Constructive bounds for a Ramsey-type problem

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Abstract

For every fixed integers r, s satisfying $2 \leq r < s$ there exists some $\epsilon = \epsilon(r, s) > 0$ for which we construct explicitly an infinite family of graphs $H_{r,s,n}$, where $H_{r,s,n}$ has n vertices, contains no clique on s vertices and every subset of at least $n^{1-\epsilon}$ of its vertices contains a clique of size r. The constructions are based on spectral and geometric techniques, some properties of Finite Geometries and certain isoperimetric inequalities.

1 Introduction

The Ramsey number $R(s, t)$ is the smallest integer n such that every graph on n vertices contains either a clique K^s of size s or an independent set of size t. The problem of determining or estimating the function $R(s, t)$ received a considerable amount of attention, see, e.g., [14] and some of its references. A more general function was first considered (for a special case) by Erdős and Gallai in [11]. Suppose $2 \leq r < s \leq n$ are integers, and let G be a K^s-free graph on n vertices. Let $f_r(G)$ denote the maximum cardinality of a subset of vertices of G that contains no copy of K^r, and define, following [12], [8]:

$$f_{r,s}(n) = \min f_r(G),$$

where the minimum is taken over all K^s-free graphs G on n vertices.

It is easy to see that for $r = 2$, we have $f_{2,s}(n) < t$ if and only if the Ramsey number $R(s, t)$ satisfies $R(s, t) > n$, showing that the problem of determining the function $f_{r,s}(n)$ extends that of determining $R(s, t)$.

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Erdős and Rogers [12] combined a geometric idea with probabilistic arguments and showed that

$$f_{s-1,s}(n) \leq O(n^{1-1/O(s^4 \log s)}).$$

This bound has been improved in several subsequent papers [8], [17], [18] and the best known bounds, proved in [17], [18], are

$$c_1 n^{\frac{1}{s-r+1}} (\log\log n)^{1-\frac{1}{s-r+1}} \leq f_{r,s}(n) \leq c_2 n^{\frac{r}{s+1}} (\log n)^{\frac{1}{r-1}} ,$$

where c_1, c_2 are positive constants depending only on r and s. Note that to place an upper bound on $f_{r,s}(n)$ one has to prove the existence of a graph with certain properties. As is the case with the problem of bounding the usual Ramsey numbers, the existence of these graphs is usually proved by probabilistic arguments. In fact there is no known *explicit* construction that provides any nontrivial upper bound for $f_{r,s}(n)$ for any value of r other than 2. By *explicit* we mean here a construction that supplies a deterministic algorithm to construct a graph with the desired properties in time polynomial in the size of the graph. It is worth noting that for the case $r = 2$, corresponding to the usual Ramsey numbers, there are several known explicit constructions; see [10], [13], [9], [1], [2], [3]. Despite a considerable amount of effort, all these constructions supply bounds that are inferior to those proved by applying probabilistic arguments. The problem of finding explicit constructions matching the best known bounds is of great interest, and may have algorithmic applications as well.

In the present note we describe two different explicit constructions providing nontrivial upper bounds for the function $f_{r,s}(n)$ in the case $r > 2$. The first one is based on a spectral technique together with some of the properties of finite geometries and implies that for every fixed r, s we have:

$$f_{r,s}(n) = O\left(n^{\frac{1}{2}+\frac{2r-3}{2s-4}}\right) .$$

The second construction is based on a geometric idea and certain isoperimetric inequalities, and shows that for every $s \geq 2$, $f_{s,s+1}(n) \leq n^{1-\epsilon(s)}$, where

$$\epsilon(s) = (1+o(1))\frac{2}{s^2(s+1)^2 \ln s}$$

and the $o(1)$ term tends to 0 as n tends to infinity.

Both constructions are explicit according to all common definitions of this notion and, in particular, provide a linear time deterministic algorithm to construct the appropriate graph as well as an algorithm that determines if two given vertices are connected using a constant number of arithmetic or bit operations on words of length $O(\log n)$, where n is the number of vertices.

In the rest of this note we describe these two constructions and prove their properties.

2 The first construction

The first construction we present applies finite geometries and the proof of its properties is based on the spectral technique used in [1] for a similar purpose, together with some additional ideas. Graphs considered in this section may have loops. Each loop contributes one to the degree of a vertex incident to it and contributes 1/2 when we count the number of edges spanned by a set of vertices.

We need the following lemma.

Lemma 2.1 *Let G be a d-regular graph on n vertices with at most one loop at each vertex and suppose that the absolute value of any eigenvalue of G but the first is at most λ. For every integer $r \geq 2$ denote*

$$s_r = \frac{(\lambda+1)n}{d}\left(1+\frac{n}{d}+\cdots+\left(\frac{n}{d}\right)^{r-2}\right) .$$

Then every set of more than s_r vertices of G contains a copy of K^r.

Proof. The proof relies on the following simple statement proved (in a slightly stronger form) in [5] (see also [6], Chapter 9, Corollary 2.6.)

Proposition 2.2 *Let $G=(V,E)$ be a d-regular graph on n vertices (with loops allowed) and suppose that the absolute value of each of its eigenvalues but the first is at most λ. Let B be an arbitrary subset of bn vertices of G and let $e(B)$ denote the number of edges in the induced subgraph of G on B. Then*

$$|e(B)-\frac{1}{2}b^2dn| \leq \frac{1}{2}\lambda bn .$$

To deduce the lemma from the above proposition we apply induction on r. Note that if S is a subset of vertices of size k, then, by the proposition above, $e(S) \geq \frac{1}{2}\frac{k^2d}{n} - \frac{1}{2}\lambda k$.

If $r=2$ and $k > (\lambda+1)n/d$ then $e(S) \geq \frac{1}{2}k\left(\frac{kd}{n}-\lambda\right) > \frac{1}{2}k$, and therefore S contains at least one non-loop edge, as needed.

Assuming the assertion of the lemma holds for all integers between 2 and r we prove it for $r+1$ (≥ 3). Since $e(S) \geq \frac{1}{2}\frac{k^2d}{n} - \frac{1}{2}\lambda k$, there exists a vertex $v \in S$ which is incident with at least $\frac{kd}{n}-\lambda$ edges in S, implying that v has at least $\frac{kd}{n}-\lambda-1$ neighbours in S other than itself. Let N denote the set of all these neighbours. By the induction hypothesis if $\frac{kd}{n}-\lambda-1 > s_r$, then N contains a copy of K^r, that together with v forms a copy of K^{r+1}. Hence, any set of more than

$$\frac{(s_r+\lambda+1)n}{d} = \frac{(\lambda+1)n}{d}\left(1+\frac{n}{d}+\cdots+\left(\frac{n}{d}\right)^{r-1}\right) = s_{r+1}$$

vertices of G contains a copy of K^{r+1}, completing the proof. $\Box$

For any integer $t \geq 2$ and for any power $q = 2^g$ of 2 let $PG(t,q)$ denote the finite geometry of dimension t over the field $GF(q)$. The interesting case for our purposes here is that of fixed t and large q. It is well known (see, e.g., [15]) that the points and hyperplanes of $PG(t,q)$ can be described as follows. Let B_t denote the set of all nonzero vectors $\bar{x} = (x_0, \ldots, x_t)$ of length $t+1$ over $GF(q)$ and define an equivalence relation on B_t by calling two vectors equivalent if one is a multiple of the other by an element of the field. The points of $PG(t,q)$ as well as the hyperplanes can be represented by the equivalence classes of B_t with respect to this relation, where a point $\bar{x} = (x_0, \ldots, x_t)$ lies in the hyperplane $\bar{y} = (y_0, \ldots, y_t)$ if and only if their inner product $\langle \bar{x}, \bar{y} \rangle = x_0y_0 + \ldots + x_ty_t$ over $GF(q)$ is zero. Let $G(t,q)$ denote the graph whose vertices are the points of $PG(t,q)$, where two (not necessarily distinct) vertices $\bar{x}$ and $\bar{y}$ as above are connected by an edge if and only if $\langle \bar{x}, \bar{y} \rangle = x_0y_0 + \ldots + x_ty_t = 0$, that is, the point represented by $\bar{x}$ lies on the hyperplane represented by $\bar{y}$. The graphs $G(t,q)$ have been considered by several authors - see, e.g., [1], [7]. It is easy to see that the number of vertices of $G(t,q)$ is $n_{t,q} = (q^{t+1}-1)/(q-1) = q^t(1+o(1))$ and that it is $d_{t,q} = (q^t-1)/(q-1) = q^{t-1}(1+o(1))-$ regular, where here and in what follows the $o(1)$ term tends to zero as q tends to infinity. It is also easy to see that the number of vertices of $G(t,q)$ with loops is precisely $d_{t,q} = (q^t-1)/(q-1)$, since the equation $x_0^2 + \ldots + x_t^2 = 0$ over $GF(q)$ is equivalent to the linear equation $x_0 + \ldots + x_t = 0$, which has exactly $q^t - 1$ nonzero solutions. This is the only place we use the fact that the field $GF(q)$ is of characteristic 2. A similar construction exists for any prime power q, but the computation in this case is (slightly) more complicated.

We claim that each copy of K^{t+2} in $G(q,t)$ contains at least one vertex represented by a vector $\bar{x}$ with $\langle \bar{x}, \bar{x} \rangle = 0$. Indeed, let $\bar{x}^1, \ldots, \bar{x}^{t+2}$ be vectors of $PG(t,q)$ such that $\langle \bar{x}^i, \bar{x}^j \rangle = 0$ for every pair $1 \leq i \neq j \leq t+2$. Since these are $t+2$ vectors in a vector space of dimension $t+1$ there are $\nu_1, \ldots, \nu_{t+2} \in GF(q)$ which are not all zero, such that $\nu_1 \bar{x}^1 + \ldots \nu_{t+2} \bar{x}^{t+2} = \bar{0}$. Multiplying this equality by $\bar{x}^i$ for every $1 \leq i \leq t+2$, we conclude that $\nu_i \langle \bar{x}^i, \bar{x}^i \rangle = 0$, and hence there exists at least one index i such that $\langle \bar{x}^i, \bar{x}^i \rangle = 0$.

The eigenvalues of $G(t,q)$ are known and easy to compute. To see this, let A be the adjacency matrix of $G(t,q)$. By the properties of $PG(t,q)$, $A^2 = AA^T = \mu J + (d_{t,q} - \mu)I$, where $\mu = (q^{t-1}-1)/(q-1)$, J is the $n_{t,q} \times n_{t,q}$ all 1-s matrix and I is the $n_{t,q} \times n_{t,q}$ identity matrix. Therefore the largest eigenvalue of of A^2 is $d_{t,q}^2$ and all other eigenvalues are $d_{t,q} - \mu$. It follows that the largest eigenvalue of A is $d_{t,q}$ and the absolute value of all other eigenvalues is $(d_{t,q} - \mu)^{1/2}$. Put $\lambda = (d_{t,q} - \mu)^{1/2} = q^{(t-1)/2}$. By Lemma 2.1 every set of $\lambda (n_{t,q}/d_{t,q})^{r-1}(1+o(1)) = n_{t,q}^{\frac{1}{2}+\frac{2r-3}{2t}}(1+o(1))$ vertices spans a copy of K^r.

Let $H(t,q)$ denote the graph obtained from $G(t,q)$ by deleting all vertices represented by vectors $\bar{x}$ with $\langle \bar{x}, \bar{x} \rangle = 0$. This is a K^{t+2}-free graph on $n = q^t$ vertices, for which the following holds:

Theorem 2.3 *For every fixed t and large enough q the graph $H(t,q)$ on $n = q^t$ vertices has the following properties:*

1. $K^{t+2} \not\subseteq G$.
2. *For every $r \geq 2$ every set of $n_0(r,t)$ vertices spans a copy of K^r, where $n_0(r,t) = n^{\frac{1}{2}+\frac{2r-3}{2t}}(1+o(1))$.*

To get a result for a general n, we can, for example, start with $H(t,q)$, where q is the minimal integer of the form $q = 2^g$ for which $q^t \geq n$. Then $2^t n \geq q^t$. Now delete from $H(t,q)$ an arbitrary subset of $q^t - n$ vertices, thus obtaining a graph on n vertices with the desired properties. This gives

Corollary 2.4 *By an explicit construction for every fixed r, t satisfying $2 \leq r < (t+1)/2$, and for every n,*

$$f_{r,t}(n) = O\left(n^{\frac{1}{2}+\frac{2r-3}{2t-4}}\right) .$$

3 The second construction

The second construction we present borrows the core idea from the construction of Erdős and Rogers [12]. Their argument is non-constructive and relies on the concentration of measure phenomenon in the high-dimensional sphere. Our construction is explicit and somewhat simpler to handle.

We begin with some notation.

Let $s \geq 2$ and k be positive integers (where s is assumed to be fixed while k tends to infinity). Denote $V = [s]^k$. Thus, the elements of V are vectors $\bar{x}$ of length k. We endow V with the normalized counting measure P, that is, $P(A) = |A|/|V|$ for every subset $A \subseteq V$. For every two vectors $\bar{x}, \bar{y} \in V$ denote by $d(\bar{x},\bar{y})$ the *Hamming distance* between $\bar{x}$ and $\bar{y}$, that is,

$$d(\bar{x},\bar{y}) = |\{1 \leq i \leq k : x_i \neq y_i\}| .$$

Also, for a set $\emptyset \neq U \subseteq V$ and a vector $\bar{x} \in V$ let

$$d(\bar{x},U) = \min\{d(\bar{x},\bar{y}) : \bar{y} \in U\}$$

denote the distance between $\bar{x}$ and U.

For every integer $\delta > 0$ define the *δ-neighbourhood* $U_{(\delta)}$ of a nonempty subset $U \subseteq V$ as

$$U_{(\delta)} = \{\bar{x} \in V : d(\bar{x},U) \leq \delta\} ;$$

thus $U_{(0)} = U$.

Define a graph $G = G(s,k)$ as follows. The vertex set of G is V, and two vectors $\bar{x}, \bar{y} \in V$ are connected by an edge in G if and only if $d(\bar{x}, \bar{y}) > k\left(1 - \binom{s+1}{2}^{-1}\right)$. Let us investigate the properties of this graph.

Proposition 3.1 *The graph G does not contain a copy of K^{s+1}.*

Proof. Suppose indirectly that $\bar{x}^1, \ldots, \bar{x}^{s+1}$ are the vertices of $K^{s+1} \subseteq G$, then according to the definition of G we have $d(\bar{x}^{i_1}, \bar{x}^{i_2}) > k(1 - \binom{s+1}{2}^{-1})$ for every pair $1 \le i_1 \ne i_2 \le s+1$. For every $1 \le j \le k$, there exists at least one pair of vertices of K^{s+1} having the same value in the j-th coordinate. Therefore, summing over all k coordinates and averaging, we obtain that there exists at least one pair of vectors $\bar{x}^{i_1}, \bar{x}^{i_2}$, that agree on at least $k\binom{s+1}{2}^{-1}$ coordinates. Thus $d(\bar{x}^{i_1}, \bar{x}^{i_2}) \le k - k\binom{s+1}{2}^{-1}$, supplying the desired contradiction. □

We next prove that every sufficiently large subset of V spans a copy of K^s. Define an *s-simplex* S to be a set of s vectors $\bar{x}^1, \ldots, \bar{x}^s \in V$ with $d(\bar{x}^{i_1}, \bar{x}^{i_2}) = k$ for every pair $1 \le i_1 \ne i_2 \le s$.

Proposition 3.2 *If $V_0 \subseteq V$ and $P[V_0] > (s-1)/s$, then V_0 contains a copy of an s-simplex.*

Proof. Clearly, each vector of V lies in the same number of s-simplices.

Choose randomly and uniformly a simplex S among all s-simplices in V. Then

$$\begin{aligned} P[S \not\subseteq V_0] &= P[\text{ at least one vertex of } S \text{ does not belong to } V_0 \text{ }] \\ &\le s\frac{|V| - |V_0|}{|V|} < s\left(1 - \frac{s-1}{s}\right) = 1. \end{aligned}$$

Hence there exists at least one s-simplex S with all vertices in V_0. □

The next step is to obtain a good isoperimetric inequality for the finite metric space (V, d). We use martingales as in, e.g., [20], [19], [21].

Lemma 3.3 *For $c > 0$ denote $\delta(c) = \lceil(\sqrt{\ln s/2} + c)\sqrt{k}\rceil$. If A is a subset of V with $P[A] \ge 1/s$, then $P[A_{(\delta)}] > 1 - e^{-2c^2}$.*

Proof. Define a function $f : V \to R$ by $f(\bar{x}) = d(\bar{x}, A)$. This function is clearly Lipschitz with constant 1, that is, $|f(\bar{x}) - f(\bar{y})| \le d(\bar{x}, \bar{y})$ for all $\bar{x}, \bar{y} \in V$. Also, $f(\bar{x}) = 0$ for every $\bar{x} \in A$. Let $Ef = X_0, X_1, \ldots, X_k = f$ be the coordinate exposure martingale with respect to f, that is, $X_i(\bar{x}) = E[f(\bar{y}) : \bar{y} \in V : \bar{y}_j = \bar{x}_j \; \forall j \leq i]$. Hence Hoeffding's inequality (see, e.g. Lemma 1.2 of [19]) implies:

$$P[X_k - X_0 < -c\sqrt{k}] \;\; < \;\; e^{-2c^2}, \qquad (1)$$
$$P[X_k - X_0 > \; c\sqrt{k}] \;\; < \;\; e^{-2c^2} \qquad (2)$$

for all $c > 0$. In particular, substituting $c = \sqrt{\ln s/2}$ in (1) and recalling that $P[A] \geq 1/s$, we see that there exists at least one point $\bar{x} \in A$, for which

$$X_k(\bar{x}) - X_0(\bar{x}) = f(\bar{x}) - X_0 \geq -\sqrt{\ln s/2}\sqrt{k} \; .$$

However, since $\bar{x} \in A$, one has $X_k(\bar{x}) = 0$, and therefore $X_0 \leq \sqrt{\ln s/2}\sqrt{k}$. Thus (2) implies that

$$P[f > (\sqrt{\ln s/2} + c)\sqrt{k}] < e^{-2c^2} \; .$$

The left-hand side of the above inequality is at least $P[V \setminus A_{(\delta)}]$ with $\delta = \delta(c)$ as in the formulation of the lemma, and hence

$$P[A_{(\delta)}] > 1 - e^{-2c^2} \; . \qquad \square$$

Remark. McDiarmid gives in [19] an isoperimetric inequality for graph product spaces (see [19], Prop. 7.12), implying directly our Lemma 3.3. We chose however to present its proof here for the sake of completeness. Moreover, it is possible to improve the inequality and obtain an asymptotically tight isoperimetric inequality. This and related results will appear in [4]. For our purpose here the present estimate suffices.

The result of the lemma can be reformulated in the following more convenient way: for every $c > 0$, if $U \subseteq V$ and $P[U] \geq e^{-2c^2}$, then $P[U_{(\delta)}] > (s-1)/s$. Indeed, assuming $P[U_{(\delta)}] \leq (s-1)/s$, denote $W = V \setminus U_{(\delta)}$, then $P[W] \geq 1/s$ and therefore $P[W_{(\delta)}] > 1 - e^{-2c^2}$, contradicting the fact that $W_{(\delta)} \cap U = \emptyset$. Define

$$c = \frac{\sqrt{k}}{2\binom{s+1}{2}} - \sqrt{\ln s/2} - 1 = \frac{\sqrt{k}}{s(s+1)}(1 + o(1)) \; , \qquad (3)$$

where the $o(1)$ term tends to 0 as k tends to infinity. Then if $P[U] \geq e^{-2c^2}$, then $P[U_{(\delta)}] > (s-1)/s$, where $\delta = \delta(c)$. Therefore, by Proposition 3.2, the set $U_{(\delta)}$ contains an s-simplex S. Let $\bar{x}^1, \ldots, \bar{x}^s$ denote its vertices. Let $\bar{y}^i \in U$ satisfy $d(\bar{x}^i, \bar{y}^i) \leq \delta(c)$, where $1 \leq i \leq s$. By the triangle inequality

$$d(\bar{y}^{i_1}, \bar{y}^{i_2}) \;\; \geq \;\; d(\bar{x}^{i_1}, \bar{x}^{i_2}) - 2\delta(c)$$

$$
\begin{aligned}
&> k - \frac{k}{\binom{s+1}{2}} + 2\sqrt{k} - 2 \\
&\geq k\left(1 - \frac{1}{\binom{s+1}{2}}\right) .
\end{aligned}
$$

This means that the vertices $\bar{y}^1, \ldots, \bar{y}^s \in U$ form a copy of K^s. Recalling (3), we see that every subset $U \subseteq V$ of size $|U| = |V|\, e^{-2c^2} = |V|\, e^{-(1+o(1))2k/(s^2(s+1)^2)}$ spans a copy of K^s.

Summing up the above, we obtain the following

Theorem 3.4 *For every fixed s and large k the graph $G = G(s,k)$ described above is a graph on $n = s^k$ vertices, having the following properties:*

1. $K^{s+1} \not\subseteq G$;
2. *Every set of at least $n^{1-\epsilon(s)}$ vertices spans a copy of K^s, where $\epsilon(s) = (1+o(1))2/(s^2(s+1)^2 \ln s)$ and the $o(1)$ term tends to 0 as k tends to infinity.*

Corollary 3.5 *By an explicit construction, for every fixed $s \geq 2$,*

$$f_{s,s+1}(n) \leq n^{1-\epsilon(s)} ,$$

where $\epsilon(s) = (1 + o(1))2/(s^2(s+1)^2 \ln s)$.

Remarks. **1.** For the case $s = 2$ the above construction coincides with the construction of Erdős [10], proving a lower bound for the Ramsey number $R(3,t)$. In this case instead of applying our approach based on isoperimetric inequalities one can calculate exactly the maximum size of a subset of vertices of G not containing an edge K^2, using the well-known result of Kleitman [16].

2. For general s, our estimate on $\epsilon(s)$ is better by a factor of 4 than the one of Erdős and Rogers.

3. A similar construction, requiring somewhat more complicated analysis, yields an improvement of the expression for $\epsilon(s)$ by a logarithmic factor. Below we give an outline of the argument, leaving all technical details to the reader.

For a given s, define

$$t = t(s) = \min\{2 \leq i \leq s : s (\mathrm{mod}\ i) < i/2\} .$$

It can be shown that $t(s)$ is at most polylogarithmic in s. Put $s = tq + r$. It follows from the definition of t that $r < t/2$. Let $V = [t]^k$. Define

$$
\begin{aligned}
\alpha_1(s) &= 1 - \frac{(r+1)\binom{q+1}{2} + (t-r-1)\binom{q}{2}}{\binom{s+1}{2}} , \\
\alpha_2(s) &= 1 - \frac{r\binom{q+1}{2} + (t-r)\binom{q}{2}}{\binom{s}{2}} .
\end{aligned}
$$

It is easy to see that $\alpha_1(s)k$ is an upper bound for the minimum Hamming distance between a pair in any family of $s+1$ vectors in V. Similarly, $\alpha_2(s)k$ is an upper bound for the minimum Hamming distance between a pair in any family of s vectors in V.

Note that since $r < t/2$ we have

$$\alpha_2(s) - \alpha_1(s) = \frac{2q(t-r-1)}{(s-1)s(s+1)} \geq \frac{1}{2s(s+1)} \ .$$

Now define a graph G with vertex set V by joining two vectors $\bar{x}, \bar{y} \in V$ by an edge if their Hamming distance exceeds $\alpha_1(s)k$. Then G is clearly K^{s+1}-free. Define an s-simplex to be a family of s vectors in G, whose mutual Hamming distances are all equal to $\alpha_2(s)k$. Assuming "nice" divisibility properties of k, we can claim that such an s-simplex indeed exists and that every $\frac{s-1}{s}|V|$ vertices of G span an s-simplex. Now the same argument as in the proof of Theorem 3.4 shows that every $|V|e^{-ck/s^4} = n^{1-c/s^4 \ln t} = n^{1-c'/s^4 \ln\ln s}$ vertices of G span a copy of K^s, where $n = |V|$ and c, c' are some absolute constants. This gives the bound

$$f_{s,s+1}(n) \leq n^{1-\frac{c_0}{s^4 \ln\ln s}} \ ,$$

where c_0 is an absolute constant.

4. The idea applied in the construction of Theorem 3.4 can be used also for obtaining constructive upper bounds for the function $f_{r,s}(n)$ for values of r other than $s-1$. The bounds obtained (as well as the above bound for $f_{s-1,s}(n)$) are considerably weaker that the ones proved in [18] by probabilistic arguments. It would be interesting to find explicit examples providing bounds closer to the last ones.

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