List coloring of random and pseudo-random graphs

Noga Alon [*] Michael Krivelevich [†] Benny Sudakov [‡]

Abstract

The *choice number* of a graph G is the minimum integer k such that for every assignment of a set $S(v)$ of k colors to every vertex v of G, there is a proper coloring of G that assigns to each vertex v a color from $S(v)$. It is shown that the choice number of the random graph $G(n, p(n))$ is almost surely $\Theta(\frac{np(n)}{\ln(np(n))})$ whenever $2 < np(n) \leq n/2$. A related result for pseudo-random graphs is proved as well. By a special case of this result, the choice number (as well as the chromatic number) of any graph on n vertices with minimum degree at least $n/2 - n^{0.99}$ in which no two distinct vertices have more than $n/4 + n^{0.99}$ common neighbors is at most $O(n/\ln n)$.

1 Introduction

A *vertex-coloring* of a graph G is an assignment of a color to each of its vertices. The coloring is *proper* if no two adjacent vertices get the same color. The *chromatic number* $\chi(G)$ of G is the minimum number of colors used in a proper coloring of it. If $\chi(G) \leq k$ we say that G is k-colorable.

A related, more complicated quantity is the *choice number* $ch(G)$ of G, introduced in [11] and [22]. This is the minimum integer k such that for every assignment of a set $S(v)$ of k colors to every vertex v of G, there is a proper coloring of G that assigns to each vertex v a color from $S(v)$. The study of this parameter received a considerable amount of attention in recent years, see, e.g., [2], [15] for two surveys.

In this paper we consider the asymptotic behavior of the choice number of the random graph $G(n, p)$, as well as its behavior for certain pseudo-random graphs. Formally, $G(n, p)$ denotes the

[*]Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: noga@math.tau.ac.il. Research supported in part by a USA Israeli BSF grant and by a grant from the Israel Science Foundation.

[†]Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: krivelev@math.tau.ac.il. Research supported in part by a Charles Clore Fellowship.

[‡]Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: sudakov@math.tau.ac.il.

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probability space whose points are graphs on a fixed set of n labeled vertices, where each pair of vertices forms an edge, randomly and independently, with probability p. The term "the random graph $G(n,p)$" means, in this context, a random point chosen in this probability space. Each graph property A (that is, a family of graphs closed under graph isomorphism) is an event in this probability space, and one may study its probability $Pr[A]$, that is, the probability that the random graph $G(n,p)$ lies in this family. In particular, we say that A holds *almost surely* (or a.s., for short), if the probability that $G(n,p)$ satisfies A tends to 1 as n tends to infinity. There are numerous papers dealing with random graphs, and the book of Bollobás [8] is an excellent extensive account of the known results in the subject proved before its publication in 1985.

Answering an old question of Erdős and Rényi, Bollobás [9] proved that the chromatic number of the random graph $G(n,1/2)$ is $(1+o(1))n/(2\log_2 n)$ almost surely. His result, together with the one of Łuczak in [19], imply that if $p(n)$ satisfies $2 < np(n) \leq n/2$, then almost surely $\chi(G(n,p(n))) = \Theta(np/\ln(np))$. (In fact, their results are more precise and supply an asymptotic formula for the typical value of $\chi(G(n,p(n)))$ in all this range.)

The asymptotic behavior of the choice number for random graphs is not that well understood. In their original paper, Erdős, Rubin and Taylor [11] conjectured that almost surely $ch(G(n,1/2)) = o(n)$. This was proved in [1]. Kahn applied the above mentioned result of Bollobás and proved that almost surely $ch(G(n,1/2)) = (1+o(1))\chi(G(n,1/2)) = (1+o(1))n/(2\log_2 n)$. His argument, described (in a slightly modified form) in [2], does not supply an estimate for the typical choice number of $G(n,p)$ for sparse random graphs. Here we prove the following result, which determines the asymptotic behavior of $ch(G(n,p))$ for all $p = p(n)$ satisfying $2 < np(n) \leq n/2$.

Theorem 1.1 *There exist two absolute positive constants c_1 and c_2 such that if $p = p(n)$ satisfies $2 < np \leq n/2$ then the choice number of the random graph $G(n,p)$ satisfies, almost surely,*

$$c_1 \frac{np}{\ln(np)} \leq ch(G(n,p)) \leq c_2 \frac{np}{\ln(np)}.$$

It is convenient to prove this theorem for relatively large values of $p(n)$ by proving that its assertion in fact holds for graphs that exhibit some (rather weak) pseudo-random properties. This is done in the following (purely deterministic) theorem, which is interesting in its own right.

Theorem 1.2 *For every δ satisfying $0 < \delta < 1/4$ there exists an $n_0 = n_0(\delta)$ such that for every $n > n_0$ and every p satisfying $\frac{1}{n^{\delta/3}} \leq p \leq 1/2$ the following holds. Let G be a graph on n vertices satisfying the following two properties:*

1. *Each vertex degree is at least $pn - n^{1-4\delta}$.*
2. *Every two distinct vertices have at most $p^2n + n^{1-4\delta}$ common neighbors.*

Then the choice number $ch(G)$ of G and its chromatic number $\chi(G)$ satisfy

$$\chi(G) \le ch(G) \le \frac{4np}{\delta \ln n}.$$

It turns out that even some very special cases of the last theorem are nontrivial and yield interesting consequences in Combinatorial Number Theory. A similar result has been proved, independently, by Vu [23].

The rest of the paper is organized as follows. In section 2 we present the proof of Theorem 1.2 and apply it to some special cases. Section 3 contains the proof of Theorem 1.1, and section 4 contains some results about the chromatic numbers of graphs with separated eigenvalues and graphs in which the neighborhood of each vertex spans relatively few edges. The final section 5 contains some concluding remarks and open problems. Throughout the paper we assume, whenever this is needed, that n is sufficiently large. We also omit all floor and ceiling signs whenever these are not crucial, to simplify the presentation.

2 Pseudo-random graphs

In this section we prove Theorem 1.2 which bounds the choice number (and hence also the chromatic number) of pseudo-random graphs. The bound provided by the theorem is tight, up to a constant factor, as shown, for example, by appropriate random graphs. We make no attempt to optimize our absolute constants here and in the rest of the paper.

The proof is rather short. We need the following lemma and its corollary.

Lemma 2.1 *Suppose $0 < \delta < 1/4$, $n \ge n_0(\delta)$ and let p satisfy $\frac{1}{n^\delta} \le p \le 1/2$. Let $G = (V, E)$ be a graph on n vertices satisfying the two properties in the assumption of Theorem 1.2. Then every $B \subset V$ of size at least $n^{1-\delta}$ contains at most $\frac{1.01}{2} p|B|^2$ edges of G.*

Proof. Let A be the adjacency matrix of G, let J be the all 1 matrix whose rows and columns are indexed by the vertices of G and put $H = A - pJ = (h_{uv})_{u,v \in V}$. An easy computation shows that the inner product of any two columns of H is relatively small. Indeed, if $N(v)$ and $N(v')$ denote the sets of all neighbors of v and v', respectively, and $v \ne v'$ then,

$$\sum_{u \in V} h_{uv} h_{uv'} = |N(v) \cap N(v')| - p(|N(v)| + |N(v')|) + np^2 \le 2n^{1-4\delta}.$$

Therefore

$$\sum_{u \in B} (\sum_{v \in B} h_{uv})^2 \le \sum_{u \in V} (\sum_{v \in B} h_{uv})^2$$

$$= \sum_{u \in V} (\sum_{v \in B} h_{uv}^2 + \sum_{v,v' \in B, v \ne v'} h_{uv} h_{uv'}) =$$

$$\sum_{v\in B}\sum_{u\in V} h_{uv}^2 + \sum_{v,v'\in B, v\neq v'}\sum_{u\in V} h_{uv}h_{uv'} \leq |B|n + |B|^2 2n^{1-4\delta}.$$

Let $e(B)$ denote the total number of edges of G contained in B. By the Cauchy-Schwartz inequality and the last estimate

$$(2e(B) - p|B|^2)^2 = (\sum_{u\in B}\sum_{v\in B} h_{uv})^2$$

$$\leq |B| \sum_{u\in B}(\sum_{v\in B} h_{uv})^2 \leq |B|^2 n + |B|^3 2n^{1-4\delta}.$$

Hence

$$e(B) \leq \frac{1}{2}p|B|^2 + \frac{1}{2}|B|\sqrt{n} + |B|^{3/2}n^{1/2-2\delta} < \frac{1.01}{2}p|B|^2,$$

where the last inequality follows from the facts that $\delta < 1/4$, $p \geq n^{-\delta}$ and n is sufficiently large. This completes the proof. □

Corollary 2.2 *Suppose* $0 < \delta < 1/4$, $n \geq n_0(\delta)$ *and let* p *satisfy* $\frac{1}{n^\delta} \leq p \leq 1/2$. *Let* $G = (V, E)$ *be a graph on* n *vertices satisfying the two properties in the assumption of Theorem 1.2. Then every subset* C *of at least* $n^{1-\delta/2}$ *vertices of* G *contains an independent set of* G *of size at least* $\frac{\delta}{3p}\ln n$.

Proof. Repeatedly choose a vertex of minimum degree in the induced subgraph of G on C, add it to the independent set and delete it and its neighbors from C. By Lemma 2.1, as long as the remaining part of C has at least $n^{1-\delta}$ vertices the minimum degree in the induced subgraph on it is at most $\frac{2.02p}{2}|C|$ and hence the number of vertices deleted from C at each such step is at most $\frac{2.02p}{2}|C| + 1$, implying that its size after such a step exceeds, say, $e^{-3p/2}|C|$. Therefore one can complete at least $\frac{\delta}{3p}\ln n$ steps successfully before the size of $|C|$ drops below $n^{1-\delta}$, completing the proof. □

Proof of Theorem 1.2. Let $G = (V, E)$ be a graph satisfying the assumptions of the theorem. For each vertex $v \in V$, let $S(v)$ be a list of at least $\frac{4np}{\delta \ln n}$ colors. Our objective is to prove that there is a proper coloring of G assigning to each vertex a color from its list. As long as there is a set C of at least $n^{1-\delta/2}$ vertices containing the same color c in their lists we can, by Corollary 2.2, find an independent set of at least $\frac{\delta}{3p}\ln n$ vertices in C, color them all by c, omit them from the graph and omit the color c from all lists. The total number of colors that can be deleted in this process cannot exceed $\frac{3np}{\delta \ln n}$ (since in each such deletion at least $\frac{\delta \ln n}{3p}$ vertices are deleted from the graph). When this process terminates, no color appears in more than $n^{1-\delta/2}$ lists, and each list still contains at least $\frac{np}{\delta \ln n} > n^{1-\delta/2}$ colors. Therefore, by Hall's theorem, we can assign to each of the remaining vertices a color from its list so that no color is being assigned to more than one vertex, thus completing the coloring and the proof. □

The above proof is clearly algorithmic in the sense that there is a polynomial time algorithm that finds, given a graph satisfying the assumptions of the theorem, and an assignment of lists of colors of the appropriate size for each vertex, a proper coloring assigning to each vertex a color from its list.

To demonstrate the applications of Theorem 1.2 consider the case $p = 1/2$. Here we can choose, say, $\delta = 1/10$ and conclude that for any graph G with a large number n of vertices in which every degree exceeds $n/2 - n^{0.6}$ and every two distinct vertices have at most $n/4 + n^{0.6}$ common neighbors, $\chi(G) \leq ch(G) \leq 20n/\ln n$. Here are a few examples illustrating this estimate.

- The obvious example that satisfies the above assumptions is the random graph $G(n, 1/2)$, which satisfies all assumptions almost surely. Here, in fact, a better estimate is known. As mentioned in the introduction, Bollobás [9] proved that almost surely the chromatic number of $G(n, 1/2)$ is $(1 + o(1))n/2 \log_2 n$, and Kahn (cf. [2]) showed that this implies a similar estimate for $ch(G(n, 1/2))$.

- Another class of examples is the well known Paley graphs G_q defined for every prime q congruent to 1 modulo 4. The vertices of G_q are all elements of the finite field Z_q and two vertices are adjacent iff their difference is a quadratic residue modulo q. These graphs are $(q-1)/2$ regular and any two distinct vertices of them have at most $(q-1)/4$ common neighbors. Thus, for any large q, by choosing, say, $\delta = 1/5$ in our theorem we conclude that the chromatic and the choice number of G_q are both at most $10q/\ln q$. We can in fact improve the constant 10 by being slightly more careful. It is not known if for any $\epsilon > 0$ and any prime $q > q_0(\epsilon)$ the chromatic number of G_q is bigger than $q^{1-\epsilon}$, and a proof of such an estimate would have far reaching number theoretic consequences.

- An additional class of pseudo-random graphs for which the case $p = 1/2$ of our theorem can be applied (and yields, as in the case of random graphs, tight bounds up to a constant factor) is the following. For any odd integer k let H_k denote the graph whose $n_k = 2^{k-1} - 1$ vertices are all binary vectors of length k with an odd number of ones except the all one vector, in which two (distinct) vertices are adjacent iff the inner product of the corresponding vectors is 1 modulo 2. It is easy to check that this graph is $(2^{k-2} - 2)$ regular, and every two distinct vertices in it have at most $2^{k-3} - 1$ common neighbors. Therefore, by our theorem, the chromatic number and the choice number of H_k are both at most, say, $10n_k/\ln n_k$ for all large k. Here, too, the constant 10 can be improved. This estimate is tight up to a constant factor since it is easy to see that the independence number of H_k is $k = (1 + o(1)) \log_2 n_k$, as the vectors corresponding to any independent set are linearly independent over Z_2, because the inner product of each of them with itself is 1 modulo 2 and the inner product of each of them with another one is 0 modulo 2.

- Finally we present examples which show that sometimes the estimate of Theorem 1.2 may be very far from the right answer. Let $q > 2$ be a prime and let F_{q^2} be the finite field with q^2

elements. The vertices of the graph G_{q^2} are all the elements of the field and two vertices are adjacent iff their difference is not a quadratic residue in F_{q^2}. These graphs are $(q^2-1)/2$ regular and any two distinct vertices of G_{q^2} have at most $(q^2-1)/4$ common neighbors. Therefore Theorem 1.2 with $p = 1/2$ supplies an $10q^2/\ln(q^2)$ upper bound on both the choice and the chromatic numbers of G_{q^2}. But the actual chromatic number of G_{q^2} is much smaller. Indeed, note that all elements of the subfield $F_q \subset F_{q^2}, |F_q| = q$ are quadratic residues and thus F_q and also all its additive cosets are independent sets in G_{q^2} and cover all its vertices. This implies that $\chi(G_{q^2}) \leq q$. In fact G_{q^2} contains a clique of size q (any multiplicative coset αF_q of F_q with $\alpha \in F_{q^2}$, quadratic non-residue), showing that $\chi(G_{q^2}) = q$.

3 Random graphs

In this section we prove Theorem 1.1. We start with the statement of some preliminary known results, and continue with a brief outline of the proof. Next we present some lemmas dealing with the properties of random graphs and conclude with the proof of the theorem. Throughout this section ϵ and δ always denote positive reals, and we always assume, whenever this is needed, that n is sufficiently large as a function of ϵ and δ.

3.1 Preliminaries

A graph is *d-degenerate* if every subgraph of it contains a vertex of degree at most d. The following is a simple, well known fact (c.f., e.g., [2]):

Proposition 3.1 *Every d-degenerate graph is $(d+1)$-choosable.* □

Let $\Delta(G)$ denote the maximum vertex degree in a graph G. The *girth* of G is the minimum length of a cycle in it. The following result of Kim [14] is one of the main ingredients of our proof.

Proposition 3.2 ([14]) *Let G be a graph with girth at least 5, then*

$$ch(G) \leq (1 + o(1)) \frac{\Delta(G)}{\ln \Delta(G)}$$

where the $o(1)$ term tends to zero as $\Delta(G)$ tends to infinity. □

3.2 An outline of the proof

In the proof of Theorem 1.1 for the random graph $G = G(n,p) = (V,E)$, we consider three possible ranges of the edge probability p. If p is large, say, $p \geq n^{-1/30}$, the result follows from Theorem 1.2.

If p is small, say $p \leq n^{-3/4-\epsilon}$, (but $np \geq 30$), then almost surely there is a relatively small set of vertices U containing all triangles and all 4-cycles of G, as well as all vertices of degree bigger than, say, $4np$. By repeatedly adding to U vertices in $V-U$ that have many neighbors in it, as long as such vertices exist, we obtain a new set U which still contains all cycles of length 3 and 4 as well as all vertices of high degree, such that no vertex in $V-U$ has too many neighbors in U. Moreover, by the properties of the random graph it can be shown that this set U is still rather small. Given , now, a list of colors assigned to each vertex, we first find, using Proposition 3.1, a proper coloring of the induced subgraph of G on U assigning to each vertex a color from its list. Next we delete from the list of each vertex in $V-U$ all the colors used to color its neighbors in U, noting that the remaining list is still large, since no vertex in $V-U$ has too many neighbors in U. Finally, we use Proposition 3.2 to properly color the remaining vertices, using the fact that the induced subgraph of G on $V-U$ has girth at least 5 and maximum degree at most $4np$.

It remains to deal with the medium values of p, say $n^{-7/8} \leq p \leq n^{-\epsilon}$. Here (if, say, $p = n^{-1/2}$) every vertex is likely to lie in many 4-cycles and hence there is no small set U as before. The trick here is to split the vertex set of the graph arbitrarily into many parts, noting that the induced subgraph on each part is now a random graph which is much sparser (as a function of the smaller number of vertices in a part). Given lists of colors, we can now split their union randomly into disjoint sets, where each set is assigned to one of our subgraphs, making sure by some standard large deviation inequalities that the list of colors of each vertex contains sufficiently many colors assigned to its subgraph. (Here we use the fact that p is medium, and not too small). The proof can now be completed by applying the arguments of the sparse case to each part separately.

The details, which require some care and some careful computation, are presented in the rest of the section.

3.3 Some properties of random graphs

We need the following simple though somewhat technical lemma.

Lemma 3.3 *If $p \geq 30/n$, then the random graph $G(n,p)$ has the following properties:*

- *(i) Almost surely every $s \leq 2n/\ln^2(np)$ vertices of G span fewer than $(4np/\ln^2(np))s$ edges. Therefore any subgraph of this graph induced by a subset $V_0 \subset V$ of size $|V_0| \leq 2n/\ln^2(np)$, is $8np/\ln^2(np)$-choosable.*

- *(ii) If $30/n \leq p \leq n^{-3/4-\epsilon}, \epsilon > 0$ then almost surely:*

  1. All but at most $n/\ln^2(np)$ vertices of G have degree at most $4np$.

2. There exists a subset $U \subset V(G)$ of size at most $n^{1-\epsilon}$ such that the induced subgraph $G[V-U]$ of G on $V-U$ has girth at least 5;

- *(iii) If $n^{\delta} \leq pn \leq n^{1-\delta}, \delta > 0$ then with probability at least $1 - e^{-\sqrt{np}}$:*

  1. Every vertex $v \in V(G)$ has degree $(1+o(1))np$;

  2. The maximum number of edge disjoint 3-cycles in G is at most $5n^3p^3$ and the maximum number of edge disjoint 4-cycles is at most $5n^4p^4$.

Proof. (i) Define $r = 4np/\ln^2(np)$. Then the probability of existence of a subset $V_0 \subset V$ violating the assertion of the lemma is at most

$$\sum_{i=r}^{2n/\ln^2(np)} \binom{n}{i}\binom{\binom{i}{2}}{ri} p^{ri} \leq \sum_{i=r}^{2n/\ln^2(np)} \left[\frac{en}{i}\left(\frac{ei}{2r}\right)^r p^r\right]^i = \sum_{i=r}^{2n/\ln^2(np)} \left[\frac{e^2np}{2r}\left(\frac{eip}{2r}\right)^{r-1}\right]^i$$
$$\leq \sum_{i=r}^{2n/\ln^2(np)} \left[\ln^2(np)\left(\frac{ei\ln^2(np)}{8n}\right)^{4np/\ln^2(np)-1}\right]^i = o(1).$$

The additional claim about the choosability now follows from Proposition 3.1.

(ii) 1. Suppose there are more than $n/\ln^2(np)$ vertices with degree at least $4np$. Denote by S a set containing exactly $n/\ln^2(np)$ such vertices. By statement (i), almost surely, the induced subgraph $G[S]$ has at most $e(G[S]) \leq (4np/\ln^2(np))|S| = 4n^2p/\ln^4(np) \leq n^2p/2\ln^2(np)$ edges. Therefore the number of edges between the sets of vertices S and $V-S$ is at least $4np|S| - 2e(G[S]) \geq 3n^2p/\ln^2(np)$. On the other hand the probability that $G(n,p)$ contains such a bipartite subgraph is at most

$$\binom{n}{\frac{n}{\ln^2(np)}}\binom{\frac{n^2}{\ln^2(np)}}{\frac{3n^2p}{\ln^2(np)}} p^{\frac{3n^2p}{\ln^2(np)}} \leq (e\ln^2(np))^{\frac{n}{\ln^2(np)}}\left(\frac{e}{3}\right)^{\frac{3n^2p}{\ln^2(np)}}$$
$$= \left[e\ln^2(np)\left(\frac{e}{3}\right)^{3np}\right]^{\frac{n}{\ln^2(np)}} = o(1).$$

This implies that almost surely there are at most $n/\ln^2(np)$ vertices in G with degree greater than $4np$.

(ii) 2. Let X_1, X_2 be the number of cycles of length 3 and 4, respectively, in the graph $G(n,p)$. Clearly the expectations satisfy $E(X_1) \leq n^3p^3 \leq n^{3/4-3\epsilon}$ and $E(X_2) \leq n^4p^4 \leq n^{1-4\epsilon}$ with room to spare. By Markov's inequality this implies that almost surely $X_1 + X_2 \leq n^{1-2\epsilon}$. Denote by U the union of all 3− and 4-cycles in G. Then the induced subgraph $G[V-U]$ has girth at least 5 and a.s. $|U| \leq 4n^{1-2\epsilon} \leq n^{1-\epsilon}$.

(iii) 1. The degree of every vertex $v \in V(G)$ is a binomially distributed random variable with parameters n and p. Therefore the result follows from the standard estimates for binomial distributions (see, e.g., [7], Appendix A).

(iii) 2. We describe the proof only for the case of 3-cycles, the case of 4-cycles can be treated similarly. Denote by X the number of 3-cycles and by X_0 the maximum number of edge disjoint 3-cycles in $G(n, p)$. By the inequality of Erdős and Tetali ([12])

$$Pr(X_0 \geq s) \leq \frac{(E(X))^s}{s!}.$$

Since the expectation $E(X)$ is at most n^3p^3 we conclude that

$$Pr(X_0 \geq 5n^3p^3) \leq \frac{(E(X))^{5n^3p^3}}{(5n^3p^3)!} \leq \left(\frac{e}{5}\right)^{n^3p^3} < e^{-\sqrt{np}}.$$

This completes the proof. $\square$

Proposition 3.4 *The random graph $G(n, p)$ with $30/n \leq p \leq n^{-4\epsilon}, 0.1 > \epsilon > 0$ contains almost surely a subset $U \subset V$ of size $(1 + o(1))n/\ln^2(np)$ such that the induced subgraph $G[V - U]$ is $\frac{2np}{\epsilon \ln(np)}$-choosable.*

Proof. Let us first consider the case $p \leq n^{-7/8}$. Then by Lemma 3.3, part (ii) (with $\epsilon = 1/8$) there exists a set $U_0 \subset V(G)$ such that $|U_0| \leq n^{1-1/8}$ and the induced subgraph $G[V - U_0]$ has girth at least 5. Denote by U the set of vertices consisting of U_0 and all vertices with degree greater than $4np$. Then by Lemma 3.3, part (ii) the size of U is bounded by $n/\ln^2(np) + n^{7/8} = (1 + o(1))n/\ln^2(np)$. Since the induced subgraph $G[V - U]$ has girth at least 5 and maximum degree at most $4np$, it follows by Proposition 3.2 that its choice number is at most $(1 + o(1))4np/\ln(np) \leq \frac{2np}{\epsilon \ln(np)}$.

Now we treat the case $n^{-7/8} \leq p \leq n^{-4\epsilon}$. Fix an arbitrary partition of the set of vertices $V(G)$ into $r = n^{1-\epsilon}p$ equal parts $V_1, \ldots, V_r$ each of size n/r. Note that for all $1 \leq i \leq r$ the induced subgraph $G[V_i]$ is a random graph $G(n/r, p)$. Let C_i^1 and C_i^2 be maximum families of edge disjoint cycles of length 3 and 4, respectively, in $G[V_i]$. Then as $p(n/r) = n^\epsilon > (n/r)^\epsilon$, by Lemma 3.3, part (iii) we get that with probability at least $1 - 3e^{-n^{\epsilon/2}}$

$$|C_i^1| \leq 5(\frac{n}{r})^3 p^3 = 5\frac{n^{1+2\epsilon}p}{r} < 5\frac{n^{1-2\epsilon}}{r},$$

$$|C_i^2| \leq 5(\frac{n}{r})^4 p^4 = 5\frac{n^{1+3\epsilon}p}{r} < 5\frac{n^{1-\epsilon}}{r}$$

and the maximum degree in the subgraph $G[V_i]$ is $(1 + o(1))np/r$. Denote by U_i the set of all vertices in V_i which belong to one of the cycles from the families C_i^1 and C_i^2. Then, with probability at least $1 - 3ne^{-n^{\epsilon/2}}$ the size of U_i is bounded by $|U_i| \leq 3|C_i^1| + 4|C_i^2| < n^{1-\epsilon/2}/r$ for all i. Also from the definition of the set U_i it follows that the induced subgraph $G[V_i - U_i]$ has girth greater than 4. Thus Proposition 3.2 implies that with probability at least $1 - 3ne^{-n^{\epsilon/2}}$ the choice number of the induced subgraph $G[V_i - U_i]$ is at most $(1+o(1))\frac{np}{r\ln(np/r)} \le \frac{np}{\epsilon r \ln(np)}$ for all $1 \le i \le r$. Put $U = \cup_{i=1}^{r} U_i$. Then almost surely $|U| \le r\frac{n^{1-\epsilon/2}}{r} = n^{1-\epsilon/2} < n/\ln^2(np)$. It remains to prove that the induced subgraph $G[V-U]$ is $\frac{2np}{\epsilon\ln(np)}$ choosable.

Indeed, given lists of colors L_v of size $\frac{2np}{\epsilon\ln(np)}$ for each vertex v, partition the set of all colors $X = \cup_{v\in V} L_v$ into r sets $X_1, \ldots, X_r$ by choosing for each color randomly and independently an index i between 1 and r and by placing it in X_i. As for all vertices $v \in V$ the random variable $|L_v \cap X_i|$ is binomially distributed with parameters $\frac{2np}{\epsilon\ln(np)}$ and $1/r$, it follows, by the standard large deviation inequality of Chernoff (cf. , e.g., [7], Appendix A), that

$$Pr\left(|L_v \cap X_i| \le \frac{np}{\epsilon r \ln(np)}\right) < e^{-\frac{np}{4\epsilon r \ln(np)}} < e^{-n^{\epsilon/2}}.$$

Therefore with positive probability no such event happens. This implies that there exists a partition of the colors into r pairwise disjoint parts with the property that $|L_v \cap X_i| \ge \frac{np}{\epsilon r \ln(np)}$ for all i and $v \in V$. Take one such partition. Since $ch(G[V_i - U]) = ch(G[V_i - U_i]) \le \frac{np}{\epsilon r \ln(np)}$, one can color the induced subgraph $G[V_i - U]$ using only colors from X_i. Since all the sets X_i are disjoint this gives a proper coloring of the vertices of the graph $G[V-U]$ from the original lists of colors. □

3.4 The proof of the theorem

We need the following result. Note that its statement is purely deterministic.

Proposition 3.5 *Suppose* $0 < \epsilon < 0.1$, $np \ge 30$ *and let* $G = (V, E)$ *be a graph on* n *vertices with the property that there exists a subset* $U_0 \subset V$ *of size* $(1+o(1))n/\ln^2(np)$ *such that the induced subgraph* $G[V - U_0]$ *is* $\frac{2np}{\epsilon\ln(np)}$*-choosable. Suppose further, that every* $s \le 2n/\ln^2(np)$ *vertices of* G *span fewer than* $(4np/\ln^2(np))s$ *edges. Then*

$$ch(G) \le \frac{3np}{\epsilon\ln(np)}.$$

Proof. First we find a subset $U \subset V$ of size $|U| \le 2n/\ln^2(np)$, such that the induced subgraph $G[V-U]$ is $\frac{2np}{\epsilon\ln(np)}$–choosable, and every vertex $v \in V - U$ has at most $10np/\ln^2(np)$ neighbors in U. A similar idea was used in the papers of Łuczak [20] and of Alon and Krivelevich [4]. (Note that the number 10 can be easily reduced, and we make no attempt to optimize the multiplicative constants here and in what follows.) To find U as above, start with $U = U_0, |U_0| = (1+o(1))n/\ln^2(np)$, such that the induced subgraph $G[V - U_0]$ is $\frac{2np}{\epsilon\ln(np)}$ choosable. The existence of U_0 follows from our assumptions. As long as there exists a vertex $v \in V - U$ having at least $10np/\ln^2(np)$ neighbors in U, add it to U and update U by defining $U := U \cup \{v\}$. This process terminates with $|U| < 2n/\ln^2(np)$

because otherwise we would get a subset $U \subset V$ of size $|U| = 2n/\ln^2(np)$, containing more than $(10 - o(1))n^2p/\ln^4(np)$ edges, thus contradicting the assumptions about G. On the other hand, as $V - U \subset V - U_0$ and $G[V - U_0]$ is $\frac{2np}{\epsilon \ln(np)}$ choosable, it follows that the induced subgraph $G[V - U]$ is also $\frac{2np}{\epsilon \ln(np)}$ choosable.

By the assumptions, the induced subgraph $G[U]$ is $8np/\ln^2(np)$-degenerate. Therefore by Proposition 3.1 its choice number is at most $\frac{8np}{\ln^2(np)} + 1 < \frac{3np}{\epsilon \ln(np)}$. Given lists of colors of size $\frac{3np}{\epsilon \ln(np)}$ for each vertex of G, first color the vertices of U. Then each vertex in $V - U$ has at most $10np/\ln^2(np)$ forbidden colors in its list as it has at most that many neighbors in U. Delete these colors from the list. The updated list still contains at least $\frac{3np}{\epsilon \ln(np)} - \frac{10np}{\ln^2(np)} \geq \frac{2np}{\epsilon \ln(np)}$ colors. Since the induced subgraph $G[V - U]$ is $\frac{2np}{\epsilon \ln(np)}$ choosable we can complete its coloring using the new lists. □

Having finished all necessary preparations, we are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us first consider the case $0.5 \geq p \geq n^{-1/30}$. Then the random graph $G(n, p)$ almost surely satisfies all the properties in the assertion of Theorem 1.2 with $\delta = 1/10$. Therefore we have that a.s.

$$ch(G(n,p)) \leq \frac{4np}{\delta \ln n} = \frac{40np}{\ln n} \leq \frac{40np}{\ln(np)}.$$

Now let $30/n \leq p \leq n^{-1/30}$. Then from Lemma 3.3 (i), Proposition 3.4 and Proposition 3.5 with $\epsilon = 1/120$, it follows that almost surely

$$ch(G(n,p)) \leq \frac{3np}{\epsilon \ln(np)} = \frac{360np}{\ln(np)}.$$

Finally if $2 < np \leq 30$, then a simple calculation similar to the one in the proof of Lemma 3.3, part (i) shows that a.s. the random graph $G(n, p)$ is 120-degenerate. Therefore its choice number is at most $121 \leq 121np/\ln(np)$.

Since the choice number of any graph is at least its chromatic number, the lower bound for $ch(G(n, p))$ follows from the known results for $\chi(G(n, p))$ (see [9] and [19]). Therefore almost surely $ch(G(n, p)) = \Theta(\frac{np}{\ln(np)})$. This completes the proof. □

Remark. The constants in the proof of Theorem 1.1 can be considerably improved by replacing the application of Theorem 1.2 with a more direct approach based on the ideas in its proof together with the properties of the random graph. Since, however, our method does not enable us to determine the best possible constant we make no attempt to optimize the constant it does provide.

4 Separated eigenvalues and sparse neighborhoods

A modification of the argument in Section 2 provides an upper bound for the chromatic number of d-regular graphs in which the second (adjacency matrix)-eigenvalue is much smaller than the first.

The maximum eigenvalue of each such graph is d, and the second one is strictly smaller than d iff the graph is connected. It is well known (see, e.g., [10], page 115 or [6]) that in graphs with a small second eigenvalue the number of edges in each set of vertices cannot be too large. This can be used to prove the following result.

Proposition 4.1 *Let G be a connected d-regular graph on n vertices in which $d+1 \le 2n/3$ and the second largest eigenvalue of the adjacency matrix is λ. Then the chromatic number of G satisfies*

$$\chi(G) \le \frac{6(d-\lambda)}{\ln(\frac{d-\lambda}{\lambda+1}+1)}.$$

Proof. Note, first, that since $d+1 < 2n/3$ and the trace of the adjacency matrix is 0 it follows that $\lambda+1>0$, as otherwise the trace, which is the sum of all eigenvalues, would have been at most $d-(n-1)<0$. Since $\ln(1+x) \le x$ for all $x>0$, it is easy to see that $6(d-\lambda)/(\ln(\frac{d-\lambda}{\lambda+1}+1)) \ge 6(d-\lambda)/(\frac{d-\lambda}{\lambda+1}) = 6(\lambda+1)$. As $\chi(G) \le d+1$, it follows that if $\frac{d-\lambda}{\lambda+1} \le 2$ then the inequality in the statement of the proposition is trivially true, as in this case the right hand side is at least $6(\lambda+1) \ge 2d+2 \ge d+1$. Thus we can assume that $\frac{d-\lambda}{\lambda+1} \ge 2$. First we prove the following estimate on the maximum size of an independent set $\alpha(G[U])$ in any induced subgraph of G on a subset $U \subset V(G)$.

Proposition 4.2 *Let G be a connected d-regular graph on n vertices in which $d+1 \le 2n/3$ and the second largest eigenvalue of the adjacency matrix is λ. Then the induced subgraph $G[U]$ of G on any subset U, $|U|=m$ contains an independent set of size at least*

$$\alpha(G[U]) \ge \frac{n}{2(d-\lambda)} \ln\left(\frac{m(d-\lambda)}{n(\lambda+1)}+1\right).$$

We need the following simple, known lemma (see, e.g., [10], [6]).

Lemma 4.3 *Let G be a d-regular graph on n vertices in which the second largest eigenvalue of the adjacency matrix is λ. Let U be a set of bn vertices of G. Then the average degree in the induced subgraph $G[U]$ is at most $db+\lambda(1-b)$.* □

Proof of Proposition 4.2. Construct an independent set I in the induced subgraph $G[U]$ of G by the following greedy procedure. Repeatedly choose a vertex of minimum degree in $G[U]$, add it to the independent set I and delete it and its neighbors from U, stopping when the remaining set of vertices is empty. Let $a_i, i \ge 0$ be the sequence of numbers defined by the following recurrence formula:

$$a_0 = m, \quad a_{i+1} = a_i - (d\frac{a_i}{n} + \lambda(1-\frac{a_i}{n})+1) = (1-\frac{d-\lambda}{n})a_i - (\lambda+1), \ \forall i \ge 0.$$

Note, that the definition of a_i together with Lemma 4.3 imply that the size of the remaining set of vertices after i iterations is at least a_i. Therefore the size of the resulting independent set I is at least the smallest index i such that $a_i \le 0$. By solving the recurrence equation we have that

$$\begin{aligned} a_i &= \left(1 - \frac{d-\lambda}{n}\right)^i \left(m + \frac{n(\lambda+1)}{d-\lambda}\right) - \frac{n(\lambda+1)}{d-\lambda} \\ &\ge e^{-2(\frac{d-\lambda}{n})i} \left(m + \frac{n(\lambda+1)}{d-\lambda}\right) - \frac{n(\lambda+1)}{d-\lambda}. \end{aligned}$$

Here we used the simple fact that if $0 \le x \le 2/3$ then $1 - x \ge e^{-2x}$. Solving the inequality $a_i \le 0$, one can show that the index i should be at least

$$i \ge \frac{n}{2(d-\lambda)} \ln\left(\frac{m(d-\lambda)}{n(\lambda+1)} + 1\right).$$

This completes the proof. □

Remark. Note that if $d = o(n)$ the above proof actually shows that any set of m vertices contains an independent set of size at least

$$(1 + o(1))\frac{n}{d-\lambda} \ln\left(\frac{m(d-\lambda)}{n(\lambda+1)} + 1\right).$$

Returning to the proof of Proposition 4.1, color the graph G as follows. As long as the remaining set of vertices U contains at least $n/\ln(\frac{d-\lambda}{\lambda+1} + 1)$ vertices, find an independent set of vertices in the induced subgraph $G[U]$ of size at least

$$\frac{n}{2(d-\lambda)} \ln\left(\frac{|U|(d-\lambda)}{n(\lambda+1)} + 1\right) \ge \frac{n}{2(d-\lambda)} \ln\left(\frac{d-\lambda}{(\lambda+1)\ln(\frac{d-\lambda}{\lambda+1}+1)} + 1\right)$$

$$\ge \frac{n}{2(d-\lambda)} \ln\left(\frac{1}{\ln(\frac{d-\lambda}{\lambda+1}+1)}\left(\frac{d-\lambda}{\lambda+1} + 1\right)\right) = \frac{n}{2(d-\lambda)}\left(\ln\left(\frac{d-\lambda}{\lambda+1}+1\right) - \ln\ln\left(\frac{d-\lambda}{\lambda+1}+1\right)\right)$$

$$\ge \frac{n}{4(d-\lambda)} \ln\left(\frac{d-\lambda}{\lambda+1} + 1\right).$$

Color all the members of such a set by a new color, delete them from the graph and continue. When this process terminates, the remaining set of vertices U is of size at most $n/\ln(\frac{d-\lambda}{\lambda+1} + 1)$ and we used at most $4(d-\lambda)/\ln(\frac{d-\lambda}{\lambda+1} + 1)$ colors so far. By Lemma 4.3 the induced subgraph $G[U]$ is at most

$$d\frac{|U|}{n} + \lambda(1 - \frac{|U|}{n}) \le \frac{d-\lambda}{\ln(\frac{d-\lambda}{\lambda+1}+1)} + \lambda \le \frac{2(d-\lambda)}{\ln(\frac{d-\lambda}{\lambda+1}+1)} - 1$$

degenerate. Thus we can complete the coloring of G by coloring $G[U]$ using at most $2(d-\lambda)/\ln(\frac{d-\lambda}{\lambda+1} + 1)$ additional colors. The total number of colors used is at most $6(d-\lambda)/\ln(\frac{d-\lambda}{\lambda+1} + 1)$. □

Remark. An easy modification of the above computation shows that if $\lambda << d << n$ then $\chi(G) \leq (1+o(1))\frac{d}{\ln(d/(\lambda+1))}$. A similar result can be proved for non-regular (but nearly regular) graphs using the second smallest eigenvalue of their Laplace matrices. We omit the details.

By a special case of Proposition 4.1, if $\lambda = O(\sqrt{d})$ then $\chi(G) = O(d/\ln d)$. There are many interesting regular graphs with this property (including all three families of examples described in Section 2 above). Other examples appear in, e.g., [17], [3].

Another variant of Proposition 4.1 is obtained by noting that if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the adjacency matrix of a d-regular graph G containing N cycles of length 4 then

$$\sum_{i=1}^{n} \lambda_i^4 = 8N + nd^2 + nd(d-1)$$

(c.f., e.g., [7], Chapter 9 for the easy argument). Therefore, if N does not exceed $d^4/8$ by much, the second eigenvalue is relatively small and one can apply Proposition 4.1 and get the following result, whose (simple) proof is left to the reader.

Proposition 4.4 *Let G be a d-regular graph on n vertices in which $2\sqrt{n} < d \leq 2n/3 - 1$ and the number of 4-cycles is at most $(d^4 + nd^2)/8$. Then the chromatic number of G satisfies*

$$\chi(G) \leq O\left(\frac{d}{\ln(d^2/n)}\right). \qquad \square$$

Here, too, there are several examples of graphs satisfying these properties including the Paley graphs and the graphs H_k described in Section 2.

An additional result that follows from the method described in Section 2 is the following.

Proposition 4.5 *For every $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon)$ such that for every $n > n_0$ and every p satisfying $n^{-1/2+\epsilon} \leq p \leq 2/3$ the following holds. Let G be a graph on n vertices satisfying the following two properties:*

1. Each vertex degree is at least $pn - n^{1-\epsilon}p$.

2. Every two distinct vertices have at most $p^2 n + n^{1-\epsilon}p^2$ common neighbors.

Then the chromatic number of G satisfy $\chi(G) \leq \frac{6np}{\epsilon \ln n}$.

We need two lemmas whose detailed proofs, which are very similar to those of Lemma 2.1 and Corollary 2.2, are omitted.

Lemma 4.6 *Suppose $0 < \epsilon$, $n \geq n_0(\epsilon)$ and let p satisfy $n^{-1/2+\epsilon} \leq p \leq 2/3$. Let $G = (V, E)$ be a graph on n vertices satisfying the two properties in the assumption of Proposition 4.5. Then every subset B of vertices of G contains at most $\frac{1}{2}|B|^2 p + \frac{1}{2}|B|\sqrt{n} + |B|^{3/2} n^{1/2-\epsilon/2} p$ edges of G.* $\square$

Lemma 4.7 *Suppose $0 < \epsilon$, $n \geq n_0(\epsilon)$ and let p satisfy $n^{-1/2+\epsilon} \leq p \leq 2/3$. Let $G = (V, E)$ be a graph on n vertices satisfying the two properties in the assumption of Proposition 4.5. Then every subset C of at least $n^{1-\epsilon/2}$ vertices of G contains an independent set of G of size at least $\frac{\epsilon}{5p} \ln n$.* $\square$

Proof of Proposition 4.5. Let $G = (V, E)$ be a graph satisfying the assumptions of the proposition. Starting with $B = V$, as long as B is of size at least $n^{1-\epsilon/2}$, find, using Lemma 4.7, an independent set of at least $\frac{\epsilon}{5p} \ln n$ vertices in B, color them all by a new color, omit them from B and continue. The total number of colors used in this process cannot exceed $\frac{5np}{\epsilon \ln n}$ (since in each such deletion at least $\frac{\epsilon \ln n}{5p}$ vertices are deleted from the graph). When this process terminates, the set of uncolored vertices B is of size at most $n^{1-\epsilon/2}$. By Lemma 4.6 the induced subgraph $G[B]$ is at most

$$|B|p + \sqrt{n} + 2|B|^{1/2} n^{1/2-\epsilon/2} p < 2n^{1-\epsilon/2} p - 1$$

degenerate. Therefore we can color all the remaining vertices of B by $2n^{1-\epsilon/2}p$ additional colors. This completes the coloring of G. The total number of colors used is at most

$$\frac{5np}{\epsilon \ln n} + 2n^{1-\epsilon/2} p < \frac{6np}{\epsilon \ln n}. \qquad \square$$

Using a different approach, based on a result of Johansson [13], we can also prove the following result about graphs with sparse neighborhoods, which strengthens the assertion of Proposition 4.5 for all $p \leq n^{-\epsilon}$.

Theorem 4.8 *Let $G = (V, E)$ be a graph on n vertices with maximum degree d in which the neighborhood $N(v)$ of any vertex $v \in V$ spans at most $d^{2-\epsilon}$ edges for some fixed $\epsilon > 0$. Then the chromatic number of G is at most $O(d/(\epsilon^3 \ln d))$.*

We omit the detailed proof. Some extensions of this result have recently been proved in [5] and [23].

5 Concluding remarks and open problems

The condition $np > 2$ in Theorem 1.1 is technical and it is easy to determine the asymptotic behavior of $ch(G(n,p))$ for the trivial cases of smaller values of p.

In the proof of Theorem 1.1 we could have used Johansson's result [13] about the choice number of triangle free graphs instead of Proposition 3.2, to get a slightly simpler argument. But the advantage of the proof as described here is that it supplies a better explicit constant.

Let $X(G)$ be a graph-theoretic function. We say that a function $X(G(n,p))$ with $p = p(n)$ is *concentrated in width* $s = s(n,p)$ if there exists a $u = u(n,p)$ so that

$$\lim_{n\to\infty} Pr(u \leq X(G(n,p)) \leq u+s) = 1.$$

The question of estimating the width of concentration of the chromatic number $\chi(G(n,p))$ was first considered by Shamir and Spencer [21], who showed that $\chi(G(n,p))$ is a.s. concentrated in an interval of length roughly $\sqrt{n}$. Their proof used martingales and the fact that the chromatic number of a graph is a *vertex Lipschitz* function, which means that $|\chi(G) - \chi(G')| \leq 1$ for any pair of graphs that differ only at the edges incident with a single vertex. It is easy to see that the choice number of a graph is also vertex Lipschitz. Therefore the method of [21] can be applied to prove that almost surely $ch(G(n,p))$ is concentrated in width roughly $\sqrt{n}$. Shamir and Spencer further proved that for every constant $\alpha > 1/2$, if $p = n^{-\alpha}$ then the chromatic number of $G(n,p)$ is almost surely concentrated in some fixed number $C(\alpha)$ of values. The same statement can be proved, in a similar manner, for the choice number of the random graph as well. Here we present an outline of the proof (which is similar to the ones in [20] and [4]) only for the case $\alpha > 3/4$. We show that in this case $ch(G(n,p))$ is a.s. *two point* concentrated.

Proposition 5.1 *Suppose* $p = n^{-3/4-\delta}$, $\delta > 0$. *Then the choice number of* $G(n,p)$ *is almost surely two-point concentrated.*

Following the arguments in [4], this can be used to show that for every integer-valued function $r(n)$ satisfying $r(n) < n^{-3/4-\delta}$ there is some $p = p(n)$ such that the choice number of $G(n,p)$ is almost surely *precisely* $r(n)$.

Proof of Proposition 5.1. Fix an arbitrarily small $\epsilon > 0$ and let $u = u(n,p,\epsilon)$ be the least integer so that

$$Pr[ch(G(n,p)) \leq u] > \epsilon.$$

We can assume that $u \geq 3$, since otherwise the probability p should be very small and the result follows from the same argument as in [20]. Define $Y(G)$ to be the minimum size of a set of vertices $S_0 \subset V(G)$ for which the induced subgraph $G[V - S_0]$ is u-choosable. This Y satisfies the vertex Lipschitz condition, since at worst one could add a vertex to S_0. Let $\mu = E(Y)$. By applying the vertex exposure martingale on $G(n,p)$ (see, e.g., [7]) we get that for every $\lambda > 0$

$$Pr[Y \leq \mu - \lambda\sqrt{n-1}] < e^{-\lambda^2/2}, \quad Pr[Y \geq \mu + \lambda\sqrt{n-1}] < e^{-\lambda^2/2}.$$

Let λ satisfy $e^{-\lambda^2/2} = \epsilon$, so that these tail events each have probability less than ϵ. By our definition, with probability at least ϵ, $G = G(n,p)$ is u-choosable and hence $Y = 0$. Therefore the first inequality implies that $\mu \leq \lambda\sqrt{n-1}$. Now substitute this in the second inequality,

$$Pr[Y \geq 2\lambda\sqrt{n-1}] \leq Pr[Y \geq \mu + \lambda\sqrt{n-1}] \leq \epsilon.$$

Thus with probability at least $1-\epsilon$, there is a set of vertices S_0 of size at most $O(\sqrt{n})$ such that the induced subgraph $G[V-S_0]$ is u-choosable.

In our case $p = n^{-3/4-\delta}, \delta > 0$. A straightforward computation, similar to the one in the proof of Lemma 3.3, part (i) shows that with probability at least $1-\epsilon$ any subset of vertices S of size at most $C\sqrt{n}$ spans in $G(n,p)$ at most $(2-\delta)|S|$ edges, provided $n > n_0(\epsilon, \delta, C)$. Hence the induced subgraph $G[S]$ is 3-degenerate. Start with $S = S_0$, and as long as there exists a vertex $v \in V - S$ having at least 2 neighbors in S, add it to S and update S by defining $S := S \cup \{v\}$. This process terminates with $|S| = O(\sqrt{n})$ because otherwise we would get a subset $S \subset V$ of size $|S| = O(\sqrt{n})$, containing more than $(2-\delta)|S|$ edges, contradicting the above mentioned property of $G(n,p)$.

Now we prove that with probability at least $1-\epsilon$ the random graph $G(n,p)$ is $u+1$-choosable. Indeed, given lists of colors of size $u+1$ for each vertex $v \in V(G)$, first color the vertices of S. This can be done, since the induced subgraph $G[S]$ is 3-degenerate. Now each vertex $v \in V - S$ has at most one forbidden color in its list as it has at most one neighbor in S. If there is such a color, omit it from the list of v. The updated list still contains at least u colors. Since the induced subgraph $G[V-S]$ is u-choosable we can complete its coloring using the updated lists. The minimality of u guarantees that with probability at least $1-\epsilon$, $ch(G) \geq u$. Altogether

$$Pr[u \leq ch(G(n,p)) \leq u+1] \geq 1 - 3\epsilon,$$

and since ϵ can be chosen arbitrarily this completes the proof. $\square$

It is worth mentioning that the two-point concentration for the chromatic number of $G(n,p)$ is known (see [20], [4]) for all $p = n^{-\alpha}, \alpha > 1/2$. It seems interesting to decide if the same result is also true for the choice number of the random graph.

Our results imply that if $2 < np(n) \leq n/2$ then the choice number and the chromatic number of $G = G(n, p(n))$ satisfy almost surely $ch(G) = \Theta(\chi(G))$. The following stronger conjecture seems plausible.

Conjecture 5.2 *If $np(n)$ tends to infinity then almost surely the choice number and the chromatic number of $G(n,p(n))$ satisfy $\chi(G) = (1+o(1))ch(G)$.*

This is known to be true for $p \geq n^{-(1/4-\epsilon)}$ for any fixed $\epsilon > 0$ (see [16]). Moreover, it is possible that almost surely these two numbers are equal precisely. This remains open. Note that the results of Bollobás [9] and of Łuczak [20] supply an asymptotic formula for the chromatic number of $G(n,p(n))$

in all this range. Note also that the results described above for the concentration of the choice number of $G(n,p)$ show that it is similar to that of the chromatic number of $G(n,p)$, thus supporting the conjecture that these two numbers may be equal or nearly equal almost surely.

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