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Note

On a conjecture of Tuza about packing and covering of triangles

Michael Krivelevich*

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000 Israel

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Abstract

Zs. Tuza conjectured that if a simple graph G does not contain more than k pairwise edge disjoint triangles, then there exists a set of at most $2k$ edges which meets all triangles in G. We prove this conjecture for $K_{3,3}$-free graphs (graphs that do not contain a homeomorph of $K_{3,3}$). Two fractional versions of the conjecture are also proved.

1. Introduction

Let G be a simple, undirected graph with vertex set $V(G) = V$ and edge set $E(G) = E$. Denote by $T = T(G) \subset E^3$ the collection of triangles of G, i.e. $(e_1, e_2, e_3) \in T$ if e_1, e_2, e_3 form a triangle in G. *A triangle packing in* G is a set of pairwise edge disjoint triangles. *A triangle edge cover* in G is a set of edges meeting all triangles. *A fractional triangle packing* is a function $f: T \to \mathbb{R}^+$ such that $\sum\{f(t): t \ni e\} \leqslant 1$ for every $e \in E$. *A fractional triangle edge cover* is a function $g: E \to \mathbb{R}^+$ such that $\sum\{g(e): e \in t\} \geqslant 1$ for every $t \in T$. We denote by $\nu_t(G)$ the maximum size of a triangle packing, by $\tau_t(G)$ the minimum size of a triangle edge cover, by $\nu_t^*(G)$ the maximum of $\sum\{f(t): t \in T\}$ over all fractional triangle packings and by $\tau_t^*(G)$ the minimum of $\sum\{g(e): e \in E\}$ over all fractional triangle edge covers. Define also the hypergraph of triangles H by $V(H) := E(G)$; $E(H) := T(G)$. Obviously,

$$\tau_t(G) = \tau(H), \qquad \tau_t^*(G) = \tau^*(H),$$

$$\nu_t(G) = \nu(H), \qquad \nu_t^*(G) = \nu^*(H),$$

*Present address: Department of Mathematics, Raymond and Beverly Sachler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel.

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where $\nu(H), \tau(H), \nu^*(H), \tau^*(H)$ are the matching number, the covering number, the fractional matching number and the fractional covering number of H, respectively (for precise definitions see, e.g. [3]).

In [5] Tuza conjectured the following.

Conjecture 1. $\tau_t(G) \leqslant 2\nu_t(G)$ for every graph G.

In [6] Tuza proved it for some classes of graphs, in particular, for planar graphs. Here we make one step further, proving the conjecture of Tuza for $K_{3,3}$-free graphs (graphs that do not contain a homeomorph of $K_{3,3}$). In the second part of the article we prove the fractional versions of Tuza's conjecture, namely

$$\tau_t(G) \leqslant 2\tau_t^*(G) \quad \text{and} \quad \nu_t^*(G) \leqslant 2\nu_t(G).$$

2. Proof of the conjecture for $K_{3,3}$-free graphs

If a graph G is not 2-connected, it can be split into two parts G_1 and G_2, which have no common triangles, and if the conjecture is valid for each part, then it is valid for G. Thus we may assume that G is 2-connected.

The key to the proof is the following result of Hall [4].

Theorem 2 (Hall [4], see also Asano [1]). *Each 3-connected component of a $K_{3,3}$-free graph is either planar or exactly the graph K_5.*

As a basis of our proof we shall use the result of Tuza and Proposition 4 below.

Theorem 3 (Tuza [6]). $\tau_t(G) \leqslant 2\nu_t(G)$ *for every planar graph G,*

Proposition 4. $\tau_t(G) \leqslant 2\nu_t(G)$ *for every subgraph G of K_5.*

This is easily verified.

Let us begin with a simple technical lemma.

Lemma 5. *Let G_1, G_2 be two graphs such that*

$$V(G_1) \cap V(G_2) = \{u, v\}$$

and assume that Conjecture 1 is true for G_1 and G_2, that is

$$\tau_t(G_1) \leqslant 2\nu_t(G_1), \tag{1}$$

$$\tau_t(G_2) \leqslant 2\nu_t(G_2). \tag{2}$$

Consider the graph $G = G_1 \cup G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. Then
(1) *if $(u, v) \notin E(G)$ then $\tau_t(G) \leqslant 2\nu_t(G)$;*
(2) *if $e_0 = (i, v) \in E(G_1) \cap E(G_2)$ and*

$$\tau_t(G_1 \backslash e_0) \leqslant 2\nu_t(G_1 \backslash e_0), \tag{3}$$

$$\tau_t(G_2 \backslash e_0) \leqslant 2\nu_t(G_2 \backslash e_0) \tag{4}$$

(i.e. Conjecture 1 is true for graphs $G_1 \backslash e_0, G_2 \backslash e_0$), then $\tau_t(G) \leqslant 2\nu_t(G)$).

Proof. (1) The statement is obvious, since G_1 and G_2 have no common triangles.
(2) Obviously,

$$\begin{aligned} &\tau_t(G) \leqslant \tau_t(G_1) + \tau_t(G_2), \\ &\nu_t(G_1) + \nu_t(G_2) - 1 \leqslant \nu_t(G) \leqslant \nu_t(G_1) + \nu_t(G_2). \end{aligned} \tag{5}$$

If $\nu_t(G) = \nu_t(G_1) + \nu_t(G_2)$, then from (1), (2), and (5) it follows that $\tau_t(G) \leqslant 2\nu_t(G)$, so we may assume that

$$\nu_t(G) = \nu_t(G_1) + \nu_t(G_2) - 1. \tag{6}$$

In fact, (6) states, that if T_1 is a maximal triangle packing in G_1 and T_2 is a maximal triangle packing in G_2 ($|T_1| = \nu_t(G_1), |T_2| = \nu_t(G_2)$), then $e_0 \in E(T_1) \cap E(T_2)$, where $E(T_i) = \{e \in E(G_i)\colon \exists t \in T_i,\ e \in t\}$, $i = 1, 2$. Hence we have

$$\nu_t(G_1 \backslash e_0) = \nu_t(G_1) - 1, \qquad \nu_t(G_2 \backslash e_0) = \nu_t(G_2) - 1.$$

It follows from (3) and (4) that

$$\tau_t(G_1 \backslash e_0) \leqslant 2\nu_t(G_1 \backslash e_0) = 2\nu_t(G_1) - 2,$$

$$\tau_t(G_2 \backslash e_0) \leqslant 2\nu_t(G_2 \backslash e_0) = 2\nu_t(G_2) - 2.$$

But $\tau_t(G) \leqslant \tau_t(G_1 \backslash e_0) + \tau_t(G_2 \backslash e_0) + 1$. Hence

$$\tau_t(G) \leqslant 2\nu_t(G_1) + 2\nu_t(G_2) - 3 < 2\nu_t(G). \qquad \square$$

Now we are ready to prove the main result of this section.

Theorem 6. *Conjecture 1 is true for $K_{3,3}$-free graphs.*

Proof. By induction on the number of vertices in G. If G is 3-connected, then the assertion follows from Theorems 2 and 3 and Proposition 4. Otherwise G contains a separating pair $\{u, v\}$. Let K be one of the connected components of $G \backslash \{u, v\}$. Denote

$$G_1 = G[V(K) \cup \{u, v\}], \qquad G_2 = G \backslash K.$$

For G_1 and G_2 the conditions of Lemma 5 are satisfied by the induction hypothesis, so for $G = G_1 \cup G_2$ it follows from Lemma 5 that

$$\tau_t(G) \leqslant 2\nu_t(G). \quad \square$$

3. Proof of the fractional versions of Conjecture 1

Our aim is to prove two fractional relaxations of Conjecture 1:

$$\tau_t(G) \leqslant 2\tau_t^*(G) \quad \text{and,} \quad \nu_t^*(G) \leqslant 2\nu_t(G),$$

where $\tau_t, \nu_t, \tau_t^*, \nu_t^*$ are defined as described in Section 1. The duality theorem of linear programming states that $\tau_t^* = \nu_t^*$ and that if $f: T \to \mathbb{R}^+$ and $g: E \to \mathbb{R}^+$ are a maximum fractional triangle packing and a minimum fractional triangle edge cover respectively, then

$$f(t) > 0 \quad \text{implies} \quad \sum \{g(e): e \in t\} = 1, \tag{7a}$$

$$g(e) > 0 \quad \text{implies} \quad \sum \{f(t): t \ni e\} = 1, \tag{7b}$$

where $t \in T$, $e \in E$.

Theorem 7. $\nu_t^*(G) \leqslant 2\nu_t(G)$.

Proof. Consider the hypergraph H of triangles. H is 3-uniform, and we can use the following result of Füredi ([2]): if an r-uniform hypergraph H does not contain a projective plane of order $r-1$ as a partial hypergraph, then $\nu^*(H) \leqslant (r-1)\nu(H)$. So we have only to check that no hypergraph of triangles contains the Fano plane (the projective plane of order 2) as a partial hypergraph. Denote the Fano plane by H_0 and its vertex set by $\{1, \dots, 7\}$. Suppose to the contrary that $H_0 \subseteq H$. For $i = 1, \dots, 7$ let $e_i \in E(G)$ be the graph edge corresponding to the vertex i in H_0. Suppose also that $(1,2,3) \in E(H_0)$, so (e_1, e_2, e_3) form a triangle in G. There are in H_0 edges, that contain the pairs (4, 1), (4, 2), (4, 3). This means that the pairs of edges $(e_4, e_1), (e_4, e_2), (e_4, e_3)$ are contained in some triangles in G, so each of these pairs is intersecting, which is impossible. We have shown that $H_0 \nsubseteq H$. $\square$

The bound on the ratio between ν_t^* and ν_t is best possible, since for $G = K_4$ we have $\nu_t^*(G) = 2$, $\nu_t(G) = 1$.

Theorem 8. $\tau_t(G) \leqslant 2\tau_t^*(G)$.

Proof. Suppose to the contrary that there exist graphs which contradict the statement, and let G be a minimal graph such that $\tau_t(G) > 2\tau_t^*(G)$. Then $\tau_t(G') \leqslant 2\tau_t^*(G')$ for every proper subgraph G' of G.

Let $f: T(G) \to \mathbb{R}^+$ be a maximum fractional triangle packing and $g: E(G) \to \mathbb{R}^+$ be a minimum fractional triangle edge cover of G. Consider two possible cases:

Case 1: $g(e) > 0$ *for every* $e \in E(G)$: Then it follows from the complementary slackness condition (7b) that

$$|E(G)| = \sum_{e \in E} 1 = \sum_{e \in E} \sum_{t \in T} f(t) = \sum_{t \in T} f(t)|t \cap E| = 3 \sum_{t \in T} f(t) = 3\tau_t^*(G),$$

so

$$\tau_t^*(G) = \frac{|E(G)|}{3}. \tag{8}$$

On the other hand, there is a bipartite graph B in G with at least $|E(G)|/2$ edges. Since B contains no triangles, $E(G) \setminus E(B)$ meets all triangles in G, so for all G

$$\tau_t(G) \leqslant \frac{|E(G)|}{2}. \tag{9}$$

Comparing (8) and (9), we conclude that $\tau_t(G) \leqslant \frac{3}{2}\tau_t^*(G)$, contradicting the assumption on G.

Case 2: *There exists* $e_0 \in E(G)$ *such that* $g(e_0) = 0$: Since G is a minimal graph which contradicts the statement, every edge in G belongs to some triangle. Suppose that $(e_0, e_1, e_2) \in T(G)$. Since g is the fractional triangle edge cover, $g(e_0) + g(e_1) + g(e_2) \geqslant 1$, but $g(e_0) = 0$, so $g(e_1) \geqslant 1/2$ or $g(e_2) \geqslant 1/2$, say, $g(e_1) \geqslant 1/2$. Consider the graph $G' = G \setminus e_1$, $V(G') = V(G)$, $E(G') = E(G) \setminus \{e_1\}$. Obviously,

$$\tau_t(G') \geqslant \tau_t(G) - 1 \tag{10}$$

(if $E_0 \subseteq E(G')$ is a triangle edge cover for G', then $E_0 \cup \{e_0\}$ is a triangle edge cover for G). Due to the choice of G for G' we have $\tau_t(G') \leqslant 2\tau_t^*(G')$. But $g': E(G') \to \mathbb{R}^+$, $g'(e) := g(e)$ for all $e \in E(G')$, is a fractional triangle edge cover for G', so

$$\tau_t^*(G') \leqslant \sum_{e \in E(G')} g'(e) = \tau_t^*(G) - g(e_1) \leqslant \tau_t^*(G) - 1/2. \tag{11}$$

It follows from (10) and (11) that

$$\tau_t(G) \leqslant \tau_t(G') + 1 \leqslant 2\tau_t^*(G') + 1 \leqslant 2(\tau_t^*(G) - 1/2) + 1 = 2\tau_t^*(G),$$

again a contradiction. □

We have no example which realizes the equality $\tau_t(G) = 2\tau_t^*(G)$, and perhaps this result is not best possible.

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