# Perfect fractional matchings in random hypergraphs

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**Abstract**

Given an $r$-uniform hypergraph $H = (V, E)$ on $|V| = n$ vertices, a real-valued function $f : E \to R^+$ is called a perfect fractional matching if $\sum_{v \in e} f(e) \leq 1$ for all $v \in V$ and $\sum_{e \in E} f(e) = n/r$. Considering a random $r$-uniform hypergraph process on $n$ vertices, we show that with probability tending to 1 as $n \to \infty$, at the very moment $t_0$ when the last isolated vertex disappears, the hypergraph $H_{t_0}$ has a perfect fractional matching. This result is clearly best possible. As a consequence, we derive that if $p(n) = (\ln n + w(n)) \Big/ \binom{n-1}{r-1}$, where $w(n)$ is any function tending to infinity with $n$, then with probability tending to 1 a random $r$-uniform hypergraph on $n$ vertices with edge probability $p$ has a perfect fractional matching. Similar results hold also for random $r$-partite hypergraphs.

## 1 Introduction

A *hypergraph* $H$ is an ordered pair $H = (V, E)$, where $V$ is a finite set (the *vertex set*) and $E$ is a family of distinct subsets of $V$ (the *edge set*). A

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hypergraph $H = (V, E)$ is *$r$-uniform* if all edges of $H$ are of size $r$. In this paper we consider only $r$-uniform hypergraphs where $r$ is fixed. A subset $M \subseteq E(H)$ is called a *matching* if every pair of edges from $M$ has an empty intersection. The maximal size of a matching in a hypergraph $H$ is called the *matching number* of $H$ and is denoted by $\nu(H)$. A matching $M$ is called *perfect* if $|M| = |V|/r$ (clearly, a perfect matching can exist only if $r$ divides $|V|$).

A *random hypergraph* $\mathcal{H}_r(n,p)$ is an $r$-uniform hypergraph with vertex set $V$ of size $|V| = n$, in which each subset $e \in \binom{V}{r}$ is chosen to be an edge of $H$ with probability $p$ (where $p$ may depend on $n$), all choices being independent. More exactly, $\mathcal{H}_r(n,p)$ is the probability space $(\Omega, P)$, where $\Omega$ is the finite set of all $r$-uniform hypergraphs on $n$ labeled vertices and the probability of each hypergraph $H = (V, E)$ from $\Omega$ equals to $p^{|E|}(1-p)^{\binom{n}{r}-|E|}$. The underlying set of $\mathcal{H}_r(n,p)$ is denoted by $\mathcal{H}_r(n)$. We define also the model $\mathcal{H}_r(n, M)$, this probability space consists of all $r$-uniform hypergraphs on $n$ labeled vertices with $M$ edges, where all such hypergraphs are equiprobable.

A *property* $Q$ of $\mathcal{H}_r(n)$ is a subset of $\mathcal{H}_r(n)$, closed under vertex permutations. The statement '$H$ has $Q$' means $H \in Q$. A property $Q$ is called *monotone* if whenever $H \in Q$ and $E(H) \subset E(H')$ then also $H' \in Q$. A function $p^* = p^*(n)$ is called a *threshold* for a property $Q$ of $\mathcal{H}_r(n)$ if $p(n)/p^*(n) \to 0$, as $n \to \infty$, implies that **whp**[1] $H \in \mathcal{H}_r(n,p)$ does not have $Q$, while $p(n)/p^*(n) \to \infty$, as $n \to \infty$, implies that **whp** $H \in \mathcal{H}_r(n,p)$ has $Q$.

One of the central problems in probabilistic combinatorics is that of determining the threshold for a perfect matching in a random $r$-uniform hypergraph on $n$ vertices (assuming $r$ divides $n$). This problem was posed by Schmidt and Shamir in [6], they managed to prove that if $p(n) = n^{-r+3/2}w(n)$, where $w(n)$ is any function tending to infinity arbitrarily slowly, then **whp**

[1]An event $\mathcal{E}_n$ happens **whp** (with high probability) if the probability of $\mathcal{E}_n$ tends to 1 as $n$ tends to infinity.

$H \in \mathcal{H}_r(n,p)$ has a perfect matching. This result has recently been improved by Frieze and Janson [4], they showed that it suffices to take $p(n) = n^{-r+4/3}w(n)$. Both papers used the second moment method and the Chebyshev inequality. Frieze and Janson (as well as others, see, e.g., Erdős ([1], Appendix B)) conjectured that the threshold function is $p^*(n) = n^{-r+1}\log n$. For the case $r = 2$ this has been proved by Erdős and Rényi [3] in 1966, but for every $r > 2$ this remains an open problem. The main difficulty in tackling it seems to originate in the lack of appropriate combinatorial tools (such as the Hall-König and Tutte theorems in graph theory).

A possible and rather natural way to make a progress in this important problem is to discuss its fractional relaxation, that is, to consider the problem of determining the threshold function for a perfect *fractional* matching. For a hypergraph $H = (V, E)$, a non-negative real-valued function $f : E \to R^+$ is called a *fractional matching* if $\sum_{v\in e} f(e) \leq 1$ for every vertex $v \in V$. Clearly, if $f$ takes only 0-1 values, then the set of all edges of positive weight forms a matching. The *value* $|f|$ of a fractional matching $f$ is $|f| = \sum_{e\in E} f(e)$. A fractional matching $f$ is called *perfect* if $|f| = |V|/r$. (Note that we do not require here that $r$ necessarily divides $n$). We will give some additional definitions and useful facts about fractional matchings in Section 2. It is easy to see that the existence of a perfect matching implies the existence of a perfect fractional matching, but not vice versa.

It turns out that this fractional relaxation of the integer problem is much more tractable, and quite precise results can be obtained about it. In order to formulate them exactly, we introduce the notion of a random hypergraph process. For a fixed integer $r \geq 1$, a *random $r$-uniform hypergraph process* on a set $V$ of size $n$ is a Markov chain $\tilde{H} = (H_t)_0^\infty$, whose states are hypergraphs from $\mathcal{H}_r(n)$. The process starts with the empty hypergraph ($E = \emptyset$) and for $1 \leq t \leq \binom{n}{r}$ the hypergraph $H_t$ is obtained from $H_{t-1}$ by an addition of an edge from $\binom{V}{r} \setminus E(H_{t-1})$, all new edges being equiprobable. Since $H_t$ has exactly $t$ edges, for $t = \binom{n}{r}$ we have a complete $r$-uniform hypergraph on $V$. For all $t > \binom{n}{r}$ we also define $H_t = H_{\binom{n}{r}}$.

Let $\tilde{\mathcal{H}}_r(n)$ be the set of all random hypergraph processes on $n$ vertices. We turn $\tilde{\mathcal{H}}_r(n)$ into a probability space by giving the same probability to each process $\tilde{H} \in \tilde{\mathcal{H}}_r(n)$. We use the notation **whp** in this space as well with the obvious meaning.

The map $\tilde{\mathcal{H}}_r(n) \to \mathcal{H}_r(n, M)$, defined by $\tilde{H} = (H_t)_0^\infty \to H_M$, is measure preserving, so the set of all hypergraphs obtained at time $M$ can be identified with $\mathcal{H}_r(n, M)$.

For a monotone non-empty property $Q$ of $\mathcal{H}_r(n)$ we refer to the time $t = t(Q, \tilde{H})$ at which it appears as the *hitting time* of $Q$:

$$t(Q, \tilde{H}) = \min\{t \geq 0 : H_t \text{ has } Q\} .$$

Now we are equipped with all necessary terminology to formulate our main result.

**Theorem 1** **whp** *a random hypergraph process $\tilde{H} \in \tilde{\mathcal{H}}_r(n)$ is such that*

$$t(H \textit{ has a perfect fractional matching }, \tilde{H}) =$$
$$t(H \textit{ has no isolated vertices }, \tilde{H}) .$$

In words, this theorem states that **whp** at the very moment $t_0$ the last isolated vertex disappears, one has a perfect fractional matching in $H_{t_0}$. This theorem yields the following result about the threshold for a perfect fractional matching in $\mathcal{H}_r(n, p)$.

**Corollary 1** *Let $w(n)$ be any function tending to infinity arbitrarily slowly as $n \to \infty$. Then*

1. *if $p = \frac{\ln n - w(n)}{\binom{n-1}{r-1}}$, then* **whp** *$H \in \mathcal{H}_r(n, p)$ has no perfect fractional matching,*

2. *if* $p = \frac{\ln n + w(n)}{\binom{n-1}{r-1}}$, *then* **whp** $H \in \mathcal{H}_r(n,p)$ *has a perfect fractional matching.*

We will prove the above theorem and corollary in the next sections.

We close this section with some notation used in the sequel. For a hypergraph $H = (V, E)$, the *degree* $d(v)$ of a vertex $v \in V$ is $d(v) = |\{e \in E : v \in e\}|$. If $V_0$ is a subset of $V$, then $H[V_0]$ stands for the induced subhypergraph of $H$ on $V_0$. Two vertices $v, u \in V$ are called *adjacent* if there exists an edge $e \in E$ such that $v, u \in e$.

All logarithms are in base $e = 2.71828\ldots$.

Throughout the paper, the parameter $n$ is assumed to tend to infinity, we also assume it to be sufficiently large if necessary, the uniformity number $r$ is kept fixed. The notation $o()$, $O()$ has its usual meaning, that is, $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$ and $f(n) = O(g(n))$ if there exists a constant $c > 0$ such that $f(n) \leq cg(n)$ for all $n$.

# 2 Fractional matchings and covers in hypergraphs

Let $H = (V, E)$ be a hypergraph. Recall that a non-negative real-valued function $f : E \to R^+$ is called a *fractional matching* with *value* $|f| = \sum_{e\in E} f(e)$ if $\sum_{v\in e} f(e) \leq 1$ for every $v \in V$. The maximum of $|f|$ over all fractional matchings of $H$ is the *fractional matching number* of $H$, denoted by $\nu^*(H)$. Similarly, a *fractional cover* of $H$ is a non-negative real-valued function $g : V \to R^+$ such that $\sum_{v\in e} g(v) \geq 1$ for every $e \in E(H)$. The *value* of $g$ is $|g| = \sum_{v\in V} g(v)$. The minimum of $|g|$ over all fractional covers of $H$ is the *fractional covering number* of $H$, denoted by $\tau^*(H)$.

It is easy to see that the above two definitions of $\nu^*(H)$ and $\tau^*(H)$ can be represented as optimal solutions of a pair of dual linear programming problems. The Duality Theorem of Linear Programming asserts that

**Proposition 1** *For every hypergraph $H = (V, E)$ the following holds true:*

1. *for every fractional cover $g$ and every fractional matching $f$ one has $|g| \geq |f|$;*

2. *$\tau^*(H) = \nu^*(H)$;*

3. *if $g$ is an optimal fractional cover of $H$ (i.e. $|g| = \tau^*(H)$) and $f$ is an optimal fractional matching of $H$ (i.e. $|f| = \nu^*(H)$), then*

$$\begin{aligned} f(e) > 0 \quad & \textit{implies} \quad \sum_{v \in e} g(v) = 1; \\ g(v) > 0 \quad & \textit{implies} \quad \sum_{v \in e} f(e) = 1 \ . \end{aligned} \tag{1}$$

(These are the so called *complementary slackness conditions*).

We will also use the following

**Proposition 2** *For every $r$-uniform hypergraph $H = (V, E)$ one has :*

1. *$\nu^*(H) \geq \nu(H)$;*

2. *if $V_0 \subseteq V$ is the set of all non-isolated vertices of $H$, then $\nu^*(H) \leq |V_0|/r$, therefore $\nu^*(H) \leq |V|/r$;*

3. *if $g : V \to R^+$ is a fractional cover of $H$, then for every subset $U \subseteq V$ the function $g' : U \to R^+$ defined by $g'(v) = g(v)$ for every $v \in U$ (that is, $g'$ is the restriction of $g$ on $U$) is a fractional cover of the hypergraph $H[U]$;*

4. *let $g : V \to R^+$ be an optimal fractional cover of $H$ and denote $V_1 = \{v \in V : g(v) > 0\}$, then $\nu^*(H) \geq |V_1|/r$.*

**Proof.** 1) Let $M \subseteq E$ be a matching of size $|M| = \nu(H)$, then its characteristic function $1_M : E \to \{0, 1\}$ is clearly a fractional matching with value $|M| = \nu(H)$.

2) Define a function $g : V \to R^+$ by $g(v) = 1/r$ for all $v \in V_0$ and $g(v) = 0$ for all $v \notin V_0$, then $g$ can be easily seen to be a fractional cover with value $|g| = |V_0|/r$, therefore $\nu^*(H) = \tau^*(H) \leq |V_0|/r$.

3) Obvious.

4) Let $f : E \to R^+$ be an optimal fractional matching of $H$. Then, by the complementary slackness conditions (1),

$$\begin{aligned} |V_1| &= \sum\{1 : v \in V_1\} = \sum_{v \in V_1} \sum_{v \in e} f(e) = \sum_{e \in E} f(e)|e \cap V_1| \\ &\leq \sum_{e \in E} f(e) r = r \sum_{e \in E} f(e) = r\nu^*(H) , \end{aligned}$$

therefore $\nu^*(H) \geq |V_1|/r$.

The reader is referred to [5] for additional information about integer and fractional matchings and covers in hypergraphs.

# 3 Hypergraph processes and random hypergraphs

The following proposition, whose proof (which we omit) can be obtained just by imitating the proof of the corresponding result for random graphs, gives us some initial intuition of what result we may expect to obtain.

**Proposition 3** *Let $w(n)$ be any function tending arbitrarily slowly to infinity as $n$ tends to infinity.*

1. *If $p = \frac{\log n + w(n)}{\binom{n-1}{r-1}}$, then* **whp** *the number of edges in a random hypergraph $H \in \mathcal{H}_r(n,p)$ is at least $(\log n + w(n))n/r - n^{1/2} \log n\, w(n)$.*
2. *If $p = \frac{\log n - w(n)}{\binom{n-1}{r-1}}$, then* **whp** *$H \in \mathcal{H}_r(n,p)$ has some isolated vertices, on the other hand, if $w(n) \leq \log\log\log n$, then the number of isolated vertices* **whp** *does not exceed $\log n$.*

3. *If* $p = \frac{\log n + w(n)}{\binom{n-1}{r-1}}$, *then* **whp** $H \in \mathcal{H}_r(n,p)$ *has no isolated vertices, on the other hand, if* $w(n) \le \log\log\log n$, *then* **whp** *there exists a vertex of degree one in* $H \in \mathcal{H}_r(n,p)$.

4. *If* $M = \lfloor (\log n - w(n))n/r \rfloor$, *then* **whp** *a random hypergraph* $H \in \mathcal{H}_r(n,M)$ *has some isolated vertices, on the other hand, if* $w(n) \le \log\log\log n$, *then the number of isolated vertices in* $H \in \mathcal{H}_r(n,M)$ **whp** *does not exceed* $\log n$.

5. *If* $M = \lfloor (\log n + w(n))n/r \rfloor$, *then* **whp** $H \in \mathcal{H}_r(n,M)$ *has no isolated vertices, on the other hand, if* $w(n) \le \log\log\log n$, *then* **whp** *there exists a vertex of degree one in* $H \in \mathcal{H}_r(n,M)$.

It follows from the above proposition that at the moment $t = \lfloor (\log n - w(n))n/r \rfloor$ a hypergraph process $\tilde{H} \in \tilde{\mathcal{H}}_r(n)$ **whp** has some isolated vertices and therefore (by Proposition 2, part 2) has no perfect fractional matching, while at the moment $t = \lfloor (\log n + w(n)n/r \rfloor$ the process $\tilde{H}$ **whp** has no isolated vertices and therefore may possibly have a perfect fractional matching. We will prove that this is indeed the situation for almost all hypergraph processes.

Though we will derive the desired result for random hypergraphs from the result about hypergraph processes, actually we are going to use the opposite direction. In order to obtain the result about hitting times in hypergraph processes, we introduce (as in [2], Ch. 7.4) two new models of random hypergraphs, namely, $\mathcal{H}_r(n,p;\ge 1)$ and $\mathcal{H}_r(n,M;\ge 1)$. Both models consist of hypergraphs with vertex set $V$ of size $n$, whose edges are coloured blue and green. To obtain a random element from $\mathcal{H}_r(n,p;\ge 1)$ we first take a random element $H$ of $\mathcal{H}_r(n,p)$ and colour its edges blue. Let $v_1,\dots,v_s$ be all isolated vertices of this blue hypergraph. For each $1 \le i \le s$ we add to $H$ at random an edge of size $r$ containing $v_i$, all such edges being equally likely. We colour these additional edges green. (In case $H \in \mathcal{H}_r(n,p)$ has no isolated vertices, we do not add green edges at all). The probability measure of $\mathcal{H}_r(n,p;\geq 1)$ is induced by the probability measure of $\mathcal{H}_r(n,p)$ in the obvious way. The random model $\mathcal{H}_r(n,M;\geq 1)$ is defined in a similar manner.

Why are these new models important for us? The answer is given by the following lemma, whose proof is shaped after that of Lemma 7.9 of [2].

**Lemma 1** *Let $Q$ be a monotone property in $\mathcal{H}_r(n)$, implying non-existence of isolated vertices. Let*

$$p = \frac{\log n - w(n)}{\binom{n-1}{r-1}} \ ,$$

*where $w(n) \to \infty$, but $w(n) \leq \log\log\log n$. If* **whp** *$H \in \mathcal{H}_r(n,p;\geq 1)$ has $Q$, then* **whp** *in $\tilde{\mathcal{H}}_r(n)$*

$$t(Q,\tilde{H}) = t(H \textit{ has no isolated vertices}, \tilde{H}) \ .$$

**Proof.** Denote by $X$ the number of blue edges in $H \in \mathcal{H}_r(n,p;\geq 1)$. Then $X$ is binomially distributed with parameters $\binom{n}{r}$ and $p$, hence (since $Q$ is monotone) if

$$M_1 = \lfloor \binom{n}{r} p \rfloor = \lfloor (\log n - w(n))n/r \rfloor \ ,$$

then **whp** a random hypergraph $H \in \mathcal{H}_r(n,M_1;\geq 1)$ has $Q$.

Let

$$M_2 = \lfloor (\log n + w(n))n/r \rfloor$$

and let $\tilde{\mathcal{H}}_*$ be the set of hypergraph processes $\tilde{H} = (H_t)_0^\infty$ for which $H_{M_1}$ has an isolated vertex, the minimal degree of $H_{M_2}$ equals to one and every edge added in times $M_1+1,\ldots,M_2$ contains at most one isolated vertex of $H_{M_1}$. It follows from Proposition 3 that **whp** $H_{M_1}$ has some isolated vertices and the number of isolated vertices does not exceed $\log n$. If an edge is added to such a hypergraph then the probability that it contains at least two isolated vertices is at most $\binom{\log n}{2}\binom{n-2}{r-2} \Big/ \left(\binom{n}{r} - M_1\right) = O(\log^2 n/n^2)$. Hence, with probability at least $1-(M_2-M_1)O(\log^2 n/n^2) = 1-o(1)$, no edge added in times $M_1+1,\ldots,M_2$ contains more than one isolated vertex of $H_{M_1}$. Therefore, $P[\tilde{\mathcal{H}}_*]=1-\delta_n$, where $\delta_n\to 0$.

Let $\mathcal{H}_*$ be the collection of hypergraphs from $\mathcal{H}_r(n,M_1;\geq 1)$ in which the blue subhypergraph has some isolated vertices and no green edge contains more than one isolated vertex of the blue subhypergraph. One can show easily that $P[\mathcal{H}_*]=1-\epsilon_n$, where $\epsilon_n\to 0$.

Define now a map $\phi:\tilde{\mathcal{H}}_*\to\mathcal{H}_*$ in the following way. Given $\tilde{H}=(H_t)_0^\infty\in\tilde{\mathcal{H}}_*$, let $\phi(\tilde{H})$ be the coloured hypergraph whose blue subhypergraph is $H_{M_1}$ and whose green edges are all the edges added after time $M_1$ and not later than $M_2$ which increased the degree of some vertex from 0 to 1. Clearly, $\phi(\tilde{\mathcal{H}}_*)=\mathcal{H}_*$ and $\phi$ is measure preserving in the sense that for every $H\in\mathcal{H}_*$ the number of hypergraph processes in $\tilde{\mathcal{H}}_*$ which are mapped by $\phi$ to $H$ is the same. Therefore for every $A\subseteq\mathcal{H}_*$ one has

$$\frac{P[A]}{P[\mathcal{H}_*]}=\frac{P[\phi^{-1}(A)]}{P[\tilde{\mathcal{H}}_*]}\ . \qquad (2)$$

Set

$$\mathcal{H}_0=\{H\in\mathcal{H}(n,M_1;\geq 1): H \text{ has } Q\}\ ,$$

then according to the above discussion $P[\mathcal{H}_0]=1-\gamma_n$, where $\gamma_n\to 0$. Therefore

$$P[\mathcal{H}_0\cap\mathcal{H}_*]\geq 1-\epsilon_n-\gamma_n\ ,$$

hence by (2) the set $\tilde{\mathcal{H}}_0=\phi^{-1}(\mathcal{H}_0\cap\mathcal{H}_*)$ satisfies

$$P[\tilde{\mathcal{H}}_0]=\frac{P[\mathcal{H}_0\cap\mathcal{H}_*]P[\tilde{\mathcal{H}}_*]}{P[\mathcal{H}_*]}\geq\frac{(1-\epsilon_n-\gamma_n)(1-\delta_n)}{1-\epsilon_n}=1-o(1)\ .$$

But if $\tilde{H}\in\tilde{\mathcal{H}}_0$, then $H_{M_1}$ has some isolated vertices (this is because $\phi(\tilde{H})\in\mathcal{H}_*$) and at the moment $t_0$ when the last isolated vertex disappears, $H_{t_0}$ has $Q$ (this is because $\phi(\tilde{H})\in\mathcal{H}_0$). Since according to the lemma conditions $Q$ can not appear before this moment, we have for every $\tilde{H}\in\tilde{\mathcal{H}}_0$

$$t(Q,\tilde{H})=t(H \text{ has no isolated vertices },\tilde{H})\ ,$$

completing the proof. $\square$

A useful fact is that the probability measure of $\mathcal{H}_r(n,p;\geq 1)$ differs only slightly from the one of $\mathcal{H}_r(n,p)$, as shown by the following lemma.

**Lemma 2** *Let $A, B$ be two disjoint subsets of $\binom{V}{r}$. Then the probability that a random hypergraph $H \in \mathcal{H}_r(n,p;\geq 1)$ satisfies $A \subseteq E(H)$, $B \cap E(H) = \emptyset$ is at most*

$$\left(p + \frac{r}{\binom{n-1}{r-1}}\right)^{|A|} (1-p)^{|B|} .$$

**Proof.** For every $e \in \binom{V}{r}$ the probability that $e$ is a green edge of $H \in \mathcal{H}_r(n,p;\geq 1)$ can be bounded from above by the probability that $e$ is chosen to be a green edge for some of its vertices, therefore

$$P[e \text{ is a green edge of } H] \leq \frac{r}{\binom{n-1}{r-1}} ,$$

and thus for every subset $A_0 \subseteq \binom{V}{r}$ one has

$$P[A_0 \text{ consists of green edges }] \leq \left(\frac{r}{\binom{n-1}{r-1}}\right)^{|A_0|} .$$

Returning to $A$ and $B$ from the lemma formulation, we denote $|A| = a$, $|B| = b$. If $A$ contains exactly $i$ green edges, where $0 \leq i \leq a$, then

$$P[A \subseteq E(H), B \cap E(H) = \emptyset] \leq p^{a-i} \left(\frac{r}{\binom{n-1}{r-1}}\right)^{i} (1-p)^b ,$$

therefore

$$P[A \subseteq E(H), B \cap E(H) = \emptyset] \ \leq \ \sum_{i=0}^{a} \binom{a}{i} p^{a-i} \left(\frac{r}{\binom{n-1}{r-1}}\right)^{i} (1-p)^b$$

$$\leq (1-p)^b \sum_{i=0}^{a} \binom{a}{i} p^{a-i} \left( \frac{r}{\binom{n-1}{r-1}} \right)^i$$
$$= (1-p)^b \left( p + \frac{r}{\binom{n-1}{r-1}} \right)^a . \qquad \square$$

**Claim 1** **whp** *every vertex of a random hypergraph* $H \in \mathcal{H}_r(n,p;\geq 1)$*, with* $p = p(n)$ *as in Lemma 1, is contained in at most one green edge.*

**Proof.** According to Proposition 3, part 2, the number of isolated vertices in the blue subhypergraph of $H$ **whp** does not exceed $\log n$. Therefore the probability of existence of a vertex $v \in V(H)$ which is incident with at least two green edges is at most

$$n\binom{\log n}{2} \left( \frac{\binom{n-2}{r-2}}{\binom{n-1}{r-1}} \right)^2 = o(1) . \qquad \square$$

Define now $Q$ as '$H$ contains a perfect fractional matching', then $Q$ is obviously monotone and Proposition 2, part 2 implies that every hypergraph having $Q$ does not contain isolated vertices, so in view of Lemma 1 it remains to prove that a random hypergraph $H \in \mathcal{H}_r(n,p;\geq 1)$ **whp** has a perfect fractional matching, where $p = p(n)$ is as in Lemma 1.

# 4 Properties of $\mathcal{H}_r(n,p;\geq 1)$

Set for the rest of the paper

$$p = \frac{\log n - \log\log\log n}{\binom{n-1}{r-1}} .$$

Also, set $\delta = 0.1$. Define

$$\begin{aligned} W_0 &= \{v \in V : d(v) < \delta \log n\}, \\ W_1 &= \{v \in V \setminus W_0 : \exists e \in E, v \in e, e \cap W_0 \neq \emptyset\}. \end{aligned}$$

($W_0$ is the set of low degree vertices, $W_1$ is the set of their neighbours.)

The following lemma states some properties of almost all hypergraphs from $\mathcal{H}_r(n, p; \geq 1)$ with $p$ as defined above. Basically, it assures that **whp** the set of low degree vertices is relatively small and can be matched and that a random hypergraph has good local expansion and matching properties.

**Lemma 3** *A random hypergraph $H = (V, E) \in \mathcal{H}_r(n, p; \geq 1)$* **whp** *has the following properties:*

**(P0)** *Every vertex is incident with at least one edge;*

**(P1)** *For every vertex $v \in V$ there exists at most one pair of edges $e_1, e_2$ such that $v \in e_1 \cap e_2$ and $|e_1 \cap e_2| \geq 2$;*

**(P2)** $|W_0| \leq n^{0.4}$;

**(P3)** *Every edge $e \in E$ intersects $W_0$ in at most one vertex;*

**(P4)** *Every two edges incident with distinct vertices of $W_0$ do not intersect each other;*

**(P5)** *Every vertex $v \in V \setminus W_0$ is incident with at most one edge intersecting $W_1 \setminus \{v\}$;*

**(P6)** *Every subset $U \subseteq V$ of size $|U| \geq n/\log\log n$ spans at least one edge;*

**(P7)** *For every subset $U \subseteq V$ of size $|U| \leq n/log \log n$ there exist at most $2|U| \log n / \log\log\log n$ edges intersecting $U$ in at least two points;*

**(P8)** *For every pair of disjoint subsets* $U_1, U_2 \subset V$ *of sizes* $|U_1| \leq n/\log\log n$, $|U_2| \leq r|U_1|$, *there exist at most* $2|U_1|\log n/\log\log\log n$ *edges intersecting both* $U_1$ *and* $U_2$;

**(P9)** *For every pair of disjoint subsets* $U_1 \subset V \setminus W_0$, $U_2 \subset V \setminus U_1$ *of sizes* $|U_1| \leq n/\log\log n$, $|U_2| \leq r|U_1|$, *there exists a set* $E_0 \subseteq E$ *of size* $|E_0| \geq |U_1|\log\log\log n$ *such that* $|e \cap U_1| = 1$, $|e \cap (W_1 \setminus U_1)| = 0$, $|e \cap U_2| = 0$ *for every* $e \in E_0$, *and also* $e_1 \cap e_2 \subset U_1$ *for every* $e_1, e_2 \in E_0$.

Two remarks are in place here. First, various bounds cited in (P0)–(P9) are not necessarily tight, but they will suffice for our purposes. Second, in the sequel we will make a direct use only of part of these properties, however they are formulated in the present form so as to make their proof easier.

We postpone the (quite technical) proof of the above lemma until Section 7. Assuming it for granted we proceed with the proof of Theorem 1.

# 5 Perfect fractional matchings in $\mathcal{H}_r(n, p; \geq 1)$

In this section we prove the following

**Lemma 4** *If* $H$ *is a hypergraph on* $n$ *vertices satisfying (P0)–(P9), then* $H$ *has a perfect fractional matching.*

In view of Lemmas 1 and 3 this lemma actually establishes Theorem 1. Note that the assertion of the above lemma is fully deterministic, that is, the lemma guarantees that a hypergraph having certain properties, *always* has a perfect fractional matching.

Here is a brief outline of lemma's proof. First, we find a matching $M$ for the vertices of $W_0$ and delete the set $V_0$ of all vertices, belonging to the edges of $M$, from the hypergraph $H$. In the remaining subhypergraph $H_1$ with vertex set $V_1$ all vertices have relatively large degree (at least $\delta \log n/2$). Now, if $g : V \to R^+$ is an optimal fractional cover of $H$, then $|g| \geq \tau^*(H[V_0]) +$

$\tau^*(H_1)$. The hypergraph $H[V_0]$ has a perfect matching, implying $\tau^*(H[V_0]) \geq |V_0|/r$. The hypergraph $H_1$ has good expansion properties. Therefore, if $g_1 : V_1 \to R^+$ is an optimal fractional cover of $H_1$ and if there exists a vertex $v_0 \in V_1$ with $g_1(v_0) = 0$, then taking $\delta \log n/2$ edges of $H_1$ intersecting each other only at $v_0$, we see that the vertices of these edges (with $v_0$ deleted) have an average weight in $g_1$ at least $1/(r-1)$ (instead of $1/r$ in a perfect fractional cover). We remove these 'heavy' vertices (which we denote by $U$) from $V_1$. The function $g_1$ restricted to the set $U_1 = V_1 \setminus U$ is a fractional cover of $H[U_1]$, therefore $\sum_{v \in U_1} g_1(v) \geq \tau^*(H[U_1])$. Now it suffices to show that $H[U_1]$ has an *almost* perfect fractional matching, that is, we have some extra room to operate, this is due to the fact that the vertices of $U$ are 'overweighted' in $g_1$. This gives us $\tau^*(H_1) \geq |V_1|/r$, implying in turn $\tau^*(H) \geq |V|/r$.

Suppose $H = (V, E)$ is a hypergraph on $n$ vertices satisfying (P0)–(P9). Let $g : V \to R^+$ be an optimal fractional cover of $H$ with value $|g| = \nu^*(H)$. By (P0), every vertex of $H$ is incident with at least one edge. Consider the vertices of $W_0$. If for every vertex $v \in W_0$ we choose arbitrarily an edge $e(v)$ containing it, then the chosen edges are pairwise disjoint as follows from (P4). This means that the set $M = \{e(v) : v \in W_0\}$ is a matching. Denote now

$$V_0 = \{v \in V : \exists e \in M, v \in e\} ,$$

that is, $V_0$ is the union of all vertices in edges of $M$. Denote also

$$\begin{aligned} V_1 &= V \setminus V_0 , \\ n_1 &= |V_1| , \\ H_0 &= H[V_0] , \\ H_1 &= H[V_1] . \end{aligned}$$

It follows from Proposition 2, part 3, that the function $g$, restricted to $V_0$ ($V_1$, resp.) is a fractional cover of the hypergraph $H_0$ ($H_1$, resp.), therefore

$$|g| = \sum_{v \in V_0} g(v) + \sum_{v \in V_1} g(v) \geq \nu^*(H_0) + \nu^*(H_1) \ .$$

Since $V_0$ is a union of edges of a matching, $H_0$ clearly contains a perfect matching and therefore (see Proposition 2, parts 1 and 2)

$$\nu^*(H_0) = \frac{|V_0|}{r} \ .$$

Hence it remains to prove that the hypergraph $H_1$ also has a perfect fractional matching, that is,

$$\nu^*(H_1) = \frac{|V_1|}{r} \ . \tag{3}$$

Note that it follows from (P0)–(P9) that the deletion of $V_0$ does not seriously affect 'nice' properties of $H_1$. This means that $H_1$ satisfies the following.

**(Q1)** Every subset $U \subseteq V_1$ of size $|U| \geq n/\log\log n$ spans an edge of $H_1$ (follows from (P6));

**(Q2)** for every pair of disjoint subsets $U_1, U_2 \subset V_1$ of sizes $|U_1| \leq n/\log\log n$, $|U_2| \leq r|U_1|$ there exists a set $E_0 \subseteq E(H_1)$ of size $|E_0| \geq |U_1| \log\log\log n$ such that $|e \cap U_1| = 1$, $|e \cap U_2| = 0$ for every $e \in E_0$ and also $e_1 \cap e_2 \subset U_1$ for every $e_1, e_2 \in E_0$ (follows from (P9));

**(Q3)** for every vertex $v \in V_1$ there exist at least $\delta \log n - 2 > \frac{\delta}{2} \log n$ edges of $H_1$, whose pairwise intersection is $\{v\}$ (follows from (P1), (P5) and the definition of $W_0$).

The proof of (3) is based on the following lemma.

**Lemma 5** *Let a sequence $\{a_i\}_0^\infty$ be defined as follows: $a_0 = \left\lceil \frac{r^2 n}{\log\log n} \right\rceil$ and $a_i = \left\lceil \frac{a_{i-1}}{1+\log\log\log n} \right\rceil$ for every $i \geq 1$. Denote*

$$k_0 = \min\{i : a_i \leq \log\log n\} \ .$$

*Then for every* $0 \le i \le k_0$ *every subset* $U \subset V_1$ *of size* $|U| = a_i$ *satisfies*

$$\nu^*(H_1[V_1 \setminus U]) > \frac{n_1 - a_i}{r} - \frac{a_i}{(r-1)r} \ .$$

**Proof.** Let first $i = 0$. Fix a subset $U \subset V_1$ of size $|U| = a_0$ and denote $U_1 = V_1 \setminus U$, we will prove that $\nu^*(H_1[U_1]) > |U_1|/r - n/\log\log n > |U_1|/r - a_0/(r-1)r$. By (Q1), every subset $U_0 \subset U_1$ of size $|U_0| \ge n/\log\log n$ spans an edge of $H_1$, therefore $U_1$ contains a matching of size more than $|U_1|/r - n/\log\log n$, which can be obtained, for example, by picking the edges one by one greedily. Therefore by Proposition 2, part 1, $\nu^*(H_1[U_1]) \ge \nu(H_1[U_1]) > |U_1|/r - n/\log\log n$.

Assuming that the assertion of the lemma holds for all indices between 0 and $i-1$ and still $a_{i-1} > \log\log n$, we prove the assertion for $a_i$. Let $U$ be a subset of $V_1$ of size $|U| = a_i$, denote $U_1 = V_1 \setminus U$. Suppose $g : U_1 \to R^+$ is an optimal fractional cover of $H_1[U_1]$ with value $|g| = \nu^*(H_1[U_1])$. Denote $U_0 = \{v \in U_1 : g(v) = 0\}$. If $|U_0| < a_i/(r-1)$, then it follows from Proposition 2, part 4, that $\nu^*(H_1[U_1]) \ge |U_1 \setminus U_0|/r > (n_1 - a_i - a_i/(r-1))/r = (n_1 - a_i)/r - a_i/(r-1)r$, as required. Thus we may assume that $|U_0| \ge a_i/(r-1)$. Fix a subset $U_0' \subseteq U$ of size $a_i/(r-1) \le |U_0'| \le n/\log\log n$. According to (Q2) with $U_0'$ and $U$ instead of $U_1$ and $U_2$, respectively, there exists a subset $E_0 \subseteq E(H_1[U_1])$ of size $|E_0| \ge |U_0'| \log\log\log n$ such that the intersection of any two edges from $E_0$ is contained in $U_0'$. Every edge $e \in E_0$ is covered by $g$, and since $U_0'$ consists of vertices of zero weight in $g$, it follows that $\sum_{v \in e \setminus U_0'} g(v) \ge 1$. Denote $T = \bigcup_{e \in E_0} e \setminus U_0'$. Since all sets $\{e \setminus U_0' : e \in E_0\}$ are pairwise disjoint, we obtain $|T| = (r-1)|E_0| \ge (r-1)|U_0'| \log\log\log n \ge a_i \log\log\log n$ and $\sum_{v \in T} g(v) \ge |E_0| = |T|/(r-1)$. (A crucial observation here is that the average weight of the vertices of $T$ in $g$ is at least $1/(r-1)$ instead of $1/r$ as in a perfect fractional cover). Denote by $T_0$ the subset of $T$ consisting of $a_{i-1} - a_i \le a_i \log\log\log n$ vertices with the largest weights in $g$. Clearly $\sum_{v \in T_0} g(v) \ge |T_0|/(r-1) = (a_{i-1} - a_i)/(r-1)$. Consider now the hypergraph $H_1[U_1 \setminus T_0]$. By Proposition 2, part 3, the function $g$

restricted to the vertices of $U_1 \setminus T_0$ is a fractional cover of $H_1[U_1 \setminus T_0]$, therefore $\sum_{v \in U_1 \setminus T_0} g(v) \geq \nu^*(H_1[U_1 \setminus T_0])$. On the other hand, since $|T_0| = a_{i-1} - a_i$, one has $\nu^*(H_1[U_1 \setminus T_0]) > (n_1 - a_{i-1})/r - a_{i-1}/(r-1)r$, by the induction hypothesis. Summing the above, we obtain

$$
\begin{aligned}
|g| &= \sum_{v \in U_1} g(v) = \sum_{v \in T_0} g(v) + \sum_{v \in U_1 \setminus T_0} g(v) \\
&> \frac{a_{i-1} - a_i}{r-1} + \frac{n_1 - a_{i-1}}{r} - \frac{a_{i-1}}{(r-1)r} \\
&= \frac{n_1}{r} - \frac{a_i}{r-1} = \frac{n_1 - a_i}{r} - \frac{a_i}{(r-1)r} . \qquad \square
\end{aligned}
$$

Returning to the proof of Lemma 4, we use essentially the same idea as in the proof of Lemma 5. Let $g_1 : V_1 \to R^+$ be an optimal fractional cover of $H_1$ with value $|g_1| = \nu^*(H_1)$. If all vertices of $V_1$ have positive weights in $g_1$, then it follows from Proposition 2, parts 2 and 4, that $\nu^*(H_1) = |V_1|/r = n_1/r$. If there exists a vertex $v_0 \in V_1$ with $g_1(v_0) = 0$, consider a maximum set $E_0$ of edges of $H_1$, whose pairwise intersection is $\{v_0\}$. According to (Q3), $|E_0| \geq \frac{\delta}{2} \log n$. Since all edges of $E_0$ are covered by $g_1$, one has $\sum_{v \in e \setminus \{v_0\}} g_1(v) \geq 1$ for every $e \in E_0$. Denote $T = \bigcup_{e \in E_0} e \setminus \{v_0\}$, then $|T| = (r-1)|E_0| \geq \frac{\delta(r-1)}{2} \log n > a_{k_0}$ and $\sum_{v \in T} g_1(v) \geq |T|/(r-1)$. Let $T_0$ be a subset of $T$, consisting of $a_{k_0}$ vertices with the largest weights in $g_1$, then $\sum_{v \in T_0} g_1(v) \geq |T_0|/(r-1) = a_{k_0}/(r-1)$. Consider the hypergraph $H_1[V_1 \setminus T_0]$. It follows from Lemma 4 that

$$
\sum_{v \in V_1 \setminus T_0} g_1(v) \geq \nu^*(H_1[V_1 \setminus T_0]) > \frac{n_1 - a_{k_0}}{r} - \frac{a_{k_0}}{(r-1)r} ,
$$

therefore

$$
\begin{aligned}
\nu^*(H_1) &= \sum_{v \in V_1} g_1(v) = \sum_{v \in T_0} g_1(v) + \sum_{v \in V_1 \setminus T_0} g_1(v) \\
&> \frac{a_{k_0}}{r-1} + \frac{n_1 - a_{k_0}}{r} - \frac{n_1 - a_{k_0}}{(r-1)r} \\
&= \frac{n_1}{r} ,
\end{aligned}
$$

obtaining a contradiction since by Proposition 2, part 2, $\nu^*(H_1) \leq n_1/r$. (Actually, we have shown that such a vertex $v_0$ with $g_1(v_0) = 0$ does not exist).

The proof of Lemma 4 and Theorem 1 has been finished. $\square$

**Proof of Corollary 1.** 1)Follows from Proposition 2, part 2, and Proposition 3, part 2;

2) It follows from Proposition 3, part 1, that **whp** the number of edges in $H \in \mathcal{H}_r(n,p)$ is at least $(\log n + w'(n))n/r$ for some function $w'(n) \to \infty$ as $n \to \infty$, then Proposition 3, part 5, and Theorem 1 imply the desired result. $\square$

# 6 Concluding remarks

Results similar to those presented above can be obtained also for random $r$-partite hypergraphs. A hypergraph $H = (V, E)$ is called *r-partite* if there exists a partition $V = V_1 \cup \ldots \cup V_r$ such that $|E \cap V_i| = 1$ for every $1 \leq i \leq r$. A *random r-partite hypergraph* $\mathcal{H}'_r(n,p)$ is an $r$-partite hypergraph with vertex set $V = V_1 \cup \ldots \cup V_r$, $|V_i| = n$, $1 \leq i \leq r$, in which each subset $e \in V_1 \times \ldots \times V_r$ is chosen to be an edge independently and with probability $p$. The corresponding probability space of hypergraph processes $\tilde{\mathcal{H}}'_r(n)$ is defined in the obvious way. A perfect fractional matching $f : E \to R^+$ in this model has value $|f| = n$.

**Theorem 2** **whp** *a random hypergraph process* $\tilde{H} \in \tilde{\mathcal{H}}'_r(n)$ *is such that*

$$t(H \text{ has a perfect fractional matching }, \tilde{H}) = t(H \text{ has no isolated vertices }, \tilde{H}) \ .$$

**Corollary 2** *Let $w(n)$ be any function tending to infinity arbitrarily slowly as $n \to \infty$. If $p = \frac{\ln n - w(n)}{n^{r-1}}$, then* **whp** *$H \in \mathcal{H}'_r(n,p)$ has no perfect fractional matching, and if $p = \frac{\ln n + w(n)}{n^{r-1}}$, then* **whp** *$H \in \mathcal{H}'_r(n,p)$ has a perfect fractional matching.*

The proof of the above theorem and corollary proceeds along the same lines as the presented proof for the model $\mathcal{H}_r(n,p)$, and is thus omitted.

The results obtained in this paper give some evidence supporting the commonly believed conjecture stating that **whp** at the very moment $t_0$ the last isolated vertex disappears, a hypergraph $H_{t_0}$ has a perfect *integer* matching. However, it seems that the methods used to prove Theorem 1 do not suffice to prove this conjecture.

# 7 Appendix: Proof of Lemma 3

**(P0)** Follows immediately from the definition of $\mathcal{H}_r(n,p;\geq 1)$;

**(P1)** Let us fix a vertex $v \in V$ and bound the probability that it violates (P1). It can be easily seen that the following three cases are the only existing possibilities.

*Case 1.* There exist three edges $e_1, e_2, e_3$ such that $\{v,u\} \subset e_1 \cap e_2 \cap e_3$ for some vertex $u \in V \setminus \{v\}$. The probability of this case is at most

$$(n-1)\binom{\binom{n-2}{r-2}}{3}\left(p+\frac{r}{\binom{n-1}{r-1}}\right)^3$$
$$= O(n^{1+3(r-2)-3(r-1)}\log^3 n) = o(n^{-1}) \ .$$

*Case 2.* There exist three edges $e_1, e_2, e_3 \in E$ such that $\{v,u\} \subset e_1 \cap e_2$ and $\{v,w\} \subset e_1 \cap e_3$ for some $u \neq w \in V \setminus \{v\}$. The probability of this case is at most

$$\binom{n-1}{2}\binom{n-2}{r-2}^2\binom{n-3}{r-3}\left(p+\frac{r}{\binom{n-1}{r-1}}\right)^3$$
$$= O(n^{2+2(r-2)+r-3-3(r-1)}\log^3 n) = o(n^{-1}).$$

*Case 3.* There exist four edges $e_1, e_2, e_3, e_4 \in E$ such that $\{v,u\} \subset e_1 \cap e_2$ and $\{v,w\} \subset e_3 \cap e_4$ for some $u \neq w \in V \setminus \{v\}$. The probability of this case does not exceed

$$\binom{n-1}{2}\binom{\binom{n-2}{r-2}}{2}^2\left(p+\frac{r}{\binom{n-1}{r-1}}\right)^4$$
$$= O(n^{2+4(r-2)-4(r-1)}\log^4 n) = o(n^{-1}) ,$$

hence the probability of the existence of a vertex $v \in V$ violating (P1) is at most $n \cdot o(n^{-1}) = o(1)$.

**(P2)** For every $v \in V$ we bound from above the probability $P[d(v) < \delta \log n]$. Denote for every $1 \le i \le \lfloor \delta \log n \rfloor$

$$s_i = \binom{\binom{n-1}{r-1}}{i}\left(p+\frac{r}{\binom{n-1}{r-1}}\right)^i (1-p)^{\binom{n-1}{r-1}-i} ,$$

the sequence $\{s_i\}$ can be easily checked to be increasing. It follows from Lemma 2 that $P[d(v) = i] \le s_i$, therefore

$$P[d(v) < \delta \log n] \le \sum_{i=1}^{\lfloor \delta \log n \rfloor} s_i \le \delta \log n \; s_{\lfloor \delta \log n \rfloor} .$$

Estimating $s_{\lfloor \delta \log n \rfloor}$, we obtain

$$\begin{aligned}
s_{\lfloor \delta \log n \rfloor} &= \binom{\binom{n-1}{r-1}}{\lfloor \delta \log n \rfloor}\left(p+\frac{r}{\binom{n-1}{r-1}}\right)^{\lfloor \delta \log n \rfloor} (1-p)^{\binom{n-1}{r-1}-\lfloor \delta \log n \rfloor} \\
&\le \left(\frac{e\binom{n-1}{r-1}}{\lfloor \delta \log n \rfloor}\;\frac{\log n - \log\log\log n + r}{\binom{n-1}{r-1}}\right)^{\lfloor \delta \log n \rfloor} \times \\
&\times \; \exp\left\{-\frac{\log n - \log\log\log n}{\binom{n-1}{r-1}}\left(\binom{n-1}{r-1} - \lfloor \delta \log n \rfloor\right)\right\} \\
&\le \left(\frac{e \log n}{\lfloor \delta \log n \rfloor}\right)^{\lfloor \delta \log n \rfloor} e^{-\log n + \log\log\log n + o(1)} \\
&= \exp\{\delta \log n + \delta \log(1/\delta)\log n - \log n + \log\log\log n + O(1)\} \\
&= O(n^{\delta + \delta \log(1/\delta) - 1}\log\log n) \le n^{-0.66} .
\end{aligned}$$

Hence

$$P[d(v) < \delta \log n] \leq \delta \log n \cdot n^{-0.66} \leq n^{-0.65} \ .$$

It follows that the expectation of the number of vertices of $H \in \mathcal{H}_r(n, p; \geq 1)$ of degree less than $\delta \log n$ does not exceed $n \cdot n^{-0.65} = n^{0.35}$ and using Markov's inequality we obtain that

$$P[|\{v \in V : d(v) < \delta \log n\}| \geq n^{0.4}] \leq \frac{n^{0.35}}{n^{0.4}} = o(1) \ ;$$

**(P3)** For every vertex $v \in W_0$ of degree $d(v) < \delta \log n$ let us choose $d(v)$ edges incident with it. (P2) asserts that **whp** $|W_0| \leq n^{0.4}$. Conditioning on this inequality, we have for every edge $e$ incident with $v$

$$P[|e \cap W_0| > 1] \leq \frac{|W_0|\binom{n-2}{r-2}}{\binom{n-1}{r-1}} \leq \frac{n^{0.4}(r-1)}{n-1} = O(n^{-0.6}) \ .$$

Therefore

$$P[\exists e \in E : |e \cap W_0| > 1] \leq |W_0| \max_{v \in W_0} d(v) \, P[|e \cap W_0| > 1] = o(1) \ ;$$

**(P4)** In view of (P3) it remains to prove that **whp** every vertex from $V \setminus W_0$ is adjacent to at most one vertex from $W_0$. Fix a vertex $v \in V \setminus W_0$. Every vertex $u \in W_0$ is adjacent to at most $d(u)(r-1) < \delta \log n(r-1)$ vertices from $V \setminus W_0$ and all these vertices are equally likely, therefore

$$P[v \text{ is adjacent to } u|(P2)] < \frac{(r-1)\delta \log n}{n - n^{0.4}} \ ,$$

and thus

$$\begin{aligned} & P[\exists v \in V \setminus W_0, u_1, u_2 \in W_0 : v \text{ is adjacent to } u_1, u_2|(P2)] \\ \leq \ & |V \setminus W_0| \binom{|W_0|}{2} P[v \text{ is adjacent to } u \in W_0|(P2)]^2 \\ = \ & O(n \cdot n^{0.4 \cdot 2} \left(\frac{logn}{n}\right)^2) = o(1) \ ; \end{aligned}$$

**(P5)** It is easy to see that if the assertion of (P5) does not hold then at least one of the following cases happens:

1) there exist vertices $u, v, w \in V$ and edges $e_1, e_2, e_3 \in E$ such that $w \in W_0$ and $u, w \in e_1$, $u, v \in e_2 \cap e_3$;

2) there exist vertices $u_1, u_2, v, w \in V$ and edges $e_1, e_2, e_3 \in E$ such that $w \in W_0$ and $u_1, u_2, w \in e_1$, $u_1, v \in e_2$, $u_2, v \in e_3$;

3) there exist vertices $u_1, u_2, v, w \in V$ and edges $e_1, e_2, e_3, e_4 \in E$ such that $w \in W_0$ and $u_1, w \in e_1$, $u_2, w \in e_2$, $u_1, v \in e_3$, $u_2, v \in e_4$;

4) there exist vertices $u_1, u_2, v, w_1, w_2 \in V$ and edges $e_1, e_2, e_3, e_4 \in E$ such that $w_1, w_2 \in W_0$ and $u_1, w_1 \in e_1$, $u_2, w_2 \in e_2$, $u_1, v \in e_3$, $u_2, v \in e_4$;

5) there exist vertices $u, v, w \in V$ (where $u$ may coincide with $w$) and edges $e_1, e_2 \in E$ such that $w \in W_0$ and $u, v, w \in e_1$, $u, v \in e_2$;

6) there exist vertices $v, w_1, w_2 \in V$ and edges $e_1, e_2 \in E$ such that $w_1, w_2 \in W_0$ and $v, w_1 \in e_1$, $v, w_2 \in e_2$;

7) there exist vertices $u, v, w \in V$ and edges $e_1, e_2, e_3 \in E$ such that $w \in W_0$ and $v, w \in e_1$, $u, w \in e_2$, $u, v \in e_3$;

8) there exist vertices $u, v, w_1, w_2 \in V$ and edges $e_1, e_2, e_3 \in E$ such that $w_1, w_2 \in W_0$ and $v, w_1 \in e_1$, $v, w_2 \in e_2$, $u, v \in e_3$.

(The cases 5)–8) cover the case when $v \in W_1$).

Note that according to the proof of (P2) for every pair of vertices $w_1, w_2 \in V$ we have $P[d(w_1) < \delta \log n] \leq n^{-0.6}$ and $P[d(w_1) < \delta \log n, d(w_2) < \delta \log n] \leq (n^{-0.6})^2 = n^{-1.2}$. Straightforward estimates show that the probability of each of the above cases is $o(1)$. Let us prove this, for example, for case 4). The probability that this case happens is at most

$$O\left(n\binom{\binom{n-1}{r-1}}{2}p^2\binom{n-1}{r-1}^2p^2n^{-1.2}\right)$$

(Choose $v$, then choose $e_3$ and $e_4$, then $e_1$ and $e_2$, and finally require that $d(w_1) < \delta \log n$, $d(w_2) < \delta \log n$). The above expression is $O\left(n^{4r-4.2}p^4\right) = o(1)$;

**(P6)**

$$P[\exists V' \subset V, |V'| = \left\lceil \frac{n}{\log\log n} \right\rceil, E(H[V']) = \emptyset]$$
$$\le \binom{n}{\left\lceil \frac{n}{\log\log n} \right\rceil} (1-p)^{\binom{\lceil \frac{n}{\log\log n} \rceil}{r}}$$
$$\le (e \log\log n)^{\frac{n}{\log\log n}} \exp\left\{ -\frac{\log n - \log\log\log n}{\binom{n-1}{r-1}} \left( \frac{n}{r \log\log n} \right)^r \right\} = o(1) \ ;$$

**(P7)** For a set $U \subset V$ of size $|U| = k \le \frac{n}{\log\log n}$ denote by $X_U$ the number of blue edges in $H$, intersecting $U$ in at least two points. The random variable $X_U$ is binomially distributed with parameters $t_k$ and $p$, where $t_k \le \binom{k}{2}\binom{n-2}{r-2}$. Then

$$P[\exists U, |U| \le \frac{n}{\log\log n}, X_U \ge \frac{|U| \log n}{\log\log\log n}]$$
$$\le \sum_{k=2}^{\lfloor \frac{n}{\log\log n} \rfloor} \binom{n}{k} \binom{t_k}{\left\lceil \frac{k \log n}{\log\log\log n} \right\rceil} p^{\lceil \frac{k \log n}{\log\log\log n} \rceil}$$
$$\le \sum_{k=2}^{\lfloor \frac{n}{\log\log n} \rfloor} \left( \frac{en}{k} \right)^k \left( \frac{e t_k p}{\left\lceil \frac{k \log n}{\log\log\log n} \right\rceil} \right)^{\lceil \frac{k \log n}{\log\log\log n} \rceil} p^{\lceil \frac{k \log n}{\log\log\log n} \rceil}$$
$$\le \sum_{k=2}^{\lfloor \frac{n}{\log\log n} \rfloor} \left[ \frac{en}{k} \left( \frac{cek \log\log\log n}{n} \right)^{\frac{\log n}{\log\log\log n}} \right]^k = \sum_{k=2}^{\lceil \frac{n}{\log\log n} \rceil} s_k \ .$$

The expression in brackets is an increasing function of $k$, which is less than 1 for every $2 \le k \le n/\log\log n$. Hence, if $a \le k \le b$, the $k$-th summand of the above sum can be estimated from above by substituting $k = b$ in the brackets and $k = a$ in the power.

Consider two intervals $2 \le k \le n^{1/2}$ and $n^{1/2} \le k \le n/\log\log n$. In the first interval we have

$$s_k \le \left[ en^{\frac{1}{2}} \left( \frac{cen^{\frac{1}{2}} \log\log\log n}{\log\log n} \right)^{\frac{\log n}{\log\log\log n}} \right]^2 = o(n^{-1}) \ ,$$

while in the second interval

$$s_k \le \left[ e \log\log n \left( \frac{ce \log\log\log n}{\log\log n} \right)^{\frac{\log n}{\log\log\log n}} \right]^{n^{\frac{1}{2}}} = o(n^{-1}) \ .$$

It follows that **whp** for every set $U \subset V$ of size $|U| = k \le n/\log\log n$ there exist at most $k \log n/\log\log\log n$ blue edges intersecting it in at least two points.

As indicated by Claim 1, **whp** every vertex of $U$ is incident with at most one green edge, therefore the green edges contribute at most $|U|$ to the total number of edges intersecting $U$ in at least two points, so this quantity **whp** does not exceed $|U| \log n/\log\log\log n + |U| < 2|U| \log n/\log\log\log n$;

**(P8)** Clearly, it suffices to prove the assertion of (P8) for the case $|U_2| = r|U_1|$.

For two disjoint sets $U_1, U_2 \subset V$ of sizes $|U_1| = k \le n/\log\log n$, $|U_2| = rk$, denote by $X_{U_1,U_2}$ the number of blue edges of $H$, intersecting both $U_1$ and $U_2$. The random variable $X_{U_1,U_2}$ is binomially distributed with parameters $t_k$ and $p$, where $t_k \le k \cdot rk\binom{n-2}{r-2}$. Then

$$\begin{aligned} & P[\exists U_1, U_2, U_1 \cap U_2 = \emptyset, |U_1| \le \frac{n}{\log\log n}, |U_2| = r|U_1|, \\ & X_{U_1,U_2} \ge |U_1| \log n/\log\log\log n] \\ \le & \sum_{k=2}^{\left\lceil \frac{n}{\log\log n} \right\rceil} \binom{n}{k} \binom{n-k}{rk} \binom{t_k}{\left\lceil \frac{k \log n}{\log\log\log n} \right\rceil} p^{\left\lceil \frac{k\log n}{\log\log\log n} \right\rceil} \ . \end{aligned}$$

In a manner quite similar to the proof of (P7), one can show that every summand of the above sum is $o(n^{-1})$. Again, by Claim 1 for every choice of $U_1, U_2$ as above the green edges contribute at most $|U_1| + |U_2| < |U_1| \log n/\log\log\log n$ to the total number of edges intersecting both $U_1$ and $U_2$;

**(P9)** It suffices to prove the assertion for the case $|U_2| = r|U_1|$.

Let us first choose $U_1$ and $U_2$, denote $k = |U_1|$. It follows from (P5) that **whp** for every vertex $v$ of $U_1$ at most one edge incident with it intersects $W_1 \setminus \{v\}$. Also, from (P7) we get that **whp** at most $2k \log n / \log\log\log n$ edges intersect $U_1$ in at least two vertices. Finally, (P8) asserts that **whp** at most $2k \log n / \log\log\log n$ edges intersect both $U_1$ and $U_2$ and therefore, assuming that (P5), (P7) and (P8) hold true and recalling that the degree of every vertex of $U_1$ is at least $\delta \log n$, we see that at least $k\,\delta \log n - k - 2k \log n / \log\log\log n - 2k \log n / \log\log\log n > \frac{1}{2}\delta k \log n$ edges have one point in $U_1$ and the remaining $r-1$ points in $V \setminus (W_1 \cup U_1 \cup U_2)$. Let us denote the set of these edges by $E_1$. Define now the following process of building a set $E_0 \subset E_1$. Initially $E_0 = \emptyset$. At each step, we inspect edges from $E_1 \setminus E_0$ and add to $E_0$ an edge $e$ if $e \cap e' \subset U_1$ for every edge $e' \in E_0$, if there exist several such edges we choose one of them arbitrarily. We proceed with this process until no edge can be added to $E_0$. Let us look at $E_0$ after the process has terminated. We claim that it satisfies the conditions of (P9). Obviously, due to the definition of $E_1$ we need only check that $|E_0| \geq |U_1| \log\log\log n$. Suppose that this is not so, this means that every edge from $E_1$ intersects the set

$$U_3 = \{v \in V \setminus U_1 : \exists e \in E_0, v \in e\}$$

in at least one point and $|U_3| = (r-1)|E_0| < (r-1)k \log\log\log n$. For a randomly chosen edge $e$ intersecting $U_1$ in exactly one vertex and contained in $V \setminus (W_1 \cup U_2)$ the probability that $e$ intersects $U_3$ is at most

$$\frac{|U_1||U_3|\binom{|V\setminus(U_1\cup U_2\cup W_1)|}{r-2}}{|U_1|\binom{|V\setminus(U_1\cup U_2\cup W_1)|}{r-1}} \leq O\left(\frac{k \log\log\log n}{n}\right) .$$

But $|E_1| > \frac{1}{2}\delta k \log n$, therefore the probability of the existence of a pair $U_1, U_2$ violating (P9) can be bounded from above by

$$\sum_{k=1}^{\lfloor \frac{n}{\log\log n} \rfloor} \binom{n}{k}\binom{n}{rk}\left(O\left(\frac{k \log\log\log n}{n}\right)\right)^{\frac{1}{2}\delta k \log n}$$

and every summand in the above sum can be shown to be $o(n^{-1})$ by methods similar to those in the proof of (P7).

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