The choosability version of Brooks' theorem — a short proof

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Abstract

We present a short and self-contained proof of the choosability version of Brooks' theorem.

The following choosability version of Brooks' theorem is due to Vizing [2] and to Erdős, Rubin and Taylor [1].

Theorem. Let $\Delta \geq 3$ be an integer, and let $G \neq K_{\Delta+1}$ be a connected graph of maximum degree at most Δ. Then G is Δ-choosable.

(The case $\Delta = 2$ should – and can easily – be treated separately for this proof.)

Proof. The proof borrows its main idea from the nice argument of Zając [3] for the classical Brooks' theorem. We proceed by induction on $n = |V(G)|$. The basic case $n \leq \Delta$ is obvious — given a list assignment L for $V(G)$, one can just choose distinct colors for all vertices of G.

For the induction step, assume we are given a graph $G = (V, E)$ on n vertices and a list assignment L for the vertices of G satisfying $|L(v)| = \Delta$ for all $v \in V$. We aim to find an L-coloring f of G, which is a choice $f(v) \in L(v)$, $v \in V$, such that no edge of G is monochromatic under f.

If G contains a vertex v with $d(v) < \Delta$, we can apply induction to color every connected component of $G - v$ from its lists in L. Let f be the obtained coloring. It can be extended to v by choosing $f(v) \in L(v) - \{f(u) : (u, v) \in E\}$. We may thus assume G is Δ-regular.

Consider first the following special case: G has a cycle C on $k < n$ vertices and a vertex on the cycle having no neighbors outside C. Since G is connected we can find two vertices u, v adjacent along C and such that v has all its neighbors in C, and u has some neighbor w outside C. By the induction hypothesis the subgraph $G - V(C)$ is Δ-choosable. Let f be an L-coloring of $G - V(C)$. We now extend f to $V(C)$. If $f(w) \notin L(u)$, we choose $f(v) \in L(v)$ arbitrarily. If $f(w) \in L(u) \cap L(v)$, we set $f(v) = f(w)$. Finally, if $f(w) \in L(u) \setminus L(v)$, the lists $L(u)$ and $L(v)$ are different, and we can find $c \in L(v) \setminus L(u)$. We then set $f(v) = c$. In all three cases:

$$|L(u) \cap \{f(v), f(w)\}| \leq 1 \,. \tag{1}$$

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Enumerate the vertices of C when moving from v to u along C as $V(C) = (v_1, \dots, v_k)$ with $v_1 = v$ and $v_k = u$. Now we pick colors for $V(C) - \{v_1\}$ in the order $v_2, \dots, v_k$. For $2 \le i \le k-1$, vertex v_i has at least one neighbor following it in this order, and thus an available color $f(v_i)$ can be found in its list $L(v_i)$. For $i = k$, at most $\Delta - 1$ distinct colors from $L(v_k)$ have been used by f on the neighbors of $v_k = u$ due to (1), and we can choose $f(v_k) \in L(v_k)$ without creating a monochromatic edge.

Now we treat the general case. Since G is connected and is not a clique, we can locate $v_1, v_2, v_3 \in V$ such that $(v_1, v_2), (v_2, v_3) \in E$, but $(v_1, v_3) \notin E$. Let $P = (v_1, v_2, v_3 \dots, v_\ell)$ be a longest path in G starting with v_1, v_2, v_3. All neighbors of v_ℓ reside on P. Let v_i be the neighbor of v_ℓ farthest from it along P. The cycle $C = (v_i, v_{i+1}, \dots, v_\ell)$ then contains all neighbors of v_ℓ. If C is not Hamiltonian then we are done by the special case considered above. We may thus assume $\ell = n$ and $i = 1$. Let v_j be a neighbor of v_2 different from v_1, v_3 (here we use the assumption $\Delta \ge 3$). Fix the following order σ on V: $\sigma = (v_1, v_3, \dots, v_{j-1}, v_n, v_{n-1}, \dots, v_j, v_2)$. As before, we can choose $f(v_1) \in L(v_1)$, $f(v_3) \in L(v_3)$ so that

$$|L(v_2) \cap \{f(v_1), f(v_3)\}| \le 1\,. \tag{2}$$

We now choose colors for the rest of V in the order of σ. Every v_i other than v_2 has at least one neighbor following it in σ, and thus we can set $f(v_i) \in L(v_i)$ without creating a monochromatic edge. Finally, when arriving to color v_2, we can allocate it a color from $L(v_2)$ distinct from the colors assigned to its neighbors, by (2). $\square$

References

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