Long cycles in critical graphs

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Abstract

It is shown that any k-critical graph with n vertices contains a cycle of length at least $2\sqrt{\log(n-1)/\log(k-2)}$, improving a previous estimate of Kelly and Kelly obtained in 1954.

1 Introduction

A graph is *k-critical* if its chromatic number is k but the chromatic number of any proper subgraph of it is at most $k-1$. For a graph G, let $L(G)$ denote the maximum length of a cycle in G, and define $L_k(n) = min\ L(G)$ where the minimum is taken over all k-critical graphs G with at least n vertices. Answering a problem of Dirac, Kelly and Kelly [3] proved that for every fixed $k > 2$ the function $L_k(n)$ tends to infinity as n tends to infinity. They also showed that $L_4(n) \leq O(\log^2 n)$, and after several improvements by Dirac and Read, Gallai [2] proved that for every fixed $k \geq 4$ there are infinitely many values of n for which

$$L_k(n) \leq \frac{2(k-1)}{\log(k-2)} \log n.$$

This is the best known upper bound for $L_k(n)$. The best known lower bound, proved in [3], is that for every fixed $k \geq 4$ there is some $n_0(k)$ such that for all $n > n_0(k)$

$$L_k(n) \geq \left(\frac{(2+o(1)) \log\log n}{\log\log\log n}\right)^{1/2}, \tag{1}$$

where the $o(1)$ term tends to 0 as n tends to infinity.

Note that the gap between the upper and lower bounds given above is exponential for fixed k, and the problem of determining the asymptotic behaviour of $L_k(n)$ more accurately is still open; see also [1], Problem 5.11 for some additional relevant references.

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In the present note we improve the lower bound given in (1) and show that in fact $L_k(n) \geq \Omega(\sqrt{\log n / \log(k-1)})$ for every n and $k \geq 4$. (Note that trivially $L_3(n) = n$.) The precise result we prove is the following.

Theorem 1 *Let G be a k-critical graph on n vertices, and let t denote the length of the longest path in it. Then*

$$n \leq 1 + \sum_{j=0}^{t-1} s(j,k) \tag{2}$$

where

$$s(j,k) = j! \text{ for } j \leq k-3 \text{ and } s(j,k) = (k-2)!(k-2)^{j-k+2} \text{ for } j \geq k-2. \tag{3}$$

Therefore, any k-critical graph on n vertices contains a path of length at least $\log(n-1)/\log(k-2)$ and a cycle of length at least $2\sqrt{\log(n-1)/\log(k-2)}$.

We note that the construction of Gallai shows that there are infinitely many values of n for which there is a k-critical graph on n vertices with no path of length greater than $\frac{2(k-1)}{\log(k-2)} \log n$, showing that the statement of the above theorem for paths is nearly tight for fixed k.

2 The Proof

Suppose $k \geq 4$, and let $G = (V, E)$ be a k-critical graph on n vertices. It is easy and well known that G is 2-connected. Fix $v_0 \in V$, and let T be a depth first search (= DFS) spanning tree of G rooted at v_0. Denote the *depth* of T, (that is, the maximum length of a path from v_0 to a leaf) by r, and recall that all non-tree edges of G are backward edges, that is, they connect a vertex of T with some ancestor of it in the tree. Call an edge uv of T, where u is the parent of v, an edge of *type* j, if the unique path in T from v_0 to u has length j. Note that the type of each edge is an integer between 0 and $r-1$.

Claim: The number of edges of type j in T is at most $s(j,k)$, where $s(j,k)$ is given in (3).

Proof: Assign to each edge $e = uv$ of type j in T, where u is the parent of v, a word S_e of length $j+1$ over the alphabet $K = \{0, 1, 2, \ldots, k-2\}$ as follows. Let $v_0, v_1, \ldots, v_j = u$ be the unique path in T from the root v_0 to u. Let F_e be a proper coloring of $G - e$ by the $k-1$ colors in K such that $F_e(v_i) \leq i$ for all $i \leq k-2$. Then $S_e = (F_e(v_0), F_e(v_1), \ldots, F_e(v_j))$. The crucial observation is the fact that if e and e' are distinct tree edges of type j, then $S_e \neq S_{e'}$. Indeed, let $e = uv$ be as above and suppose $e' = u'v'$ is another edge of type j, where u' is the parent of v'. Let w be the lowest common ancestor of u and u' (which may be u itself, if $u = u'$), and suppose $S_e = S_{e'}$. Then the two colorings F_e and $F_{e'}$ coincide on the tree path from v_0 to w. Let y be the vertex following w on the tree-path from v_0 to v and let T_y be the subtree of T rooted at y. Define a coloring H of G as follows; for each vertex z of G, $H(z) = F_e(z)$ if $z \notin T_y$, and $H(z) = F_{e'}(z)$ if $z \in T_y$. It is easy to check that since the only edges of G connecting T_y with the rest of the graph lead from T_y to the path from v_0 to w, the coloring H is a proper coloring of G with $k-1$ colors. This contradicts the assumption that the chromatic number of G is k, and hence proves the required fact. Since every word S_e corresponds to a proper coloring of a path of length $j+1$ in which the color of vertex number i is at most i (for all $0 \leq i \leq j$), the number of possible distinct words is at most $j!$ for $j \leq k-3$, and at most $(k-2)!(k-2)^{j-k+2}$ if $j \geq k-2$. This completes the proof of the Claim.

By the above claim, the total number, $n-1$, of edges of T satisfies $n-1 \leq \sum_{j=0}^{r-1} s(j,k)$. Since r is the depth of the tree, G contains a path of length r, showing that $t \geq r$ and hence implying (2). As $k \geq 4$, the right-hand-side of (2) is easily checked to be at most $1+(k-2)^{t-1}$, implying that the maximum length of a path in G is at least $\log(n-1)/\log(k-2)$. Since, as mentioned before, G is 2-connected, it follows, by a theorem of Dirac (cf., e.g., [4]), that it contains a cycle of length at least $2\sqrt{t}$, completing the proof. □

Remark 1. It is easy to check that the above theorem implies that if $k \geq 4$ then any k-critical graph G on n vertices contains an odd cycle of length at least $\sqrt{\log(n-1)/\log(k-2)}$. Indeed, let C be a longest cycle in G. If it is odd, the desired result follows, by Theorem 1. Otherwise, let A be an odd cycle in G. If A and C are vertex disjoint, there are, by the 2-connectivity of G, two internally disjoint paths from A to C providing an odd cycle containing at least half of C. A similar argument gives the same conclusion if A and C share only one common vertex. If they have more common vertices, split the edges of A not in C into paths that intersect C only in their ends. Then, there is such a path whose union with C is not 2-colorable (as otherwise the union of A and C would have been 2-colorable). Thus, in this case too we obtain an odd cycle containing at least half of C, and the required result follows from Theorem 1. Note that this shows that any large k-critical graph contains a large 3-critical subgraph. The problem of deciding if every large k-critical graph contains a large s critical graph for other values of $k > s \geq 3$, which is mentioned in [1], Problem 5.6, remains open.

Remark 2. A very simple proof of the fact that any 2-connected graph G containing a path P of length at least $2s^2$ contains a cycle of length at least $2s$ is as follows. If the distance in G between the two ends x and y of the path is at least s, then the union of two internally disjoint paths between x and y forms a cycle of length at least $2s$. Otherwise, consider a shortest path between x and y, and list its intersection points with the path P. Then the distance along P between some two such consecutive intersection points must be at least $2s^2/s = 2s$, providing, again, the required cycle. Although the proof in [4] gives a slightly better constant, the above argument is much simpler.

References

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