An improved bound on the minimal number of edges in color-critical graphs

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Abstract

It is proven that for $k \geq 4$ and $n > k$ every k-color-critical graph on n vertices has at least $\left(\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)}\right) n$ edges, thus improving a result of Gallai from 1963.

A graph G is k-color-critical (or simply *k-critical*) if $\chi(G) = k$ but $\chi(G') < k$ for every proper subgraph G' of G, where $\chi(G)$ denotes the chromatic number of G. (See, e.g., [2] for a detailed account of graph coloring problems). Consider the following problem: given k and n, what is the minimal number of edges in a k-critical graph on n vertices? It is easy to see that every vertex of a k-critical graph G has degree at least $k-1$, implying $|E(G)| \geq \frac{k-1}{2}|V(G)|$. Gallai [1] improved this trivial bound to $|E(G)| \geq \left(\frac{k-1}{2} + \frac{k-3}{2(k^2-3)}\right)|V(G)|$ for every k-critical graph G (where $k \geq 4$), which is not a clique K_k on k vertices. In this note we strengthen Gallai's result by showing

Theorem 1 *Suppose $k \geq 4$, and let $G = (V, E)$ be a k-critical graph on more than k vertices. Then*

$$|E(G)| \geq \left(\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)}\right)|V(G)| \ .$$

In the first non-trivial case $k = 4$ we get $|E(G)| \geq \frac{11}{7}|V(G)|$, compared to the estimate $|E(G)| \geq \frac{20}{13}|V(G)|$ of Gallai.

Let us introduce some definitions and notation (we follow the terminology of [4]). If $G = (V, E)$ is a k-critical graph, then the *low-vertex subgraph* of G, denoted by $L(G)$, is the subgraph of G, induced by all vertices of degree $k-1$. The *high-vertex subgraph* of G, which we denote by $H(G)$, is the subgraph of G induced by all vertices of degree at least k in G. Brooks' theorem implies that if $k \geq 4$ and $G \neq K_k$, then $H(G) \neq \emptyset$. A maximal by inclusion connected subgraph B of a graph G such that every two edges of B are contained in a cycle of G is called a *block* of G. A connected graph all of whose blocks are either complete graphs or odd cycles is called a *Gallai tree*, a *Gallai forest* is a graph all of whose connected components are Gallai trees. A *k-Gallai forest (tree)* is a Gallai forest (tree), in which all vertices have degree at most $k-1$.

Our proof utilizes results of Gallai [1] and Stiebitz [5], describing the structure of color-critical graphs. Gallai proved the following fundamental result.

Lemma 1 ([1], Satz E.1) *If G is a k-critical graph then its low-vertex subgraph $L(G)$ is a k-Gallai forest (possibly empty).*

Using induction on the number of vertices, it follows from the above statement that

Lemma 2 ([1], Lemma 4.5) *Let $k \geq 4$. Let $G = (V, E) \neq K_k$ be a k-Gallai forest. Then*

$$|E(G)| \leq \left(\frac{k-2}{2} + \frac{1}{k-1}\right) |V(G)| - 1 \ . \tag{1}$$

The second ingredient of our proof is the following result of Stiebitz.

Lemma 3 ([5]) *Let G be a k-critical graph. Then the number of connected components of its high-vertex subgraph $H(G)$ does not exceed the number of connected components of its low-vertex subgraph $L(G)$.*

Proof of Theorem 1. Let $L(G)$ and $H(G)$ be the low-vertex and the high-vertex subgraphs of G, respectively. Denote $n_L = |V(L(G))|$, $n_H = |V(H(G))|$, $n = |V(G)| = n_L + n_H$. By Brooks' theorem $n_H > 0$.

Let r be the number of connected components of $H(G)$, then trivially

$$|E(H(G))| \geq n_H - r \ . \tag{2}$$

Also, by Lemma 3, the number of connected components of $L(G)$ is at least r. Now the crucial observation is that each connected component of $L(G)$ is itself a k-Gallai tree, therefore the estimate (1) is valid for it too. We infer that

$$|E(L(G))| \leq \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n_L - r \ . \tag{3}$$

Indeed, if $G_1 = (V_1, E_1), \ldots, G_{r'} = (V_{r'}, E_{r'})$ are the connected components of $L(G')$, where $r' \geq r$, then by Lemma 1

$$|E_i| \leq \left(\frac{k-2}{2} + \frac{1}{k-1}\right) |V_i| - 1 \ , \quad i = 1, \ldots, r' \, .$$

Summing the above inequalities over $1 \leq i \leq r'$, we get (3).

Using (2) and (3), the number of edges of G can be bounded from below as follows:

$$\begin{aligned} |E(G)| &= \sum_{v \in V(L(G))} d(v) - |E(L(G))| + |E(H(G))| \\ &\geq (k-1)n_L - \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n_L + r + n_H - r \\ &= n + \frac{k^2 - 3k}{2(k-1)} n_L \ . \end{aligned} \tag{4}$$

On the other hand, it follows from the definition of $L(G)$ and $H(G)$ that

$$\begin{aligned} |E(G)| &= \frac{1}{2}\sum_{v\in V(G)} d(v) = \frac{1}{2}\left(\sum_{v\in V(L(G))} d(v) + \sum_{v\in V(H(G))} d(v)\right) \\ &\geq \frac{1}{2}((k-1)n_L + kn_H) = \frac{k}{2}n - \frac{1}{2}n_L \ . \end{aligned} \tag{5}$$

Multiplying (5) by $(k^2-3k)/(k-1)$ and summing with (4) we get

$$\left(1+\frac{k^2-3k}{k-1}\right)|E(G)| \geq \left(1+\frac{k}{2}\frac{k^2-3k}{k-1}\right)n \ ,$$

or

$$|E(G)| \geq \left(\frac{k-1}{2}+\frac{k-3}{2(k^2-2k-1)}\right)n \ ,$$

as claimed. □

A more detailed treatment of the problem, containing lower and upper bounds on the minimal number of edges in a k-critical graph on n vertices with additional restrictions imposed, and some applications of these bounds to Ramsey-type problems and problems on random graphs, will appear in a forthcoming paper [3].

References

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