The critical bias for the Hamiltonicity game is $(1 + o(1))n/\ln n$

Michael Krivelevich *

March 9, 2010

Abstract

We prove that in the biased $(1 : b)$ Hamiltonicity Maker-Breaker game, played on the edges of the complete graph K_n, Maker has a winning strategy for $b(n) \leq \left(1 - \frac{30}{\ln^{1/4} n}\right) \frac{n}{\ln n}$, for all large enough n.

1 Introduction

A *Maker-Breaker game* is a triple (H, a, b), where $H = (V, E)$ is a hypergraph with vertex set V, called the board of the game, and edge set E, a family of subsets of V called winning sets. The parameters a and b are positive integers, related to the so called game bias. The game is played between two players, called Maker and Breaker, who change turns occupying previously unclaimed elements of V; Maker claims a elements in his turn, Breaker answers by claiming b elements. We assume that Breaker moves first. The game ends when all board elements have been claimed by either of the players. (In the very last move, if the board does not contain enough elements to claim for the player whose turn is now, that player claims all remaining elements of the board.) Maker wins if and only if he has occupied one of the winning sets $e \in E$ by the end of the game, Breaker wins otherwise, i.e., if he manages to occupy at least one element of ("to break into") every winning set by the end of the game. The most basic case is when $a = b = 1$, which is the so called *unbiased* game. Here we will be concerned with $1 : b$ games.

*School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: krivelev@post.tau.ac.il. Research supported in part by a USA-Israel BSF grant, by a grant from the Israel Science Foundation and by a Pazy Memorial Award.

It is quite easy to see that Maker-Breaker games are *bias monotone*. This is to say that if the game $(H, 1, b)$ is Maker's win, then $(H, 1, b')$ is Maker's win as well for every integer $b' < b$. This allows to define the *critical bias* of the game H, which is the maximum possible value of bias b for which Maker still wins the $1 : b$ game played on H (if the 1:1 game is Breaker's win, we say that the critical bias in this case is zero).

We refer the reader to a recent monograph [2] of Beck for extensive background on positional games in general and on Maker-Breaker games in particular.

The subject of this paper is the *Hamiltonicity game* played on the edge set of the complete graph K_n. In this game, players take turns in claiming unoccupied edges of K_n. Maker's aim is to construct a Hamilton cycle, and thus the family of winning sets coincides with the family of (the edge sets of) graphs on n vertices containing a Hamilton cycle. The research on biased Hamiltonicity games has a long and illustrious history. Already in the very first paper about biased Maker-Breaker games back in 1978, Chvátal and Erdős [5] treated the unbiased Hamiltonicity game and showed that Maker wins it for every sufficiently large n. (Chvátal and Erdős showed in fact that Maker wins within $2n$ rounds. Later the minimum number of steps required for Maker to win this game was shown to be at most $n + 2$ by Hefetz et al. in [7], and finally the optimal $n + 1$ by Hefetz and Stich [8].) We would like to mention that Chvátal and Erdős also proved in their paper that for $b(n) \geq (1 + \epsilon)n/ \ln n$, where $\epsilon > 0$ is an arbitrary small constant, Breaker can isolate a vertex in the $1 : b$ game played on K_n, i.e., to claim all $n - 1$ edges incident to it.

Chvátal and Erdős conjectured that there is a function $b(n)$ tending to infinity such that Maker can still build a Hamilton cycle if he plays against bias $b(n)$. Their conjecture was verified by Bollobás and Papaioannou [4] who proved that Maker is able to build a Hamilton cycle even if Breaker's bias is as large as $\frac{c \ln n}{\ln \ln n}$, for some constant $c > 0$. Beck improved greatly on this [1] and showed that Maker wins the Hamiltonicity game provided Breaker's bias is at most $\left(\frac{\ln 2}{27} - o(1)\right) \frac{n}{\ln n}$. In view of the above mentioned Chvátal-Erdős theorem about isolating a vertex, Beck's result established that the order of magnitude of the critical bias in the Hamiltonicity game is $n/ \log n$. Krivelevich and Szabó [9] improved upon Beck's result and showed that the critical bias $b(n)$ for the Hamiltonicity game is at least $(\ln 2 - o(1))n/ \ln n$.

In a relevant development, Gebauer and Szabó showed recently in [6] that the critical bias for the connectivity game on K_n (where Maker wins if and only if he creates a spanning tree from his edges by the end of the game) is asymptotically equal to $n/ \ln n$. We will rely extensively on some of their results and approaches here.

It was widely believed that the critical bias for the Hamiltonicity game on K_n is asymptotically equal to $n/ \ln n$ as well. This conjecture has even attained the (somewhat dubious) honor to be stated as one of the most "humiliating open problems" of the subject by Beck in his book [2] (see Chapter 49 there).

2 The result

In this paper we resolve the above stated conjecture. Here is our result:

Theorem 1 *Maker has a strategy to win the* $(1:b)$ *Hamiltonicity game played on the edge set of the complete graph* K_n *on* n *vertices in at most* $14n$ *moves, for every* $b \le \left(1 - \frac{30}{\ln^{1/4} n}\right) \frac{n}{\ln n}$*, for all large enough* n*.*

The constants and the error term expression of the above theorem are clearly not optimal and can be improved somewhat by a more careful implementation of our arguments. We find however little reason to pursue this goal.

3 Notation

Our basic notation is quite standard and follows closely that of most of graph theory books. In particular, for a graph $G = (V, E)$ and a vertex subset $U \subset V$, we denote by $N_G(U)$ the external neighborhood of U in G, i.e., $N_G(U) = \{v \in V \setminus U : v \text{ has a neighbor in } U\}$. We systematically omit rounding signs for the sake of clarity of presentation. The underlying parameter n is assumed to be large enough where necessary.

Let

$$\begin{aligned} \delta_0 = \delta_0(n) &= \frac{6}{\ln^{1/2} n}\,, \\ \delta = \delta(n) &= \frac{15}{\ln^{1/4} n}\,, \\ \epsilon = \epsilon(n) &= \frac{30}{\ln^{1/4} n}\,, \\ k_0 = k_0(n) &= \delta_0 n = \frac{6n}{\ln^{1/2} n}\,. \end{aligned}$$

For a positive integer k, a graph $G = (V, E)$ is a *k-expander* if $|N_G(U)| \ge 2|U|$ for every subset $U \subset V$ of at most k vertices.

Given a graph G, a non-edge $e = (u, v)$ of G is called a *booster* if adding e to G creates a graph G', which is Hamiltonian or whose maximum path is longer than that of G. Boosters advance a graph towards Hamiltonicity when added; adding sequentially n boosters clearly brings a graph to Hamiltonicity.

4 Tools

The following lemma, that can be traced back to a seminal work of Pósa [11], is used quite frequently in papers on Hamiltonicity and on extremal problems involving paths and cycles.

Lemma 1 *Let G be a connected non-Hamiltonian k-expander. Then at least $(k+1)^2/2$ non-edges of G are boosters.*

Proof. See, e.g., Lemma 8.5 of [3] or Corollary 2.10 of [10]. □

Although k-expanders are not necessarily connected, their connected components are guaranteed to be of a relatively large size, as shown in the following easy lemma.

Lemma 2 *Let $G = (V, E)$ be a k-expander. Then every connected component of G has size at least $3k$.*

Proof. If not, let V_0 be the vertex set of a connected component of G of size less than $3k$. Choose an arbitrary subset $U \subseteq V_0$ of cardinality $|U| = \min\{|V_0|, k\}$, clearly $|U| > |V_0|/3$. Since G is a k-expander, it follows that $|N_G(U)| \geq 2|U|$. On the other hand, $N_G(U) \subseteq V_0$, implying $|V_0| \geq |U| + |N_G(U)| \geq 3|U|$ – a contradiction. □

Now we can describe the main tool of our proof, a recent result of Gebauer and Szabó, who analyzed in [6] the biased minimum degree game. For our goals, it will suffice to specialize their analysis to the game where Maker's goal is to reach a graph of minimum degree at least 12. Here is Maker's strategy employed by Gebauer and Szabó. Maker and Breaker play a $1 : b$ game on the edges of the complete graph K_n on n vertices. For a current position of the game (with some edges of K_n having been claimed by Maker and some other by Breaker), we denote by $deg_M(v)$ and $deg_B(v)$ the degrees of a vertex v in Maker's graph and in Breaker's graph, respectively. The *danger* $dang(v)$ of a vertex v with respect to the current position of the game is defined as $dang(v) := deg_B(v) - 2b \cdot deg_M(v)$.

Strategy S:
As long as there is a vertex of degree less than 12 in Maker's graph, Maker chooses a vertex v of degree less than 12 in his graph with the largest danger value $dang(v)$ (breaking ties arbitrarily) and claims an **arbitrary** unclaimed edge e containing v.

If Maker claims an edge e due to a vertex v in the above strategy, we say that e *is chosen by* v. Gebauer and Szabó proved the following statement about it.

Theorem 2 *([6], Theorem 1.2): In a* $\left(1 : \frac{(1-\epsilon)n}{\ln n}\right)$*-game played on the edge set of the complete graph* K_n *on* n *vertices, strategy* S *guarantees Maker minimum degree at least 12 in his graph.*

In our argument we will need more than the above statement – it will be essential for us that Maker is able, for every vertex v of the graph, to reach degree at least 12 at v when a substantial part of the edges at v is still unclaimed. Fortunately, the proof of Gebauer and Szabó gives this as well, as stated in the lemma below.

Lemma 3 *In a* $\left(1 : \frac{(1-\epsilon)n}{\ln n}\right)$*-game played on the edge set of the complete graph* K_n *on* n *vertices, strategy* S *guarantees that for every vertex* $v \in [n]$ *Maker has at least 12 edges incident to* v *before Breaker accumulates at least* $(1-\delta)n$ *edges at* v*.*

The proof of Lemma 3 is a straightforward modification of the proof of Theorem 1.2 of [6]. More specifically, the argument of [6] can be used to analyze a slightly different game in which Breaker wins if he accumulates at least $(1-\delta)n$ edges at a vertex whose Maker degree is still less than 12. Then in the analysis the danger of the last vertex v_g before Breaker's last move is now at least $(1-\delta)n - 12 - b$. Finally, one checks that in the relevant calculations the danger of the original set I_{g-1} before the game started still comes out positive. We refer the reader to [6] for further details.

5 The proof

In this section we prove our main result, Theorem 1. Maker's strategy is composed of three stages. At the first stage, he creates a k_0-expander in a linear number of moves. At the second stage, Maker makes sure his graph is connected in at most $O(n/k_0)$ moves. Finally, he turns his graph into a Hamiltonian one, using at most n further moves.

Stage 1 – creating an expander.

Let us go back to the Gebauer-Szabó winning strategy S for the minimum degree 12 game. As it turns out, this strategy not only guarantees minimum degree 12 or more in Maker's graph, but has enough flexibility in it to allow Maker to pursue an even more important goal – that of creating quickly a good expander from its edges. First observe that as long as the game is played at this stage, Maker increases by one the degree of a vertex whose current degree in his graph is still less than 12. Therefore, Maker wins this game in at most $12n$ moves. More importantly, while describing strategy S, we stressed that at each round Maker is allowed to choose an edge e incident to its vertex of minimum degree v arbitrarily. We can utilize this freedom of choice by specifying that Maker claims each time a **random** edge e incident to

v. This random choice of Maker allows us to prove that he has a strategy to create a good expander quickly.

Lemma 4 *Maker has a strategy to create a k_0-expander in at most $12n$ moves.*

Proof. Maker augments the strategy S described above by choosing at each round a random edge incident to a vertex. Here is his strategy S'.

> **Strategy S':**
> As long as there is a vertex of degree less than 12 in Maker's graph, Maker chooses a vertex v of degree less than 12 in his graph with the largest danger value $dang(v)$ (breaking ties arbitrarily) and claims a **random** unclaimed edge e containing v.

The game lasts till the minimum degree in Maker's graph is at least 12. As we argued above, the game duration does not exceed $12n$. Since the game analyzed is a perfect information game with no chance moves, it is enough to prove that Maker's strategy succeeds to create a k_0-expander with positive probability. (We will in fact prove that his strategy succeeds with probability approaching 1.)

So suppose that Maker's graph is not a k_0-expander. Then there is a subset A of size $|A| = i \leq k_0$ in Maker's graph M after the end of Stage 1 such that $N_M(A)$ is contained in a set B of size at most $2i-1$. Since the minimum degree in Maker's graph is 12, we can assume that $i \geq 5$; more importantly, there are at least $6i$ edges of Maker incident to A. Consider one such edge $e = (u, v)$ and assume that e was chosen by $v \in A \cup B$ in the course of the game. Notice crucially that, by Lemma 3, when choosing e Breaker's degree at v was at most $(1-\delta)n$, while Maker's degree at v was at most 11. Therefore at that point of the game, there were at least $\delta n - 12$ unclaimed edges incident to v. The probability that at that point Maker chose an edge at v whose second endpoint belongs to $A \cup B$ is thus at most $\frac{|A\cup B|-1}{\delta n-12}$, regardless of the history of the game so far. It follows that the probability that all these $6i$ edges incident to A will end up entirely in $A \cup B$ is at most $\left(\frac{3i-2}{\delta n-12}\right)^{6i}$. Summing over all relevant values of i, we derive that the probability that Maker's strategy fails to create a k_0-expander is at most

$$\sum_{5\leq i\leq k_0} \binom{n}{i}\binom{n-i}{2i-1}\left(\frac{3i-2}{\delta n-12}\right)^{6i} \leq \sum_{5\leq i\leq k_0}\left[\frac{en}{i}\left(\frac{en}{2i}\right)^2\left(\frac{4i}{\delta n}\right)^6\right]^i$$
$$= \sum_{5\leq i\leq k_0}\left[4^5e^3\left(\frac{i}{n}\right)^3\frac{1}{\delta^6}\right]^i .$$

Denote the i-th term of the above sum by $g(i)$. Then for $5 \le i \le \sqrt{n}$ we have $g(i) \le \left(O(1)(\ln n)^{3/2}n^{-3/2}\right)^6 = o(1/n)$, while for $\sqrt{n} \le i \le k_0$ we can estimate $g(i) \le \left(\frac{4^5e^3\delta_0^3}{\delta^6}\right)^{\sqrt{n}} = o(1/n)$ as well. This implies that Maker's strategy fails with negligible probability, and thus with positive probability (and in fact almost surely) he creates a k_0-expander in the first $12n$ moves.

Stage 2 – creating a connected expander.

If Maker's graph M is not yet connected by the end of Stage 1, he can turn it easily into such in very few moves. Indeed, M is a k_0-expander and therefore by Lemma 2 all connected components of M are of size at least $3k_0$. In the next $n/(3k_0)-1$ rounds at most, Maker claims an arbitrary edge between two of its connected components. Observe that there are at least $9k_0^2 = 324n^2/\ln n$ edges of the complete graph between any two such components, and Breaker has at most $(12n + n/(3k_0)) \cdot b < 13n^2/\ln n$ edges claimed on the board altogether. Therefore, Breaker cannot block Maker from achieving his goal. Stage 2 lasts at most $n/(3k_0) - 1 < n$ rounds.

Stage 3 – completing a Hamilton cycle.

Recall that by the end of Stage 1 Maker has created a k_0-expander. Clearly, his graph at every subsequent round inherits this expansion property. Also, after Stage 2 Maker's graph is already connected. But then by Lemma 1 at any round of Stage 3 Maker's graph is either already Hamiltonian, or has at least $k_0^2/2$ boosters. Maker goes on to add a booster after a booster in the next n rounds at most, till finally he reaches Hamiltonicity. Breaker is helpless – he just does not have enough edges on the board to block all of Maker's boosters during these rounds. Indeed, the game lasts altogether at most $12n + n + n = 14n$ rounds, during which Breaker puts on the board at most $14n \cdot b \le 14n^2/\ln n$ edges – less than $k_0^2/2$ boosters of Maker. Hence, at any round of Stage 3 there is an available booster with respect to the current Maker's graph – which he happily claims. □

6 Concluding remarks

We have essentially resolved the biased Hamiltonicity game on the complete graph K_n by proving that the critical bias $b(n)$ is asymptotic to $n/\ln n$.

The method we employed in our proofs (creating quickly a good expander first) is quite general and has a clear potential to be applicable to other biased combinatorial games as well. For example, it can be used to show that Maker can create a c-connected spanning graph G in the $1 : b$ game on K_n for any constant c, or even for a growing function $c = c(n)$, as long as the bias $b(n)$ satisfies $b(n) \leq (1 - o(1))n/\ln n$. This would provide an alternative proof of the corresponding results of Gebauer and Szabó [6] and in fact would strengthen their assertions.

Finally, let us mention that the strategy we used in our argument is random. It would be very interesting to provide a deterministic (explicit) Maker's strategy for winning the Hamiltonicity game close to the critical bias.

Acknowledgement. The author wishes to thank Dan Hefetz and the referee of the paper for their helpful remarks.

References

[1] J. Beck, *Random graphs and positional games on the complete graph*, Annals of Discrete Mathematics 28 (1985) 7–13.

[2] J. Beck, **Combinatorial Games**, Cambridge University Press, Cambridge, 2008.

[3] B. Bollobás, **Random graphs**, 2nd ed., Vol. 73, Cambridge University Press, Cambridge, 2001.

[4] B. Bollobás and A. Papaioannou, *A biased Hamiltonian game*, Congressus Numerantium, 35 (1982), 105–115.

[5] V. Chvátal and P. Erdős, *Biased positional games*, Annals of Discrete Mathematics 2 (1978), 221–228.

[6] H. Gebauer and T. Szabó, *Asymptotic random graph intuition for the biased connectivity game*, Random Structures and Algorithms 35 (2009), 431–443.

[7] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, *Fast winning strategies in Maker-Breaker games*, Journal of Combinatorial Theory Series B 99 (2009), 39-47

[8] D. Hefetz and S. Stich, *On two problems regarding the Hamilton cycle game*, The Electronic Journal of Combinatorics, Vol 16 (1) (2009), publ. R28.

[9] M. Krivelevich and T. Szabó, *Biased positional games and small hypergraphs with large covers*. Electronic Journal of Combinatorics, Vol. 15 (1) (2008), publ. R70.

[10] M. Krivelevich, E. Lubetzky and B. Sudakov, *Hamiltonicity thresholds in Achlioptas processes*, Random Structures and Algorithms, to appear.

[11] L. Pósa, *Hamiltonian circuits in random graphs*, Discrete Mathematics 14 (1976), no. 4, 359–364.