Approximating coloring and maximum independent sets in 3-uniform hypergraphs

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1 Introduction

Approximate coloring problems and problems of approximating the independence number of a graph are between the most important problems in Combinatorial Optimization. Many positive and negative results have been obtained recently, the state of the art is reflected in [6].

In contrast, much less is known on the hypergraph versions of these problems. Krivelevich and Sudakov [11] developed a coloring algorithm with approximation ratio $O(n(\log\log n/\log n)^2)$ for k-uniform hypergraphs on n vertices. Algorithms for coloring k-uniform 2-colorable hypergraphs have been proposed in [2, 5, 11]. On the negative side, it is easy to show that for every fixed $k \geq 3$, approximating the chromatic number of a k-uniform hypergraph is at least as hard as the corresponding problem for graphs [9, 11]. Very recently, Guruswami, Håstad and Sudan showed [8] that for any constant c it is NP-hard to color 4-uniform 2-colorable hypergraphs in c colors. Naturally, this result stresses the importance of developing good approximation algorithms for coloring 2-colorable hypergraphs. The only paper on approximating the independence number of uniform hypergraphs we are aware of is [9], whose main result is significantly weaker than those known for graphs.

Here we propose approximation algorithms for coloring and independence set problems in 3-uniform hypergraphs. It appears that the 3-uniform case stands apart from other uniformity numbers $k \geq 4$, as in this case the powerful machinery of Semidefinite Programming can be applied to produce better approximation algorithms (see [2] for a relevant discussion). In Section 2 we discuss an algorithm for finding a large independent set in hypergraphs on n vertices and with independent set of size at least γn, for a constant $\gamma > 0$. Then in Section 3 we use this algorithm as a subroutine of an algorithm for coloring 3-uniform 2-colorable hypergraphs in $\tilde{O}(n^{1/5})$ colors, thus improving the $\tilde{O}(n^{9/41})$ algorithm from [11] and the previous results from [2], [5]. Finally, for some values of γ, we show how to improve our results from Section 2 using the Local Ratio approach.

Due to space limitations we present only brief outlines of obtained results and their proofs. All the details will appear in a full version of the paper, to be published elsewhere.

2 Finding large independent sets

Here we discuss an algorithm for the maximum independent set problem with a promise. We will assume that an input hypergraph H on n vertices contains an independent set of size γn. The performance of our algorithm depends on γ. The graph version of this problem has been tackled by Boppana and Halldórsson [4] using the subgraph exclusion argument, and then by Alon and Kahale [1] based on the Lovász θ-function.

THEOREM 2.1. *Let H be a 3-uniform hypergraph on n vertices, m edges and with an independent set of size at least γn, for some constant $\gamma > 0$. There exists a polynomial time algorithms which finds in H an independent set of size* $\tilde{\Omega}(\min(n, n^{3-3\gamma}/m^{2-3\gamma}))$.

Proof. We first formulate a Semidefinite Programming relaxation of the maximum independence set problem as follows:

$$\max \sum_i \frac{1 - v_0^t v_i}{2}$$

$$\text{s.t. } ||v_0|| = ||v_i|| = 1,\ 1 \leq i \leq n$$

$$v_i^t v_j + v_i^t v_k + v_j^t v_k \leq v_0^t(v_i + v_j + v_k), \{i,j,k\} \in E(H).$$

Next we compute its optimal solution in terms of vectors in R^n and finally, using rounding techniques similar to those exploited in [10] and [7] together with some additional ideas, find a large independent set.

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Corollary 2.1. *If $\gamma > 1/2$ and H is a 3-uniform hypergraph on n vertices with independent set of size at least γn, then there is a polynomial time algorithm which finds in H an independent set of size $\tilde{\Omega}(\min(n, n^{6\gamma-3}))$.*

Proof. Note that in Theorem 2.1 the value of m is always at most $O(n^3)$.

3 Coloring 2-colorable hypergraphs

Theorem 3.1. *Let H be a 3-uniform 2-colorable hypergraph on n vertices. Then there is a polynomial time algorithm which colors H in $\tilde{O}(n^{1/5})$ colors.*

Proof. The algorithm is obtained by combining Corollary 2.1 and the ideas from [11]. Below we give its brief outline.

The algorithm uses the routine *Semidef* of [11] for finding an independent set of size $\tilde{O}(\frac{n^{9/8}}{m^{1/8}})$ in a 3-uniform 2-colorable hypergraph on n vertices and with $m \geq n$ edges. As usually, to produce a coloring of H, it is enough to be able to find an independent set of size $\tilde{\Omega}(n^{4/5})$.

Step 1. If there is a pair of vertices u, v so that $d(u,v) \geq n^{4/5}$ and $N(u,v)$ is an independent set, color it by a fresh color. Otherwise, delete all edges of H passing through u, v and replace them by an edge (u, v). Repeat. Denote the resulting hypergraph by H_1.

Step 2. If $|E(H_1)| = O(n^{13/5})$, apply *Semidef*. Otherwise, find greedily a subhypergraph H_2 of H_1 with minimal degree $\Omega(n^{4/5})$ and all codegrees $d(u,v) \leq 1$.

Step 3. For $v \in V(H_2)$, we denote by $N(v)$ the set of vertices adjacent to v in H_2. As explained in [11], for a 2-coloring c of H, there exists a vertex v so that $|N(v) \setminus C(v)| \geq (2/3)|N(v)|$, where $C(v)$ is a color class of v. Thus $N(v)$ spans in H a subhypergraph $H(v)$ with $\Omega(n^{4/5})$ vertices, satisfying $\alpha(H(v)) \geq (2/3)|V(H(v))|$. Then Corollary 2.1, with $\gamma = 2/3$, is used to find an independent set of size $\tilde{\Omega}(n^{4/5})$ inside $N(v)$.

4 Using the Local Ratio approach

Using the Local Ratio approach developed in [3], we can improve the algorithm of Theorem 2.1 for some values of γ. Note that one can exclude all fixed subhypergraphs $H_0 \subset H$, for which $\alpha(H_0)/|V(H_0)| < \gamma$. For example, by doing this first and then applying our first algorithm we obtain the following result.

Theorem 4.1. *There is a polynomial time algorithm which finds an independent set of size $\tilde{\Omega}(n^{2/3})$ in a 3-uniform hypergraph H on n vertices and with an independent set of size at least $(3/5 + o(1))n$.*

Proof. First exclude all copies of the hypergraph $H_0 = \{(1,2,3),(1,2,4),(1,2,5),(3,4,5)\}$. Then, if some $u, v \in V(H)$, $d(u,v) \geq n^{2/3}$, then $N(u,v)$ is an independent set of the desired size. Otherwise, $|E(H)| = O(n^{8/3})$, and the algorithm of Theorem 2.1 can be applied.

The above result is a partial case of a more general statement which appear in a full version of the paper.

5 Conclusion

An interesting question which remains open is to determine if there exists a polynomial time algorithm which finds an independent set of size at least n^{ϵ}, for some $\epsilon > 0$, in a 3-uniform hypergraph on n vertices and with an independent set of size γn, where $\gamma \leq 1/2$.

Also, it would be very interesting to develop good approximation algorithms for coloring/independent set problems for k-uniform hypergraphs with $k \geq 4$.

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