) with a small enough error term, we decompose G_0 into $m \leq (1 - \alpha/3)k$ linear forests $A_1, \ldots, A_m$.

For each $i \in [m]$, pair A_i with F_i^B. Let Q_i be the set of edges of A_i incident to an endpoint of F_i^B, and define $\widehat{F}_i := F_i^B \cup (A_i \setminus Q_i)$. This is a linear forest: after removing Q_i, no endpoint of F_i^B is incident to an edge of A_i.

It remains to cover $Q := \bigcup_{i \leq m} Q_i$, using the unused forests $F_{m+1}^B, \ldots, F_k^B$. We first note that $\Delta(Q) \leq 4$. Indeed, a vertex $x \in N(B)$ is an endpoint in at most two forests F_i^B, because the unique edge from x to B is used at most twice; in each corresponding forest A_i, at most two edges incident to x are deleted. Thus $d_Q(x) \leq 4$. Every vertex outside $N(B)$ has at most one neighbour in $N(B)$, by the same short path/cycle exclusion, and hence has degree at most one in Q.

We now apply Lemma 2.1. Let

$$\mathcal{F}_{\mathrm{rem}} := \{F_{m+1}^B, \ldots, F_k^B\}, \qquad H_1 := \bigcup_{F \in \mathcal{F}_{\mathrm{rem}}} F, \qquad H_2 := Q.$$

We apply the lemma with $\mathcal{F} = \mathcal{F}_{\mathrm{rem}}$, $d = 4$, and $\ell = 2$.

Let us verify its hypotheses. The collection $\mathcal{F}_{\mathrm{rem}}$ covers $E(H_1)$ by definition, and every edge of H_1 appears in at most two forests of $\mathcal{F}_{\mathrm{rem}}$, since every edge incident to B appears in at most two forests F_j^B. Also, by the preceding paragraph, $\Delta(H_2) = \Delta(Q) \leq 4$.

It remains to check the degree condition for H_1 at vertices of $V(H_2)$. If $x \in V(Q) \cap N(B)$, then x has a unique neighbour in B, because the stars from B are vertex-disjoint. Hence $d_{H_1}(x) \leq 1$. If $x \in V(Q) \setminus N(B)$, then $x \notin B$ and x is incident to no edge of H_1, since H_1 consists only of edges between B and $N(B)$. Thus $d_{H_1}(x) \leq 1$ for every $x \in V(H_2)$. Therefore every vertex of $V(H_2)$ has degree at most 4 in both H_1 and H_2.

Finally, $|\mathcal{F}_{\mathrm{rem}}| = k - m \geq \alpha k/3 \to \infty$, so for n large $|\mathcal{F}_{\mathrm{rem}}| \geq 4d\ell + 1 = 33$. Thus Lemma 2.1 gives $k - m$ linear forests covering $H_1 \cup Q$. Together with $\widehat{F}_1, \ldots, \widehat{F}_m$, this gives k linear forests covering all edges of G.

For the remaining regime, suppose $p = o(1/n)$. Then whp G is a forest. For each component T of G, greedily edge-colour T using colours $[\,\Delta(G)\,]$. Pair the colour classes as $\{1, 2\}, \{3, 4\}, \ldots$. In each component, the union of two colour classes has maximum degree at most two and is acyclic, hence is a linear forest. Taking the union over all components, each pair of colours gives a linear forest in G. Thus G decomposes into at most $\lceil \Delta(G)/2 \rceil$ linear forests. □

7 Concluding remarks

We close by explaining why the sparse-range proof also gives Theorem 1.2. The only extra point is that τ_2 is a random time, while the estimates in Section 4 are stated for a fixed binomial random graph.

Let $\xi = \xi(n) \to \infty$ grow sufficiently slowly, and put $m_\pm = \frac{n}{2}(\log n + \log\log n \pm \xi)$. By the standard hitting-time theorem for minimum degree two, whp $m_- \leq \tau_2 \leq m_+$. We claim that, whp, every graph G_m with $m_- \leq m \leq m_+$ satisfies all deterministic hypotheses used in the proof of Theorem 4.1, except possibly the condition $\delta(G_m) \geq 2$.

Indeed, throughout this interval the edge density is $\Theta(\log n/n)$. The estimates used in Section 4 are Chernoff-type degree estimates, expansion and joining estimates, and first-moment bounds excluding short local configurations around exceptional vertices. The same estimates hold uniformly in the random graph process. To see this, put $G^- = G_{m_-}$ and $G^+ = G_{m_+}$. Since $G^- \subseteq G_m \subseteq G^+$ for every $m \in [m_-, m_+]$, monotone lower-bound properties are inherited from G^-, and monotone upper-bound or exclusion properties are checked in G^+. For the non-monotone exceptional sets, we control fixed supersets: the low-degree set at any time is contained in the low-degree set of G^-, while the high-degree set at any time is contained, after slightly weakening constants, in a high-degree set defined using G^+. The usual first-moment estimates show that these fixed sets are small and well separated in G^+. Thus the deterministic hypotheses of the sparse construction hold simultaneously throughout the window.

Now consider G_{τ_2}. On the high-probability event addressed above, G_{τ_2} satisfies all deterministic hypotheses of the sparse construction, and the remaining hypothesis $\delta(G_{\tau_2}) \geq 2$ holds by definition of τ_2. Therefore the construction from Section 4 applies verbatim to G_{τ_2}: it produces exactly $\lceil \Delta(G_{\tau_2})/2 \rceil$ linear forests covering all edges of G_{τ_2}, and then extends them to Hamilton cycles using the reserved random structure. This gives a tight Hamilton cover of G_{τ_2} whp. The algorithmic statement follows from the same implementation discussed in the appendix.

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A Algorithmic aspects

In this appendix we record the algorithmic form of the proof. The only purpose of this appendix is to justify polynomial-time constructivity; we do not reprove the probabilistic estimates. The algorithm is allowed to fail if a required auxiliary object cannot be found. On the whp event analysed in the corresponding existential proof, each such object exists and the algorithmic subroutines below find it.

A.1 General algorithmic ingredients

We shall use the following standard algorithmic facts throughout.

(i) Maximum matchings, perfect matchings in bipartite graphs, network flows, f-factors, decompositions of an even graph into 2-factors, and the elementary greedy or augmenting-path searches used in the proof can all be carried out in polynomial time.

(ii) For the applications of the Lovász local lemma, we use the Moser–Tardos resampling algorithm [34].

(iii) The approximate linear-arboricity theorem used in Theorem 2.2 may be implemented in randomized polynomial time. Indeed, the proof of Ferber–Fox–Jain [15] gives a probabilistic polynomial-time algorithm for the corresponding decomposition into linear forests. This is more than sufficient for the applications of Theorem 2.2 in Theorem 4.1.

(iv) In every application of the Friedman–Pippenger/extendability mechanism inside a sparse random reservoir, we use the algorithmic form of the Draganić–Krivelevich–Nenadov roll-back method [13]. In particular, the prescribed vertex-disjoint connecting paths required in Lemma 3.7 and in the analogous extension lemma of [11] can be found in deterministic polynomial time, with essentially the same parameter choices as in the corresponding existential argument. After the connections have been found, the remaining Hamilton path/cycle in the reservoir is found by the algorithmic version of the Hamiltonicity result in [14] for robust expanders.

(v) The Hamilton decomposition theorem for dense regular robust expanders due to Kühn and Osthus is algorithmic in the form needed here: the decomposition can be found in polynomial time [30]. Since dense regular graphs of minimum degree larger than $(1/2 + \gamma)n$ are robust expanders, this applies to the dense even-regular remainder appearing in Theorem 5.2.

The next two lemmas record the algorithmic form of the Hamilton-packing results used to prove the covering results in the intermediate regimes. We only indicate the points which are not completely formal from the published proofs.

Lemma A.1 (Algorithmic Krivelevich–Samotij packing result)**.** *The Hamilton-packing theorem of Krivelevich–Samotij [29] can be implemented by a randomized polynomial-time algorithm in its stated range of p. That is, given $G \sim G(n,p)$, the algorithm whp outputs $\lfloor \delta(G)/2 \rfloor$ edge-disjoint Hamilton cycles.*

Proof sketch. Most steps in the proof are already constructive: the pruning steps are explicit, the required bipartite regular subgraphs are found by standard max-flow formulations for bipartite k-factors, regular bipartite graphs are decomposed into perfect matchings by repeated matching, and the final rotation-extension stage is implemented by the usual Pósa rotation search.

The only point which is not literally algorithmic as written occurs in the proof of Lemma 3.6 of [29], just before Claim 3.11, where the proof chooses a uniformly random k-regular bipartite subgraph and a uniformly random ordered decomposition of it. Exact sampling from these objects is unnecessary. In the notation of that proof, let $X = G[A_1^1, A_2^1]$ be the cross graph. Randomly relabel the two bipartition classes, apply any fixed deterministic polynomial-time routine which finds the required almost-spanning k-factor of the relabelled graph and decomposes it into k perfect matchings, and then undo the relabelling. Finally, as in the original proof, complete each matching N_i uniformly on the uncovered vertices to a perfect matching N_i' of the complete bipartite graph.

The binomial law of the cross graph, even after conditioning on the invariant success event that the required k-factor exists, is invariant under side-preserving relabellings. Therefore, for each fixed i, the completed matching N_i' obtained by the relabelling procedure is an exact uniform perfect matching between A_1^1 and A_2^1. It is also independent of the previously exposed internal matching M_i, since M_i is determined by the inside graphs and N_i' by the independent cross graph and auxiliary randomness. This uniform marginal distribution is precisely the input used in Claim 3.12 of [29]; no uniformity of the whole k-regular subgraph, nor independence between the different N_i', is used. Thus the path-cover construction and the subsequent Hamilton-cycle extension are randomized polynomial-time with the same whp success probability as in the published proof. □

Lemma A.2 (Algorithmic Knox–Kühn–Osthus 2-factor selection)**.** *The Hamilton-packing theorem from Knox–Kühn–Osthus [27] can be implemented in randomized polynomial time. In particular, every invocation of the 2-factor selection step in the proof of Knox–Kühn–Osthus [27], in particular Lemmas 26 and 29, Corollary 27 and the use of these results inside Lemma 47 of that paper, can be implemented by a randomized polynomial-time algorithm, with failure probability n^{-A} for any fixed $A > 0$.*

Proof. We describe the replacement for Lemma 26 of [27]; the remaining uses are identical. In that lemma one orients an even-regular graph and obtains an r_B-regular bipartite graph B. A perfect matching in B corresponds to a directed 2-factor in the original graph. Let $\mathcal{P}(B)$ denote the set of perfect matchings of B, and let $\mathcal{P}_{\mathrm{bad}}(B)$ be the set of those matchings whose corresponding 2-factor has too many cycles. The proof of Lemma 24 in [27] gives the stronger estimate

$$|\mathcal{P}_{\mathrm{bad}}(B)| \leq \frac{\ell_0}{n}\left(\frac{r_B}{n}\right)^n n!,$$

where $\ell_0/n = o(1)$ in the parameter range of the lemma. On the other hand, the Egorychev–Falikman lower bound gives

$$|\mathcal{P}(B)| \geq \left(\frac{r_B}{n}\right)^n n!.$$

Consequently a uniformly random perfect matching of B is bad with probability at most $\ell_0/n = o(1)$.

We now sample from $\mathcal{P}(B)$ using the fully polynomial almost-uniform sampler of Jerrum–Sinclair–Vigoda [26]. Choose total variation error $\delta = n^{-A-3}$. A sample is bad with probability at most $\ell_0/n + \delta$, which is less than $1/2$ for n large. The algorithm samples a perfect matching, converts it to the corresponding directed 2-factor, counts its cycles in linear time, and accepts exactly when the required cycle bound holds. Repeating $O(A \log n)$ times gives failure probability at most n^{-A-2}.

Lemma 29 of [27] is handled in the same way. The preliminary random labelling of the edges of the prescribed matching, and the random partition used in Lemma 28, have checkable degree conditions and succeed with probability $1 - o(1)$; the completions to perfect matchings or pseudomatchings are standard polynomial-time matching problems. Once the auxiliary bipartite graph has been formed, the same Lemma 24 counting estimate and the same almost-uniform sampling-and-rejection procedure produce the required matching. The odd-order pseudomatching case is reduced in the proof of Lemma 29 to the same bipartite matching problem by treating the two neighbours of the unique degree-2 vertex as a single matched pair.

Corollary 27 of [27] applies Lemma 26 repeatedly to decompose an even-regular graph into 2-factors. Since there are only polynomially many iterations, the above error probability and a union bound give the claimed n^{-A} failure probability. Lemma 47 then uses these 2-factors together with explicit rotation-extension and reservoir operations; replacing each 2-factor selection by the sampler above makes the whole lemma randomized polynomial time. □

A.2 The sparse range

We first consider the range handled by Theorem 4.1, namely from the Hamiltonicity threshold up to $C \log n/n$ for a fixed constant C. The proof of Theorem 4.1 is already constructive once the ingredients in Appendix A.1 are used. The algorithm computes the exceptional sets B, S, W, constructs the B-cherry forests and the S-local forests by the explicit procedures in the proof, obtains the middle linear forests by the algorithmic form of Theorem 2.2, and performs the merging step of Lemma 2.1 greedily. All these operations are polynomial-time graph operations.

It remains only to extend each final linear forest to a Hamilton cycle. The proof applies Lemma 3.7. For the algorithmic version, the partition of the unused set S into reservoirs is produced by the same random choice, or by Moser–Tardos when the Local lemma formulation is used. Steps 1 and 2 of Lemma 3.7 are explicit searches for edges or common neighbours; if such an edge or vertex is not found, the algorithm returns failure. Step 3 is a prescribed disjoint-path embedding in a sparse expander, so we use the Draganić–Krivelevich–Nenadov algorithmic roll-back version of the Friedman–Pippenger method which works for robust expanders, which applies in our setting. Step 4 is again a constructive spanning path-finding problem in the remaining expander/reservoir, handled by the algorithmic embedding routine in [14]. Hence Theorem 4.1 is algorithmic.

A.3 The previously known covering range

We next explain why the range covered by the previously known Hamilton-cover results is algorithmic as well.

For $C \log n/n \leq p \leq n^{-2/3}$, Draganić–Glock–Munhá Correia–Sudakov [11] prove the optimal covering theorem by first using an optimal packing of $\lfloor \delta(G)/2 \rfloor$ edge-disjoint Hamilton cycles, then decomposing the leftover graph into well-behaved linear forests, and finally extending each such forest to a Hamilton cycle. The packing input is algorithmic: in the very sparse part one uses Lemma A.1, while from the polylogarithmic range onward one uses Lemma A.2. The leftover decomposition uses the same polynomial-time ingredients as above, in particular matching routines and the algorithmic approximate linear-arboricity theorem. The final forest-extension step is a simpler version of the sparse-reservoir path embedding as in Lemma 3.7, and is implemented using the Draganić–Krivelevich–Nenadov algorithmic roll-back method. Thus the theorem of [11] is algorithmic in this range.

For $\log^{117} n/n \leq p \leq 1 - n^{-1/8}$, Hefetz–Kühn–Lapinskas–Osthus [24] prove the exact cover theorem using the packing/decomposition technology of Knox–Kühn–Osthus [27]. The deterministic pseudorandom part of that proof consists of polynomial-time operations once the required 2-factors and Hamilton cycles are found. By Lemma A.2, the permanent-based 2-factor choices in the Knox–Kühn–Osthus argument can be made by almost-uniform sampling and rejection, so the HKLO cover theorem is randomized polynomial-time in the whole of its range. Since $n^{-2/3} \gg \log^{117} n/n$, the DGMS and HKLO ranges overlap and together cover all p from a sufficiently large constant times $\log n/n$ up to $1 - n^{-1/8}$.

A.4 The very dense range

Finally consider $p = 1 - q$ with $q \leq n^{-1/8}$ and $qn^2 \to \infty$, the range handled by Theorem 5.2. The proof is constructive. The regularising multigraph of Lemma 5.3 is obtained by finding the required matching in the auxiliary deficiency graph, which is a polynomial-time matching problem. Decomposing this multigraph into matchings is a standard polynomial-time edge-colouring/matching task, and the prescribed matching-to-Hamilton-cycle step in the proof is explicit: contract the edges of the prescribed matching into directed blocks, find the required Hamilton path or cycle in the dense auxiliary graph by any standard polynomial-time Dirac type argument, and then expand the blocks. After these Hamilton cycles have been removed, the remaining even-regular dense graph satisfies the hypotheses of Theorem 5.1; by the algorithmic Kühn–Osthus theorem cited in Appendix A.1, its Hamilton decomposition is found in polynomial time.