Choosability in random hypergraphs

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Abstract

The choice number of a hypergraph $H = (V, E)$ is the least integer s for which for every family of color lists $\mathcal{S} = \{S(v) : v \in V\}$, satisfying $|S(v)| = s$ for every $v \in V$, there exists a choice function f so that $f(v) \in S(v)$ for every $v \in V$, and no edge of H is monochromatic under f. In this paper we consider the asymptotic behavior of the choice number of a random k-uniform hypergraph $H(k, n, p)$. Our main result states that for every $k \geq 2$ and for all values of the edge probability $p = p(n)$ down to $p = O(n^{-k+1})$ the ratio between the choice number and the chromatic number of $H(k, n, p)$ does not exceed $k^{1/(k-1)}$ asymptotically. Moreover, for large values of p, namely, when $p \geq n^{-(k-1)^2/(2k)+\epsilon}$ for an arbitrary positive constant ϵ, the choice number and the chromatic number of $H(k, n, p)$ have almost surely the same asymptotic value.

1 Introduction

A *hypergraph* H is an ordered pair $H = (V, E)$, where V is a finite set, called the *vertex set* of H, and E is a family of distinct subsets of V (the *edge set*). A hypergraph is *k-uniform*, if all edges have cardinality k. Thus, for $k = 2$ the notion of a k-uniform hypergraph coincides with the familiar notion of a graph.

A *random k-uniform hypergraph* $H(k, n, p)$ is a k-uniform hypergraph on n labeled vertices $V = \{1, \ldots, n\}$, where each k-subset of V is chosen to be an edge of H independently and with probability $p = p(n)$. Note that for $k = 2$ we get the well studied model $G(n, p)$ of random graphs. In this paper we will study asymptotic properties of $H(k, n, p)$, that is, the number of vertices n will tend to infinity, while the uniformity parameter k will be kept fixed. A hypergraph property A holds *almost surely* in $H(k, n, p)$, or a.s. for brevity, if the probability that a hypergraph H drawn from $H(k, n, p)$ has A tends to 1 as n tends to infinity.

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Following the notation adopted in the paper [9] of Krivelevich and Sudakov, for every integer $1 \le \gamma \le k-1$, we define a *γ-independent set* in a k-uniform hypergraph $H=(V,E)$ as a subset $I \subseteq V$ such that every edge of H intersects I in at most γ vertices. The *γ-independence number* $\alpha_\gamma(H)$ is the maximal size of a γ-independent set in H. A *γ-coloring* of H is a partition of the vertex set V into γ-independent sets (colors). The *γ-chromatic number* $\chi_\gamma(H)$ is the minimal number of colors in a γ-coloring of H. The most popular choices for γ are $\gamma = k-1$, in which case we get what is usually called the *weak chromatic number* of H, and $\gamma = 1$, corresponding to the *strong chromatic number* of H. Comparing the situation again with the case of graphs, note that for $k=2$ both notions of the weak and the strong chromatic number reduce to the notion of the chromatic number of a graph, one of the most central concepts in graph theory.

In this paper we will discuss choosability properties of random hypergraphs. Fix a parameter $1 \le \gamma \le k-1$. Given a k-uniform hypergraph $H=(V,E)$ and a family of color lists $\mathcal{S} = \{S(v) : v \in V\}$, we say that H is *$\mathcal{S}$-choosable*, if there exists a choice function f, acting on the vertices of H, so that $f(v) \in S(v)$ for every vertex $v \in V$, and for each color $c \in \bigcup_{v\in V} S(v)$, the set $U_c = \{v \in V : f(v) = c\}$ forms a γ-independent set. H is called *s-choosable*, if it is $\mathcal{S}$-choosable for every family of color lists $\mathcal{S} = \{S(v) : v \in V\}$, meeting the restriction $|S(v)| = s$ for every $v \in V$. Finally, the *choice number of H*, denoted by $ch(H)$, is the least s for which H is s-choosable. In this paper we will concentrate mostly on the case $\gamma = k-1$ and therefore will usually omit the value of γ. Thus, our definition of the choice number is a generalization of the notion of a weak chromatic number. We would like to note here that the notation of this paper is different from this of [14], where the case $\gamma = 1$ is discussed. Again, for the case $k=2$ we get the definition of the choice number of a graph, introduced in the seventies independently by Erdős, Rubin and Taylor [7] and by Vizing [12].

It is easy to see that for every hypergraph H we have $ch(H) \ge \chi(H)$. Thus it is quite natural to compare the asymptotic behavior of the choice number of random uniform hypergraphs with that of the chromatic number. Extending the results of Bollobás [5] and Łuczak [10] for the graph case, Shamir [11] and Krivelevich and Sudakov [9] established the asymptotic value of the chromatic number of $H(k,n,p)$ for all values of p in the range $Cn^{-k+1} \le p(n) \le 0.9$. Their results can be formulated in the following form. Let

$$t_0 = t_0(n,p) = \max\left\{ t : \binom{n}{t}(1-p)^{\binom{t}{k}} \ge 1 \right\}.$$

Then a.s. in $H(k,n,p)$ one has

$$\chi(H) = (1+o(1))\frac{n}{t_0},$$

where the $o(1)$ term tends to 0 as the product $n^{k-1}p$ tends to infinity. It is a routine exercise to figure out the asymptotic behavior of the chromatic number from the above definition of t_0. It gives the following results: when $p(n)$ is an absolute constant, then a.s. in $H(k,n,p)$, $\chi(H) =$

$(1+o(1))n/(k!\ln n/\ln(1/(1-p)))^{1/(k-1)}$; and when $p(n)=o(1)$, then denoting $d^*=(k-1)\binom{n-1}{k-1}p$, one gets that a.s. $\chi(H)=(1+o(1))(d^*/(k\ln d^*))^{1/(k-1)}$.

Recently there has been a considerable amount of interest in choosability properties of random graphs. Kahn (see [1]) showed that for every constant edge probability p, the choice number and the chromatic number of $G(n,p)$ have almost surely the same asymptotic value. His result has been extended by Krivelevich [8] to all values of p, satisfying $p(n)\geq n^{-1/4+\epsilon}$, for any $\epsilon>0$. Alon, Krivelevich and Sudakov [2] and independently Vu [13] proved that for all values of p, the choice number and the chromatic number of $G(n,p)$ have asymptotically the same order of magnitude.

However, not much has been known about choosability in random hypergraphs. The only paper about this subject we are aware about is that of the second author [14], in which the asymptotic behavior of the choice number of random hypergraphs for the case $\gamma=1$ is addressed.

The goal of this paper is to close the gap between the graph and the hypergraph case. We prove the following two main results.

Theorem 1 *Let $k\geq 2$ be a fixed integer. Let the edge probability $p=p(n)$ satisfy $C/n^{k-1}\leq p(n)\leq 0.9$, where $C>0$ is a large enough constant. Then almost surely in $H(k,n,p)$*

$$ch(H)\leq(1+o(1))k^{\frac{1}{k-1}}\chi(H)\ ,$$

where the $o(1)$ term tends to 0 as $n^{k-1}p$ tends to infinity.

Theorem 2 *Let $k\geq 2$ be an integer. Let $0<\epsilon<(k-1)^2/(2k)$. If the edge probability $p=p(n)$ satisfies $n^{-(k-1)^2/(2k)+\epsilon}\leq p\leq 0.9$, then almost surely in $H(k,n,p)$*

$$ch(H)=(1+o(1))\chi(H)\ .$$

Hence the choice number and the chromatic number of random hypergraphs have the same order of magnitude for all values of p. Note that it follows from Theorem 1 that for the graph case $k=2$ the ratio between the choice number and the chromatic number is bounded asymptotically by 2 – the best known factor (see bounds in [2] and [13]).

The idea of the proofs of both theorems above is essentially an adaptation of the main idea of [8] to the hypergraph case. Some additional ideas will be required to prove the results for all values of p.

The paper is organized as follows. In the next section we prove technical lemmas needed for the proof of the main results. The proof of Theorems 1 and 2 is given in Section 3. The final Section 4 is devoted to concluding remarks.

Throughout the paper we omit routinely the floor and the ceiling signs for the sake of convenience. All logarithms are natural. We denote

$$d=d(n,p)=n^{k-1}p$$

and assume d large enough whenever needed.

2 Preliminaries

In this section we provide a technical background for proving Theorems 1 and 2. The section is divided into three subsections. In the first of them we deal with the distribution of edges in random hypergraphs. In the second subsection we discuss the distribution of independent sets. Finally, the third subsection shows how the Local Lemma can be applied to show choosability of hypergraphs from color lists satisfying certain conditions.

2.1 Edge distribution

Four lemmas of this subsection are aimed to show essentially that the edge distribution in random uniform hypergraphs does not deviate much from its expected behavior. All possibly strangely looking and somewhat cumbersome expressions in the formulations of these statements are designed so as to fit together smoothly in the course of proving Theorems 1 and 2.

Lemma 2.1 *For every constant $c > 0$ a.s. in $H(k,n,p)$ every subset $W \subset V$ of size $|W| \leq cn \log^3 d/d$ spans at most $(\log d)^{3k}|W|$ edges.*

Proof. The probability of the existence of a subset $W \subset V$ violating the lemma can be estimated from above by

$$\begin{aligned}
\sum_{s=k}^{\frac{cn\log^3 d}{d}} \binom{n}{s}\binom{\binom{s}{k}}{(\log d)^{3k}s} p^{(\log d)^{3k}s} &\leq \sum_{s=k}^{\frac{cn\log^3 d}{d}} \left[\frac{en}{s}\left(\frac{es^{k-1}p}{(\log d)^{3k}}\right)^{(\log d)^{3k}}\right]^s \\
&\leq \sum_{s=k}^{\frac{cn\log^3 d}{d}} \left[\frac{np^{\frac{1}{k-1}}}{(\log d)^{3k(\log d)^{3k}}}\left(O(1)s^{k-1}p\right)^{(\log d)^{3k}-\frac{1}{k-1}}\right]^s \\
&= \sum_{s=k}^{\frac{cn\log^3 d}{d}} \left[\frac{d^{\frac{1}{k-1}}}{(\log d)^{3k(\log d)^{3k}}}\left(O(1)s^{k-1}p\right)^{(\log d)^{3k}-\frac{1}{k-1}}\right]^s .
\end{aligned}$$

Denote the s-th summand in the last sum by a_s. Then, if $s^{k-1}p \leq 1/\log n$, we have $a_s \leq (O(1)s^{k-1}p)^s \leq (O(1)/\log n)^s$. If $s^{k-1}p \geq 1/\log n$, then $s^{k-1}d \geq n^{k-1}/\log n$, thus implying $s \geq n^{1/2}$ or $d \geq n^{1/3}$. But then, recalling the assumption $s \leq cn\log^3 d/d$, we get

$$a_s \leq \left[\frac{d^{\frac{1}{k-1}}}{(\log d)^{3k(\log d)^{3k}}}\left(O(1)\log d\right)^{3(k-1)(\log d)^{3k}}\right]^s \leq \left(\frac{1}{\log d}\right)^{s\log d} = o(\frac{1}{n}) .$$

Therefore $\sum_{s=k}^{cn\log^3 d/d} a_s = o(1)$. □

Lemma 2.2 *Let $p(n)$ satisfy $p(n) \leq n^{-k+1+k/(k+1)+o(1)}$. Then for every constant $c > 0$ a.s. in $H(k,n,p)$ every subset $W \subset V$ of size $|W| \leq cn\log^4 d/d^{1+1/k}$ spans at most $(2k/(2k^2-4k+1))|W|$ edges.*

Proof. Denote $B = 2k/(2k^2-4k+1) = 1/(k-2+1/(2k))$. The probability that the lemma's assertion fails is at most

$$\sum_{s=k}^{\frac{cn\log^4 d}{d^{1+1/k}}} \binom{n}{s}\binom{\binom{s}{k}}{Bs} p^{Bs} \le \sum_{s=k}^{\frac{cn}{d^{1+1/k}}} \left(\frac{en}{s}\right)^s \left(\frac{es^{k-1}p}{B}\right)^{Bs} \le \sum_{s=k}^{\frac{cn\log^4 d}{d^{1+1/k}}} \left[\frac{O(1)n}{s}\left(s^{k-1}p\right)^B\right]^s .$$

Denote the s-th summand of the last sum by a_s. In the case $s \le \log n$ we get $a_s \le (O(1)np^B(\log n)^{(k-1)B})^s$. But $np^B \le n^{1+(1+k/(k+1)-k+o(1))B}$. Note that $1+(1+k/(k+1)-k)B = 1-((k^2-k-1)/(k+1))B = -(k-1)/(2k^3-2k^2-3k+1) < 0$. Therefore is this case $a_s = o(1/\log n)$. If $s \ge \log n$, we can estimate a_s as follows:

$$a_s \le \left[\frac{O(1)d^{1+\frac{1}{k}}}{\log^4 d}\left(\frac{n^{k-1}p(\log d)^{4(k-1)}}{d^{(1+\frac{1}{k})(k-1)}}\right)^B\right]^s = \left((\log d)^{O(1)}d^{1+\frac{1}{k}-(k-1-\frac{1}{k})B}\right)^s .$$

But $1+1/k-(k-1-1/k)B = (-k+1)/(2k^3-4k^2+k) < 0$. Hence is this case $a_s \le ((\log d)^{O(1)}d^{-\Omega(1)})^s = o(1/n)$. Thus we derive $\sum_{s=k}^{cn/d^{1+1/k}} a_s = o(1)$. □

Lemma 2.3 *A.s in $H(k,n,p)$ for every subset $W \subset V$ of size $|W| \le n/\log^4 d$, the spanned subhypergraph $H[W]$ contains at most $(5k\log^3 d/d)|W|$ vertices of degree at least $d/log^3 d$.*

Proof. Denote $d_0 = d/log^3 d$. Assume that some subset $W \subset V$ violates the assertion of the lemma. Then there exists a subset $U \subset W$ of size $|U| = (5k\log^3 d/d)|W|$ such that all vertices of U have degree at least d_0 in $H[W]$. But then at least $d_0|U|/k$ edges are incident to U in $H[W]$. Denote $t = t(s) = (5k\log^3 d/d)s$. Then the probability of existence of a set W violating the lemma is at most

$$\begin{aligned}\sum_{s=k}^{\frac{n}{\log^4 d}} \binom{n}{s}\binom{s}{t}\binom{t\binom{s-1}{k-1}}{\frac{d_0 t}{k}} p^{\frac{d_0 t}{k}} &\le \sum_{s=k}^{\frac{n}{\log^4 d}} \left(\frac{en}{s}\right)^s 2^s \left(\frac{eks^{k-1}p}{d_0}\right)^{\frac{d_0 t}{k}} \\ &= \sum_{s=k}^{\frac{n}{\log^4 d}} \left(\frac{6n}{s}\right)^s \left(\frac{O(1)s^{k-1}p\log^3 d}{d}\right)^{5s} = \sum_{s=k}^{\frac{n}{\log^4 d}} \left[O(1)\frac{n}{s}\left(\frac{s^{k-1}\log^3 d}{n^{k-1}}\right)^5\right]^s \\ &= \sum_{s=k}^{\frac{n}{\log^4 d}} \left[O(1)\left(\frac{s}{n}\right)^{5(k-1)-1}\log^{15} d\right]^s \le \sum_{s=k}^{\frac{n}{\log^4 d}} \left[O(1)\left(\frac{s}{n}\right)^4 \log^{15} d\right]^s .\end{aligned}$$

Denote the s-th summand of the last sum by a_s. If $s \le n^{1/2}$, then $a_s \le O(1)\log^{15} d/n^2 = o(1/n)$. If $s \ge n^{1/2}$, then recalling that $s \le n/\log^4 d$, we get $a_s \le (O(1)/\log d)^{n^{1/2}} = o(1/n)$. □

Lemma 2.4 *For every constant $c > 0$ a.s. in $H(k,n,p)$ for every subset $W \subset V$ of size $|W| = cn\log^3 d/d$ the number of vertices $v \in V \setminus W$ for which $deg(v,W) \ge d^{1/k}$ does not exceed $n\log^4 d/d^{1+1/k}$.*

Proof. If the lemma fails, there exist disjoint subsets U, W of sizes $|U| = n\log^4 d/d^{1+1/k}$, $|W| = cn\log^3 d/d$, for which the number of edges intersecting both U and W is at least $|U|d^{1/k}/k$. Denote $s = cn\log^3 d/d$, $t = n\log^4 d/d^{1+1/k}$. Then the probability of such event can be bounded from above by

$$\begin{aligned}\binom{n}{s}\binom{n}{t}\binom{stn^{k-2}}{\frac{td^{1/k}}{k}}p^{\frac{td^{1/k}}{k}} &\leq \left(\frac{en}{s}\right)^{2s}\left(\frac{eksn^{k-2}p}{d^{1/k}}\right)^{\frac{td^{1/k}}{k}}\\ &= \left(\frac{O(1)d}{\log^3 d}\right)^{\frac{O(1)n\log^3 d}{d}}\left(\frac{O(1)n^{k-1}p\log^3 d}{d^{1+1/k}}\right)^{\frac{n\log^4 d}{kd}}\\ &\leq d^{\frac{O(1)n\log^3 d}{d}}\left(\frac{O(1)\log^3 d}{d^{1/k}}\right)^{\frac{n\log^4 d}{kd}} = o(1)\ . \qquad \square\end{aligned}$$

2.2 Large independent sets

In this subsection we show that almost surely every sufficiently large subset of vertices of $H(k,n,p)$ contains a large independent set. In the dense case (i.e. when the edge probability $p = p(n)$ is high enough), we can show that a. s. every large enough subset of vertices contains an independent set of an asymptotically optimal size. This is done by a quite standard by now application of the Janson inequality. In the general case we prove that a. s. every subset of size $n/\ln^4 d$ spans an independent set, whose size is smaller than the independence number by a factor asymptotically not exceeding $k^{1/(k-1)}$. This result is achieved by analyzing the performance of the greedy algorithm for finding independent sets.

In this section we use the notation $[x]_r = x(x-1)\dots(x-r+1)$.

Let us define now the following quantities. Let

$$\alpha_0 = \alpha_0(n,p) = \max\left\{\alpha : \binom{\frac{n}{\log^6 d}}{\alpha}(1-p)^{\binom{\alpha}{k}} > 1\right\}\ , \qquad (1)$$

$$\alpha_1 = \alpha_1(n,p) = \max\left\{\alpha : \binom{\alpha}{k-1} \leq \frac{\alpha_0 - k + 1}{k\alpha_0}\binom{\alpha_0}{k-1}\right\}\ , \qquad (2)$$

$$\alpha_2 = \alpha_2(n,p) = \max\left\{\alpha : \binom{\frac{n}{\ln^4 d}}{\alpha}(1-p)^{\binom{\alpha}{k}} \geq (100)^{\alpha}n^8\right\}\ . \qquad (3)$$

We would like to note that α_0 and α_2 are equal $(1+o(1))t_0$, where t_0 is defined in the Introduction. It follows readily from the above mentioned results of Shamir [11] and of Krivelevich and Sudakov [9], that almost surely α_1 and α_2 satisfy $\alpha_1 = (1-o(1))\alpha(H)/k^{1/(k-1)}$ and $\alpha_2 = (1-o(1))\alpha(H)$.

Lemma 2.5 *There exists a constant $C > 0$ so that if the edge probability $p = p(n)$ satisfies $\frac{C}{n^{k-1}} \leq p \leq 0.9$, then a. s. in $H(k,n,p)$ every subset V_0 of size $|V_0| \geq n/\ln^4 d$ contains an independent set of size α_1.*

Proof. Let

$$m = \frac{n}{\ln^4 d} \ .$$

Observe that by definition (1) of α_0

$$\binom{m}{\alpha_0}(1-p)^{\binom{\alpha_0}{k}} \geq \frac{[\frac{n}{\log^4 d}]_{\alpha_0}}{[\frac{n}{\log^6 d}]_{\alpha_0}} \geq (100 \log d)^{\alpha_0}, \tag{4}$$

provided d is sufficiently large.

Since there are $\binom{n}{m} \leq (en/m)^m$ subsets of m elements, it suffices to show that for a fixed subset V_0 of size $|V_0| = m$

$$Pr[V_0 \text{ contains an independent set of size } \ \alpha_1] \geq 1 - o((en/m)^{-m}) = 1 - o((e \log^4 d)^{-m}). \tag{5}$$

Set $t = m/2\alpha_1$. It is clear, by the greedy algorithm, that if V_0 fails to contain an independent set of size α_1, then it should contain t pairwise disjoint sets $A_1, A_2, \ldots, A_t$ satisfying the following two properties:

(i) Each A_i is independent and $|A_i| < \alpha_1$;

(ii) For any $v \in V_0 \backslash \cup_{i=1}^t A_i$ and any $1 \leq i \leq t$, there is an edge in $A_i \cup v$ containing v.

We call a collection $\mathcal{A} = \{A_1, \ldots, A_t\}$ *bad* if A_i are pairwise disjoint sets, $|A_i| < \alpha_1$ and (ii) holds. To prove (5), it suffices to show that

$$Pr[V_0 \text{ contains a bad collection}] = o((e \log^4 d)^{-m}). \tag{6}$$

First we estimate the chance that a fixed collection $\mathcal{A} = \{A_1, \ldots, A_t\}$, where A_i are pairwise disjoint and $|A_i| < \alpha_2$, is bad. We say that a point $v \in B = V_0 \backslash \cup_{i=1}^t A_i$ is *stuck* if for all $1 \leq i \leq t$, $A_i \cup v$ contains an edge containing v. The probability that v is stuck is

$$\prod_{i=1}^t (1 - (1-p)^{\binom{|A_i|}{k-1}}) \leq \exp(-\sum_{i=1}^t (1-p)^{\binom{|A_i|}{k-1}}) \leq \exp(-t(1-p)^{\binom{\alpha_1}{k-1}}) \ . \tag{7}$$

To make $\mathcal{A}$ bad, all $v \in B$ should be stuck. Because $|B| \geq m/2$, it follows from (7) that

$$Pr[\mathcal{A} \ \text{ bad}] \leq e^{-t(1-p)^{\binom{\alpha_1}{k-1}} m/2}, \tag{8}$$

since the events that the points in B are stuck are mutually independent. By (4)

$$(\frac{em}{\alpha_0})^{\alpha_0}(1-p)^{\binom{\alpha_0}{k}} \geq \binom{m}{\alpha_0}(1-p)^{\binom{\alpha_0}{k}} \geq (100 \log d)^{\alpha_0}.$$

The above inequality and the definition of α_1 yield

$$\frac{em}{\alpha_0}(1-p)^{\binom{\alpha_1}{k-1}} \geq \frac{em}{\alpha_0}(1-p)^{\binom{\alpha_0}{k-1}(\alpha_0 - k + 1)/(k\alpha_0)} = \frac{em}{\alpha_0}(1-p)^{\binom{\alpha_0}{k}/\alpha_0} \geq 100 \log d. \tag{9}$$

Since $t = m/2\alpha_1$, it follows from (8) and (9) that

$$Pr[\mathcal{A} \text{ bad}] \le e^{-4m \log d}. \tag{10}$$

On the other hand, the number of collections in concern is at most

$$\left(\sum_{s=1}^{\alpha_1 - 1} \binom{m}{s} \right)^t \le \binom{m}{\alpha_1}^t \le (em/\alpha_1)^{\alpha_1 t} \le (6d)^{\alpha_1 t} \le e^{m \log d}, \tag{11}$$

because $em/\alpha_1 \le 6d$ (trivial) and $\alpha_1 t \le m/2$. Inequalities (10) and (11) show that the probability that there is a bad collection is at most $e^{-3m \log d}$, which satisfies the requirement of (6). This completes the proof. $\square$

Lemma 2.6 *For every constant $\epsilon > 0$, if the edge probability $p = p(n)$ satisfies $n^{-(k-1)^2/2k+\epsilon} \le p \le 0.9$, then a. s. in $H(k, n, p)$ every subset V_0 of size $|V_0| \ge n/\log^4 d$ contains an independent set of size α_2.*

Proof. Denote again $m = n/\log^4 d$. Definition (3) of α_2 and the inequality $(em/\alpha_2)^{\alpha_2} \ge \binom{m}{\alpha_2}$ imply

$$\frac{em}{\alpha_2}(1-p)^{\binom{\alpha_2}{k}/\alpha_2} \ge 100 n^{8/\alpha_2} \ge 100. \tag{12}$$

The probability that a fixed set V_0 of size m does not contain an independent set of size α_2 is at most $e^{-\mu^2/(\mu+\Delta)}$, according to Janson's inequality (see, e.g., [3], Ch.8), where

$$\mu = \binom{m}{\alpha_2}(1-p)^{\binom{\alpha_2}{k}},$$

$$\Delta = \mu^2 \sum_{j=k}^{\alpha_2 - 1} \binom{\alpha_2}{j} \binom{m-j}{\alpha_2 - j} \binom{m}{\alpha_2}^{-1} (1-p)^{-\binom{j}{k}}.$$

Let a_j denote the term with index j in the above sum. One can choose a sufficiently small positive constant ϵ and a sufficiently large positive constant K so that

$$m^{1-\epsilon} \ge 2\alpha_2^{2-\epsilon} \text{ and } \binom{\alpha_2/K}{k-1} \le \binom{\alpha_2}{k} \epsilon/\alpha_2.$$

Assume that $j \le \alpha_2/K$. By a routine simplification, estimate (12) and the definition of ϵ

$$\begin{aligned} \frac{a_j}{a_{j+1}} &= \frac{(m-j)(j+1)}{(\alpha_2 - j)^2}(1-p)^{\binom{j}{k-1}} \ge \frac{m}{\alpha_2^2}(1-p)^{\binom{\alpha/K}{k-1}} \ge \frac{m}{\alpha_2^2}(1-p)^{\binom{\alpha}{k}\epsilon/\alpha_2} \\ &\ge \frac{m}{\alpha^2}(\frac{\alpha_2}{m})^{\epsilon} \ge 2. \end{aligned} \tag{13}$$

Moreover, if $\alpha_2/K \leq j \leq \alpha_2/5$, then

$$\begin{aligned}
\frac{a_j}{a_{j+1}} &= \frac{(m-j)(j+1)}{(\alpha_2-j)^2}(1-p)^{\binom{j}{k-1}} \\
&\geq \frac{m}{K\alpha_2}(1-p)^{\binom{\alpha_2/5}{k-1}} \geq \frac{m}{K\alpha_2}(1-p)^{\binom{\alpha_2}{k}/2\alpha_2} \\
&\geq \frac{m}{K\alpha_2}(100n^{8/\alpha_2}\alpha_2/em)^{1/2} \geq (\frac{m}{K^2\alpha_2})^{1/2} \geq 2, \qquad (14)
\end{aligned}$$

given that α_2 and m, n are sufficiently large. Inequalities (13) and (14) yield that for all $j \leq \alpha_2/5$ one has

$$\frac{a_j}{a_{j+1}} \geq 2,$$

which implies that

$$\sum_{j=k}^{\alpha_2/5} a_j \leq 2a_k.$$

To estimate the rest of the sum, consider a_j with $j > \alpha_2/5$. Again, after a routine simplification, we have

$$a_j = \frac{[\alpha_2]_j^2}{j![m]_j}(1-p)^{-\binom{j}{k}} \leq \frac{\alpha_2^{2j}}{(j/e)^j(m/2)^j}(1-p)^{\binom{j}{k}} \leq (10e\frac{\alpha_2}{m}(1-p)^{-\binom{j}{k}/j})^j. \qquad (15)$$

using the fact $\alpha_2/j < 5$. Furthermore, since $j < \alpha_2$,

$$10e\frac{\alpha_2}{m}(1-p)^{-\binom{j}{k}/j} \leq 10e\frac{\alpha_2}{m}(1-p)^{-\binom{\alpha_2}{k}/\alpha_2} \qquad (16)$$

It follows from (12) and (16) that

$$a_j \leq (n^{-8/\alpha_2})^j \leq n^{-8/5},$$

since $j > \alpha_2/5$. Together we have (taking into account that in this range of p, $\alpha_2 \leq n^{1/2}$)

$$\sum_{j=k}^{\alpha_2-1} a_j \leq 2a_k + \alpha_2 n^{-8/5} = 2a_k + o(n^{-1}). \qquad (17)$$

Since $p \leq 0.9$, $a_k \leq 10\alpha_2^{2k}/k!m^k$. Recalling that $\alpha_2 = \Omega((\frac{1}{p})^{1/(k-1)}) = \Omega(n^{(k-1)/2k-\epsilon/(k-1)})$, and $m = n/\log^4 d$, it follows that

$$a_k = O(n^{-1-\delta}), \qquad (18)$$

for some positive constant $\delta = \delta(\epsilon, k)$. Together (17) and (18) imply that $\sum_{j=k}^{\alpha_2-1} a_j = o(n)$ which in turn shows that $\mu^2/(\mu+\Delta) = \omega(n)$. Therefore, the probability that a fixed set V_0 does not contain an independent set of the required size is $e^{-\omega(n)}$. Since there are less than 2^n possibilities to choose V_0, our proof is complete. □

2.3 Applying the Local Lemma

In this subsection we show how the Lovász Local Lemma [6] can be applied to prove the existence of a coloring from given lists. This application of the Local Lemma is very similar in spirit to that in [8].

Lemma 2.7 *Let $H = (V, E)$ be a k-uniform hypergraph. Suppose that a family of color lists $\mathcal{S} = \{S(v) : v \in V\}$ is given with all lists $S(v)$ of cardinality l. For a color c denote $U_c = \{v \in V : c \in S(v)\}$. If for each color c and each vertex $v \in U_c$ the degree of v in the induced subhypergraph $H[U_c]$ does not exceed $(l/6)^{k-1}$, then H is $\mathcal{S}$-choosable.*

Proof. For every vertex $v \in V$ choose a color $c \in S(v)$ uniformly at random, making independent choices for different vertices. For an edge e and a color c such that c appears in the color lists of all the vertices of e let $I_{e,c}$ be the event "all vertices of e are colored c". As $|S(v)| = l$ for every v, we get $Pr[I_{e,c}] = l^{-k}$. The event $I_{e,c}$ is independent of all other events $I_{c',e'}$ but those for which $e \cap e' \neq \emptyset$ and $e' \subseteq U_{c'}$. The number of such events can be estimated from above by

$$\sum_{v \in e} \sum_{c' \in S(v)} d_{H[U_{c'}]}(v) \leq kl \left(\frac{l}{6}\right)^{k-1} = \frac{kl^k}{6^{k-1}}.$$

Since for $k \geq 2$ we have $k6^{1-k} < 1/e$, by the so-called symmetric version of the Local Lemma, with positive probability none of the events $I_{e,c}$ takes place. But this means exactly that there exists a proper coloring of H from $\mathcal{S}$. $\square$

3 Proofs of Theorems 1 and 2

Having finished all technical preparations, we can now present the proofs of Theorems 1 and 2. These theorems are immediate consequences of the following two *deterministic* statements.

Theorem 3 *Let n be an integer and let p satisfy $0 < p < 1$. Denote $d = n^{k-1}p$ and assume that d is large enough. Let $H = (V, E)$ be a k-uniform hypergraph on n vertices, having the properties stated in Lemmas 2.1–2.5. Let $\alpha_1 = \alpha_1(n, p)$ be as defined in (2). Then*

$$ch(H) \leq \frac{n}{\alpha_1} + \left(\frac{d}{\log^2 d}\right)^{\frac{1}{k-1}}.$$

Theorem 4 *Let n be an integer and let p satisfy $0 < p < 1$. Denote $d = n^{k-1}p$ and assume that d is large enough. Let $H = (V, E)$ be a k-uniform hypergraph on n vertices, having the properties stated in Lemmas 2.1–2.4, 2.6. Let $\alpha_2 = \alpha_2(n, p)$ be as defined in (3). Then*

$$ch(H) \leq \frac{n}{\alpha_2} + \left(\frac{d}{\log^2 d}\right)^{\frac{1}{k-1}}.$$

Indeed, it is easy to derive from the bounds on the chromatic number of $H(k,n,p)$ in [11], [9] and our definitions (2), (3) of α_1, α_2, that a.s. in $H(k,n,p)$

$$\begin{aligned}\left(\frac{d}{\log^2 d}\right)^{\frac{1}{k-1}} &= o(\chi(H))\,;\\ \frac{n}{\alpha_1} &= (1+o(1))k^{\frac{1}{k-1}}\chi(H)\,;\\ \frac{n}{\alpha_2} &= (1+o(1))\chi(H)\,.\end{aligned}$$

Therefore Theorem 3 and 4 imply Theorems 1 and 2, respectively.

A crucial part of the proofs of both above theorems is the following lemma.

Lemma 3.1 *Let n be an integer and let p satisfy $0 < p < 1$. Denote $d = n^{k-1}p$ and assume that d is large enough. Let $H = (V,E)$ be a k-uniform hypergraph on n vertices which has properties stated in Lemmas 2.1–2.4. Suppose that for a subset $U \subseteq V$ a family of color lists $\mathcal{S} = \{S(v) : v \in U\}$ is given with all lists $S(v)$ of cardinality $|S(v)| = (d/\log^2 d)^{1/(k-1)}$. Suppose further that no color appears in more than $n/\log^4 d$ lists. Then the induced subhypergraph $H[U]$ is $\mathcal{S}$-choosable.*

Proof. For a color $c \in \bigcup_{v\in U} S(v)$ denote $U_c = \{v \in U : c \in S(v)\}$. We say that a vertex $v \in U$ and a color c form a *dangerous pair* if $c \in S(v)$ and the degree of v in the induced subhypergraph $H[U_c]$ is at least $d/log^3 d$. A vertex $v \in U$ is called *dangerous* if it participates in at least $(d/\log^2 d)^{1/(k-1)}/2$ dangerous pairs. Here is an (oversimplified) outline of the proof. First we show that the set W of dangerous vertices is small. Then the set T of all vertices having many neighbors in W is even smaller. We show that then a.s. both T and W span a small number of edges. We start our coloring procedure by first coloring T using its sparseness. Then we color W using sparseness arguments again. Finally, we destroy all dangerous pairs and color the rest of U by applying Lemma 2.7.

Let us first estimate from above the number of dangerous vertices in U. Note that $\sum_{c\in\bigcup_{v\in U} S(v)} |U_c| = |U|(d/\log^2 d)^{1/(k-1)} \le n(d/\log^2 d)^{1/(k-1)}$. According to Lemma 2.3, a color c participates in at most $(5k\log^3 d/d)|U_c|$ dangerous pairs. Therefore the total number of dangerous pairs does not exceed

$$\sum_{c\in\bigcup_{v\in U} S(v)} \frac{5k\log^3 d}{d}|U_c| = \frac{5k\log^3 d}{d} \sum_{c\in\bigcup_{v\in U} S(v)} |U_c| \le \frac{5kn\log^3 d}{d}\left(\frac{d}{\log^2 d}\right)^{\frac{1}{k-1}}\ .$$

Recalling the definition of a dangerous vertex, we conclude that the total number of dangerous vertices is at most $10kn\log^3 d/d$.

Denote by W the set of all dangerous vertices. Let also $T_0 = \{v \in U\setminus W : deg(v,W) \ge d^{1/k}\}$. By Lemma 2.4 the cardinality of T_0 satisfies $|T_0| \le n\log^4 d/d^{1+1/k}$. If $n < d^{1+1/k}/\log^4 d$, the set T_0 is empty and we can skip to the next step in the proof. Otherwise, we may assume $n \geq d^{1+1/k}/\log^4 d$, thus implying $p \leq n^{-k+1+k/(k+1)+o(1)}$.

Next, we find a subset $T \subset U$ of size $|T| = O(n \log^4 d/d^{1+1/k})$ including T_0 and such that every vertex $v \in U \setminus T$ has small degree into T. We start with an arbitrary subset $T \subset U$ of size $|T| = n \log^4 d/d^{1+1/k}$ including T_0. As long as there exists a vertex $v \in U \setminus T$ with $deg(v,T) \geq 5k$ (where $deg(v,T)$ is the number of edges incident to both v and T), we add to T the union of some $5k$ edges containing v and intersecting T. Note that each time we add to T $5k$ edges and at most $5k(k-2)+1$ vertices. This process stops after at most $n \log^4 d/d^{1+1/k}$ iterations, because otherwise we get a subset T of cardinality at most $(5k(k-2)+2)n \log^4 d/d^{1+1/k}$ with at least $5kn \log^4 d/d^{1+1/k}$ edges inside. But then $|E(H[T])|/|T| \geq 5k/(5k(k-2)+2) > 1/(k-2+1/(2k))$, thus contradicting the statement of Lemma 2.2.

Invoking Lemma 2.2 once again, we notice that for every subset $T' \subseteq T$, the subhypergraph $H[T']$ spans at most $(2k/(2k^2-4k+1))|T'|$ edges and therefore has a vertex of degree at most $2k^2/(2k^2-4k+1) \leq 8$. Thus $H[T]$ is 8-*degenerate*. As every t-degenerate hypergraph can be easily proven by induction to be $(t+1)$-choosable (see, e.g. [1]), we can find a proper coloring for $H[T]$, using given lists $\{S(v) : v \in T\}$.

Now, for every $v \in U \setminus T$ we delete from $S(v)$ all those colors which have been used to color the neighbors of v in T. As by our construction of T we have $deg(v,T) < 5k$, we delete at most $5k(k-1)$ colors from $S(v)$. The turn of W to be colored has arrived. Recall that $|W| \leq 10kn \log^3 d/d$. Using a similar argument as before along with Lemma 2.1, we conclude that $H[W]$ is $k(\log d)^{3k}$-degenerate and hence $(k(\log^d)^{3k}+1)$-choosable. As every list $S(v)$ has still at least $(d/\log^2 d)^{1/(k-1)} - 5k(k-1) \gg k(\log d)^{3k}+1$, there is definitely a proper choice of colors for $H[W]$ as well.

Finally, for every $v \in U \setminus (T \cup W)$ we delete from $S(v)$ the colors of all neighbors of v in W. From our definition of T_0 and from the fact $T_0 \subseteq T$ it follows that a vertex $v \in U \setminus (T \cup W)$ has at most $(k-1)d^{1/k}$ neighbors inside W. Therefore, we delete at most that many colors from $S(v)$. Also, for every $v \in U \setminus (T \cup W)$ we delete from $S(v)$ all those colors c with whom v forms a dangerous pair. As all dangerous vertices were placed in W, we delete at most $(d/\log^2 d)^{1/(k-1)}/2$ colors from each list. Accumulating all deletions above, we conclude that for each $v \in U \setminus (T \cup W)$, the list $S(v)$ still has at least $(d/\log^2 d)^{1/(k-1)}/3$ colors. If it has more colors, we choose an arbitrary subset of cardinality $(d/\log^2 d)^{1/(k-1)}/3$ and drop the rest.

Now we are in position to apply the Local Lemma. The hypergraph $H_1 = H[U \setminus (T \cup W)]$ and the family of lists $\mathcal{S} = \{S(v) : v \in U \setminus (T \cup W)\}$, where the list $S(v)$ are as defined in the previous paragraph, are easily seen to satisfy the conditions of Lemma 2.7 with $l = (d/\log^2 d)^{1/(k-1)}/3$. Then it follows from Lemma 2.7 that H_1 is $\mathcal{S}$-choosable. This completes the coloring of $H[U]$. The proof of Lemma 2.1 is finished. $\square$

Proof of Theorem 3. Given a family of color lists $\mathcal{S} = \{S(v : v \in V\}$ with all $|S(v)| = n/\alpha_1 + (d/\log^2)^{1/(k-1)}$, we need to show that H is $\mathcal{S}$-choosable.

First, as long as there exists a color c so that c appears in at least $n/\log^4 d$ lists $S(v)$, we find an independent set I of size $|I| = \alpha_1$ in the subset $U_c = \{v \in V : c \in S(v)\}$. Such an independent set exists by Lemma 2.5. We then color all vertices of I by c, delete I and remove c from all lists of the remaining vertices.

Let U be the set of vertices, still uncolored after the above described first phase. As the first phase was repeated at most n/α_1 times, all lists $\{S(v) : v \in U\}$ still have $(d/\log^2 d)^{1/(k-1)}$ colors. Moreover, each color appears in less than $n/\log^4 d$ lists. Then by Lemma 3.1 there exists a proper choice of colors for the vertices of U from the lists $\{S(v) : v \in U\}$. $\Box$

Proof of Theorem 4. The proof is essentially identical to the above presented proof of Theorem 3, with the only difference being that instead of using Lemma 2.5, here we apply Lemma 2.6.

Given a family of color lists $\mathcal{S} = \{S(v) : v \in V\}$ for the vertices of H, we color H in two stages. First, as long as there is a color c, appearing in at least $n/log^4 d$ lists, we find an independent set I of cardinality $|I| = \alpha_2$ in $U_c = \{v \in V : c \in S(v)\}$ (such a set exists by Lemma 2.6), color I by c, discard I and remove c from all color lists.

Denote now by U the set of still uncolored vertices. Each vertex $v \in U$ still has $(d/\log^2 d)^{1/(k-1)}$ colors in its list $S(v)$. Also, each color participates in less than $n/\log^4 d$ lists. Then we can complete a coloring of H by applying Lemma 3.1. $\Box$

4 Concluding remarks

We have proven that for every uniformity number $k \geq 2$, essentially for all values of the edge probability $p = p(n)$ the choice number of the random k-uniform hypergraph $H(k, n, p)$ has almost surely the same order of magnitude as its chromatic number. Moreover, for the dense case the asymptotic values of these two parameters of the random hypergraph coincide.

Note that our proof of Theorem 1 is algorithmic in the following sense. Given a typical hypergraph from the probability space $H(k, n, p)$ and a family of lists of colors $\mathcal{S} = \{S(v) : v \in V\}$, where each color list has cardinality given by the bound on the choice number of Theorem 1, there exists a polynomial time randomized algorithm which finds with high probability a proper coloring $f : V(H) \to \bigcup_{v \in V} S(v)$, satisfying $f(v) \in S(v)$ for every $v \in V$. Indeed, the argument of Lemma 2.5 for finding a large independent set is based on the greedy algorithm and is thus constructive. Lemma 2.7 can be converted to a polynomial time randomized algorithm using a standard technique for the "algorithmization" of the Local Lemma, developed by Beck in [4]. The proof of Theorem 2 relies on the existential Lemma 2.6 for finding almost optimal independent sets and is thus non-constructive. There is no apparent obstacle to extending the scope of Theorem 2 to smaller values of the edge probability p. We thus conjecture that for *all* values of p, the choice number and the chromatic number of $H(k, n, p)$ have almost surely the same asymptotic value. A more challenging problem is to estimate from above the *absolute* difference between these two parameters or even prove (or disprove) that almost surely the choice number and the chromatic number are equal *exactly*.

Finally, we note that the methods presented in this paper can be used to show that for other values of γ, the γ-choice number and the γ-chromatic number of $H(k, n, p)$ are almost surely of the same order, i.e. differ by a constant factor.

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