A Lower Bound on the Density of Sphere Packings via Graph Theory

Michael Krivelevich
Tel Aviv University
Tel Aviv 69978, Israel
krivelev@post.tau.ac.il

Simon Litsyn
Tel Aviv University
Tel Aviv 69978, Israel
litsyn@eng.tau.ac.il

Alexander Vardy
University of California, San Diego
9500 Gilman Drive, La Jolla, CA 92093, U.S.A.
vardy@kilimanjaro.ucsd.edu

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Abstract

Using graph-theoretic methods we give a new proof that for all sufficiently large n, there exist sphere packings in $\mathbb{R}^n$ of density at least $cn2^{-n}$ exceeding the classical Minkowski bound by a factor linear in n. This matches up to a constant the best known lower bound on the density of sphere packings due to Ball. However, our proof is very different from the earlier constructions of Minkowski, Hlawka, Rogers, and Ball. Moreover, this proof makes it possible to describe the points of such a packing with complexity $\exp(n \log n)$, which is significantly lower than in the other approaches.

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1. Introduction

A sphere packing $\mathcal{P}$ in $\mathbb{R}^n$ is a collection of non-intersecting open spheres of equal radii, and its density $\Delta(\mathcal{P})$ is the fraction of space covered by their interiors. Let

$$\Delta_n \stackrel{\text{def}}{=} \sup_{\mathcal{P}\subset\mathbb{R}^n} \Delta(\mathcal{P})$$

where the supremum is taken over all packings in $\mathbb{R}^n$. A celebrated theorem of Minkowski states that $\Delta_n \geqslant \zeta(n)/2^{n-1}$ for all $n \geqslant 2$. Since $\zeta(n) = 1 + o(1)$, the asymptotic behavior of the Minkowski bound [9] is given by $\Omega(2^{-n})$. Asymptotic improvements of the Minkowski bound were obtained by Rogers [10], Davenport and Rogers [4], and Ball [2], all of them being of the form $\Delta_n \geqslant cn2^{-n}$ where $c > 0$ is an absolute constant. The best currently known lower bound on Δ_n is due to Ball [2], who showed that there exist lattice packings with density at least $2(n-1)2^{-n}\zeta(n)$. In this note, we use results from graph theory to prove the following theorem.

Theorem 1. *For all sufficiently large n, there exists a sphere packing $\mathcal{P}_n \subset \mathbb{R}^n$ such that*

$$\Delta(\mathcal{P}_n) \geqslant 0.01n2^{-n} \tag{1}$$

Moreover, the spheres in $\mathcal{P}_n$ can be described using a deterministic procedure whose complexity is at most $O(2^{\gamma n \log_2 n})$ for an absolute constant γ.

Although the constant $c = 0.01$ in (1) is not as high as in the bounds of Rogers $(c = 0.74)$, Davenport-Rogers ($c = 1.68$), and Ball $(c = 2)$, Theorem 1 still provides an improvement upon the Minkowski bound by the same linear in n factor. With some effort, the constant in Theorem 1 can be increased by at least a factor of 10. However, the main merit of our proof is not so much in the result itself, but rather in the approach we use. Our argument is essentially different from all those previously employed, including the constructions of Minkowski [9], Hlawka [5], Rogers [10], Davenport and Rogers [4], and Ball [2]. Instead of relying directly on the geometry of numbers, we apply powerful graph-theoretic tools: specifically, we use lower bounds [1, 3] on the independence number of locally sparse graphs. A similar approach has been recently used by Jiang and Vardy [7] in an asymptotic improvement of the classical Gilbert-Varshamov bound in coding theory.

We find it remarkable that all the existing approaches to the Minkowski problem, while being very different in nature, result in the same linear in n improvement over the classical Minkowski bound. At this stage, we do not have a satisfactory explanation of this rather puzzling phenomenon; perhaps it indicates an inherent difficulty in breaking the linear barrier.

2. Graph-theoretic proof of the Minkowski bound

First, let us define two cubes in $\mathbb{R}^n$ — a smaller cube K_0 of side s_n and a larger cube K_1 of side $s_n + 2r_n$. Specifically

$$K_0 \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^n \;:\; |x_i| \leqslant \frac{s_n}{2} \right\} \qquad \text{and} \qquad K_1 \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^n \;:\; |x_i| \leqslant \frac{s_n}{2} + r_n \right\} \tag{2}$$

where r_n and s_n are, so far, arbitrary functions of n, except that we assume that $r_n, s_n \in 2\mathbb{Z}$. Next, define a graph $\mathcal{G}_n$ as follows: the vertices of $\mathcal{G}_n$ are given by $V(\mathcal{G}_n) = \mathbb{Z}^n \cap K_0$, and $\{u,v\} \in E(\mathcal{G}_n)$ iff $d(u,v) < 2r_n$, where $d(\cdot,\cdot)$ is the Euclidean distance in $\mathbb{R}^n$. Let $\mathcal{I}$ be a *maximal* independent set in $\mathcal{G}_n$ — that is, $\mathcal{I}$ is such that every vertex of $V(\mathcal{G}_n) \setminus \mathcal{I}$ is adjacent to at least one vertex in $\mathcal{I}$. By the definition of $E(\mathcal{G}_n)$, spheres of radius r_n about the points of $\mathcal{I}$ do not overlap. Moreover, all such spheres lie inside the cube K_1. Since K_1 tiles $\mathbb{R}^n$, we conclude that there exists a sphere packing $\mathcal{P}_n$ with density

$$\Delta(\mathcal{P}_n) \;=\; \frac{|\mathcal{I}|\, V_n(r_n)^n}{\mathrm{Vol}(K_1)} \;=\; \frac{|\mathcal{I}|\, V_n(r_n)^n}{(s_n + 2r_n)^n} \tag{3}$$

where V_n is the volume of a unit sphere in $\mathbb{R}^n$. Let d_n denote the maximum degree of a vertex in $\mathcal{G}_n$. It is well-known (and obvious, for any graph G) that

$$|\mathcal{I}| \;\geqslant\; \frac{|V(\mathcal{G}_n)|}{d_n + 1} \;=\; \frac{(s_n+1)^n}{d_n + 1} \tag{4}$$

Let $\mathcal{S}_n(r)$ denote the open sphere of radius r about the origin $\mathbf{0}$ of $\mathbb{R}^n$. If the ratio s_n/r_n is sufficiently large, as we shall assume, then $d_n + 1$ is just the number of points of $\mathbb{Z}^n$ contained in $\mathcal{S}_n(2r_n)$. Thus we can roughly estimate $d_n + 1$ as simply the volume $V_n 2^n (r_n)^n$ of $\mathcal{S}_n(2r_n)$. Combining this estimate with (3), (4), and taking $s_n = 2n^2 r_n$, we obtain

$$\Delta(\mathcal{P}_n) \;\gtrsim\; \frac{(s_n)^n V_n (r_n)^n}{V_n 2^n (r_n)^n (s_n + 2r_n)^n} \;=\; \frac{1}{2^n \left(1 + \frac{2r_n}{s_n}\right)^n} \;\geqslant\; \frac{1 + o(1)}{2^n}$$

Alternatively, a precise bound on d_n can be derived as follows. With each point $v \in \mathbb{Z}^n$, we associate the unit cube

$$K(v) \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^n \;:\; -\frac{1}{2} < x_i - v_i \leqslant \frac{1}{2} \right\}$$

Such cubes are fundamental domains of $\mathbb{Z}^n$; hence, they do not intersect. The length of the main diagonal of $K(v)$ is $\sqrt{n}$, which implies that $d(x,v) \leqslant \sqrt{n}/2$ for all $x \in K(v)$. Let $D_n = \mathcal{S}_n(2r_n) \cap \mathbb{Z}^n$, so that $d_n + 1 = |D_n|$. It follows by the triangle inequality that if $v \in D_n$, then $d(x,\mathbf{0}) < 2r_n + \sqrt{n}/2$ for all $x \in K(v)$. Hence

$$\bigcup_{v \in D_n} K(v) \;\subset\; \mathcal{S}_n\big(2r_n + \sqrt{n}/2\big) \tag{5}$$

Expressing the volume of $\cup_{v \in D_n} K(v)$ as $|D_n| = d_n + 1$, this implies that the maximum degree of a vertex in $\mathcal{G}_n$ is bounded by

$$d_n + 1 \ \leqslant \ V_n \left(2r_n + \sqrt{n}/2\right)^n \tag{6}$$

Combining (6) with (3) and (4), then taking $r_n = 2n^2$ and $s_n = 2n^4$ (say), proves that the density of $\mathcal{P}_n$ is at least

$$\Delta(\mathcal{P}_n) \ \geqslant \ \frac{(s_n)^n V_n (r_n)^n}{(s_n + 2r_n)^n V_n \left(2r_n + \sqrt{n}/2\right)^n} \ = \ \frac{1}{2^n \left(1 + \frac{2r_n}{s_n}\right)^n \left(1 + \frac{\sqrt{n}}{4r_n}\right)^n} \ = \ \frac{1 + o(1)}{2^n} \tag{7}$$

Asymptotically, (7) coincides with the Minkowski bound on Δ_n. Since a maximal independent set in $\mathcal{G}_n$ can be found in time $O\left(|V(\mathcal{G}_n)|^2\right)$ and $|V(\mathcal{G}_n)| = (s_n + 1)^n = \left(2n^4 + 1\right)^n$, the bound in (7) also reproduces the result of Litsyn and Tsfasman [8, Theorem 1].

3. Asymptotic improvement using locally sparse graphs

We next take $\mathcal{I}$ to be an independent set of *maximum size* in $\mathcal{G}_n$ and consider bounds on $|\mathcal{I}| = \alpha(\mathcal{G}_n)$ that are sharper than the trivial bound in (4). Let T_n denote the number of triangles in $\mathcal{G}_n$. Then it is known [1, 3, Lemma 15, p. 296] that

$$\alpha(\mathcal{G}_n) \ \geqslant \ \frac{|V(\mathcal{G}_n)|}{10 d_n} \left(\log_2 d_n \ - \ \tfrac{1}{2} \log_2 \left(\frac{T_n}{|V(\mathcal{G}_n)|} \right) \right) \tag{8}$$

Now let t_n be the smallest integer with the property that for all $v \in V(\mathcal{G}_n)$, the subgraph of $\mathcal{G}_n$ induced by the neighborhood of v has at most t_n edges. Then it follows from (8) that

$$\alpha(\mathcal{G}_n) \ \geqslant \ \frac{(s_n + 1)^n}{10 d_n} \left(\log_2 d_n \ - \ \tfrac{1}{2} \log_2 \left(\frac{t_n}{3} \right) \right) \tag{9}$$

Thus to obtain an asymptotic improvement upon the Minkowski bound it would suffice to prove that $t_n = o(d_n^2)$. Before diving into the technical details of the proof, let us explain intuitively why we expect to get an improvement by a factor that is linear in n.

Let us pick two points x and y uniformly at random in $\mathcal{S}_n(2r_n)$. The relevant question is: what is the probability that $d(x, y) < 2r_n$? It is a rather standard fact in high-dimensional geometry that (regardless of the value of r_n) this probability behaves as e^{-cn} for large n. Therefore, we should expect that only an exponentially small fraction of pairs of points of $\mathbb{Z}^n$ lying within a sphere of radius $2r_n$ centered at some $z \in V(\mathcal{G}_n)$ are adjacent in $\mathcal{G}_n$. In other words, we expect that $t_n \lesssim d_n^2 / e^{cn}$ which, in view of (9), immediately leads to the desired $\Theta(n)$ improvement factor. We derive a rigorous upper bound on t_n next.

Consider the neighborhood of $\mathbf{0} \in V(\mathcal{G}_n)$, and let $\mathcal{H}_n$ denote the subgraph of $\mathcal{G}_n$ induced by this neighborhood. As in Section 2, we assume that the ratio s_n/r_n in (2) is sufficiently large so that $V(\mathcal{H}_n) = (\mathbb{Z}^n \backslash \{\mathbf{0}\}) \cap \mathcal{S}_n(2r_n)$. It is then obvious that $t_n = |E(\mathcal{H}_n)|$, so

$$2t_n = \sum_{x \in V(\mathcal{H}_n)} \deg(x) \tag{10}$$

where $\deg(x)$ denotes the degree of x in $\mathcal{H}_n$. Write $S_1 = \mathcal{S}_n(2r_n)$ and let S_2 be the sphere of radius $2r_n$ about $x \in V(\mathcal{H}_n)$. Then $\deg(x)$ is just the number of points of $\mathbb{Z}^n$ in $S_1 \cap S_2$. Using the same argument as in (5), we thus have

$$\deg(x) \leqslant \mathrm{Vol}(S_1' \cap S_2') \tag{11}$$

where $S_1' = \mathcal{S}_n(2r_n + \sqrt{n}/2)$ and S_2' is the sphere of radius $2r_n + \sqrt{n}/2$ about x. Clearly, the right-hand side of (11) depends on x only via its distance to the origin. Hence, define

$$\rho \stackrel{\text{def}}{=} 2r_n + \frac{\sqrt{n}}{2} \qquad \text{and} \qquad \delta_x \stackrel{\text{def}}{=} \frac{d(x,\mathbf{0})}{2\rho} \tag{12}$$

with $\delta_x \in (0, 1/2)$ for all x. It is not difficult to write down a precise expression for the volume of $S_1' \cap S_2'$ in terms of ρ and δ_x. Let $\theta = \cos^{-1} \delta_x$. Then $\mathrm{Vol}(S_1' \cap S_2')$ is twice the volume of a spherical sector of angle 2θ (shaded in Fig. 1) minus twice the volume of a right cone of the same angle (cross-hatched in Fig. 1). Thus

$$\mathrm{Vol}(S_1' \cap S_2') = \frac{2\rho^n V_{n-1}}{n} \left((n-1) \int_0^\theta (\sin\varphi)^{n-2} d\varphi \; - \; \delta_x (\sin\theta)^{n-1} \right) \tag{13}$$

However, rather than estimating the integral $\int_0^\theta (\sin\varphi)^{n-2} d\varphi$ in (13), we will use the following simple bound (without compromising much in the asymptotic quality of the obtained result). It is easy to see (cf. Fig. 2) that $S_1' \cap S_2'$ is contained in a cylinder of height $2\rho - d(x,\mathbf{0})$ whose base is an $(n-1)$-dimensional sphere of radius $\rho \sin\theta$. Hence

$$\mathrm{Vol}(S_1' \cap S_2') \leqslant 2\rho(1-\delta_x) V_{n-1} \rho^{n-1} (\sin\theta)^{n-1} \leqslant (1-\delta_x^2)^{n/2} \, n V_n \rho^n \tag{14}$$

where the second inequality follows from the fact that $2V_{n-1} \leqslant nV_n$ for all n. Now let U_k denote the set of all $x \in V(\mathcal{H}_n)$ such that $d^2(x,\mathbf{0}) = k$. Clearly $d^2(x,\mathbf{0})$ is an integer in the range $1 \leqslant d^2(x,\mathbf{0}) \leqslant 4r_n^2 - 1$. We thus break the sum in (10) into two parts

$$2t_n = \sum_{x \in V(\mathcal{H}_n)} \deg(x) = \sum_{k=1}^{1/4 r_n^2 - 1} \sum_{x \in U_k} \deg(x) + \sum_{k = 1/4 r_n^2}^{4r_n^2 - 1} \sum_{x \in U_k} \deg(x) \tag{15}$$

and bound each part separately. As it turns out, crude upper bounds on $|U_k|$ suffice in each case. For the first double-sum in (15), we use the fact that $\deg(x) \leqslant d_n \leqslant V_n \rho^n$ for all $x \in V(\mathcal{H}_n)$ by (6). Therefore, applying once again the method of (5), we have

$$\sum_{k=1}^{\frac{r_n^2}{4}-1} \sum_{x \in U_k} \deg(x) \leqslant V_n \rho^n \sum_{k=1}^{\frac{r_n^2}{4}-1} |U_k| \leqslant V_n \rho^n \, \mathrm{Vol}\left(\mathcal{S}_n\left(\frac{r_n + \sqrt{n}}{2} \right) \right) \leqslant \left(\frac{1}{2} \right)^n V_n^2 \rho^{2n} \tag{16}$$

where the last inequality assumes $r_n \geqslant \sqrt{n}/2$, so that $\rho \geqslant r_n + \sqrt{n}$. In fact, henceforth, let us take $r_n = 2n^2$ as in Section 2. Then $k \geqslant n^4$ in the second sum of (15), and we can bound

$|U_k|$ as follows: $|U_k| \leqslant \mathrm{Vol}\big(\mathcal{S}_n(\sqrt{k}+\sqrt{n}/2)\big) \leqslant V_n k^{n/2}\big(1+\sqrt{n}/(2n^2)\big)^n \leqslant 2V_n k^{n/2}$. Combining this with the bounds (11) and (14) on $\deg(x)$, we have

$$\sum_{k=\frac14 r_n^2}^{4r_n^2-1} \sum_{x\in U_k} \deg(x) \leqslant nV_n\rho^n \sum_{k=\frac14 r_n^2}^{4r_n^2-1} |U_k| \left(1-\delta_k^2\right)^{n/2} \leqslant 2^{n+1} n V_n^2 \rho^{2n} \sum_{k=\frac14 r_n^2}^{4r_n^2-1} \left(\delta_k^2\,(1-\delta_k^2)\right)^{n/2}$$

where $\delta_k \stackrel{\text{def}}{=} \sqrt{k}/(2\rho)$. Now, the function $f(\delta) = \delta^2(1-\delta^2)$ attains its maximum in the interval $[0, \frac12]$ at $\delta = \frac12$. Putting this together with (15) and (16), we finally obtain the desired upper bound on t_n, namely

$$t_n \leqslant \left(\frac{1}{2}\right)^{n+1} V_n^2 \rho^{2n} + 2^n n V_n^2 \rho^{2n} \sum_{k=\frac14 r_n^2}^{4r_n^2-1} \left(\frac{3}{16}\right)^{n/2} \leqslant \left(\frac{\sqrt{3}}{2}\right)^n n V_n^2\, \rho^{2n+2} \tag{17}$$

Recall that $d_n \leqslant V_n\rho^n$ by (6). Substituting this bound together with (17) in the bound (9) on the independence number of $\mathcal{G}_n$ produces

$$\begin{aligned}\alpha(\mathcal{G}_n) &\geqslant \frac{(s_n+1)^n}{10V_n\rho^n}\left(\log_2(V_n\rho^n) - \frac{1}{2}\log_2\left(\left(\frac{\sqrt{3}}{2}\right)^n V_n^2\rho^{2n}\right) - \frac{1}{2}\log_2\left(n\rho^2\right)\right)\\ &= \frac{(s_n+1)^n}{10\,V_n\,(2r_n+\sqrt{n}/2)^n}\left(\frac{n\,\log_2(2/\sqrt{3})}{2} - \frac{\log_2\big(n(2r_n+\sqrt{n}/2)^2\big)}{2}\right)\end{aligned}$$

where we have used the definition of ρ in (12). Finally, using (3) with $|\mathcal{I}| = \alpha(\mathcal{G}_n)$ while taking $r_n = 2n^2$ and $s_n = 2n^4$ as before, we obtain the following bound

$$\begin{aligned}\Delta(\mathcal{P}_n) &\geqslant \frac{n}{20}\left(\frac{s_n}{s_n+2r_n}\right)^n \left(\frac{r_n}{2r_n+\sqrt{n}/2}\right)^n \left(\log_2(2/\sqrt{3}) - \frac{\log_2\big(n(2r_n+\sqrt{n}/2)^2\big)}{n}\right)\\ &= \frac{\log_2(2/\sqrt{3})}{20}\, n2^{-n}\left(1+o(1)\right)\end{aligned} \tag{18}$$

Since $\log_2(2/\sqrt{3})/20 = 0.0103..$, this establishes (1). Now, given any graph $G = (V, E)$, there is a deterministic algorithm [6] that finds an independent set $\mathcal{I}$ in G whose size is lower bounded by (8) in time $O(d_{\text{av}}|E| + |V|)$, where d_{av} is the average degree of G. In the case of the graph $\mathcal{G}_n$, this reduces to $O\big(V_n^2(s_n+1)^n(2r_n+\sqrt{n}/2)^{2n}\big)$. With $r_n = 2n^2$ and $s_n = 2n^4$, the expression $V_n^2(s_n+1)^n(2r_n+\sqrt{n}/2)^{2n}$ behaves as $(64\pi e n^7)^n$ for large n. This completes the proof of Theorem 1, with the value of γ given by $7+\epsilon$. However, note that our choice of the values $r_n = 2n^2$ and $s_n = 2n^4$ was motivated primarily by the notational convenience of having $r_n, s_n \in 2\mathbb{Z}$. In fact $r_n = n^{1.5+\epsilon}$ and $s_n = n^{2.5+\varepsilon}$ would suffice, as can be readily seen from (7) and (18). Thus the value of γ can be taken as $4.5+\epsilon$.

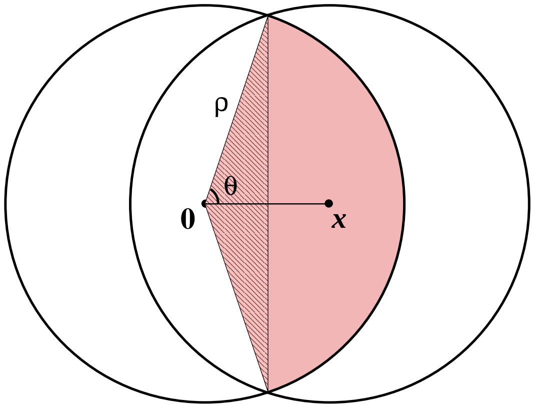

Figure 1

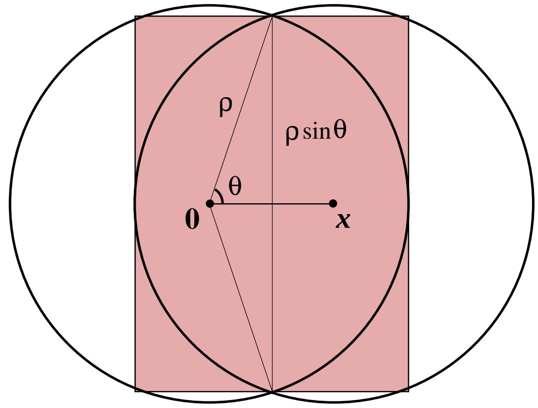

Figure 2

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