K^s-FREE GRAPHS WITHOUT LARGE K^r-FREE SUBGRAPHS

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ABSTRACT. The main result of this paper is that for every $2 \le r < s$ and n sufficiently large there exist graphs of order n, not containing a complete graph on s vertices, in which every relatively not too small subset of vertices spans a complete graph on r vertices. Our results improve on previous results of Bollobás and Hind.

1 Introduction

Let $2 \le r < s \le n$ be natural numbers. Let G be a graph on n vertices not containing a K^s, a complete graph on s vertices, as an induced subgraph. What is the largest size of a subgraph of G, which does not contain a K^r, $2 \le r < s$, as an induced subgraph? In other words, the problem is to compute the function

$$f_{r,s}(n) = \min_{G^n \not\supseteq K^s} \max\{|V_0|, V_0 \subseteq V(G), K^r \not\subseteq G[Vo]\}.$$

Note that for $r = 2$ the problem of determining $f_{r,s}(n)$ is that of determining certain Ramsey numbers, so the exact determination of $f_{r,s}$(n) seems to be hopeless in general, and the main efforts have been devoted to understand the asymptotic behavior of $f_{r,s}(n)$.

To the best of our knowledge this problem was first addressed by Erdős and Gallai ([4]). They showed that $f_{p,n-p-2}(n) = 2p - 2$, or, in other words, if every $2p - 2$ vertices of G_n contain a K^p, then $K^{n-p-2} \subset G_n$, and this bound is tight, as shown by a complement of a complete bipartite graph $K^{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Erdős and Rogers ([7]) considered the case of s fixed, $r = s - 1$ and n tending to infinity and showed that there exist graphs of order n, not containing a K^s, such that every induced subgraph of G of order $n^{1-\epsilon(s)}$ contains a K^{s-1}, where $\epsilon(s) \sim c/s^4 \log s$ for large values of s. The next step was made about thirty years after by Bollobás and Hind ([3]), they used sophisticated arguments to show that

$$n^{1/(s-r+1)} \le f_{r,s}(n) \le n^{(s-3)/(s-2)+2/(s+1)(s-2)+\epsilon},$$

and for a particular case of $r = 3, s = 4$

$$(2n)^{1/2} \le f_{3,4}(n) \le n^{7/10+\epsilon}.$$

For s and n tending to infinity and a fixed r new results were obtained recently by Linial and Rabinovich ([9]), but we do not cite their results here. For possible extensions for hypergraphs the reader is referred to [5], [10].

In this paper we concentrate on the case of r and s fixed and n tending to infinity. Our main aim is to improve on the upper bounds for the function $f_{r,s}(n)$ given by

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Bollobás and Hind. We make extensive use of probabilistic methods, the monograph of Alon and Spencer ([2]) may serve as a general reference to the subject.

Throughout the text we will use the symbol 'c' as a generic symbol for various constants (probably having different values).

2 An example: $f_{3,4}(7) = 4$

As mentioned above, the exact determination of values of $f_{r,s}(n)$ is very hard in general and it seems possible only for some values of r, s, n. As an illustration we show one such case which can be easily solved completely: $f_{3,4}(7) = 4$.

First let us prove that $f_{3,4}(7) \geq 4$, or, in other words, if every four vertices of a graph G of order 7 contain a triangle, then $K^4 \subset G$. If all vertices of G are of degree at most three and G does not contain a K^4, then according to Brooks' theorem G is three-colourable, so there exists in G an independent set of vertices W of size three. But then W together with any vertex of $V \setminus W$ does not contain a triangle - a contradiction to our assumption about G. Hence there exists a vertex $v_0 \in V(G)$ of degree at least four. The neighbourhood of v_0 $\Gamma(v_0) = \{v \in V(G) : (v, v_0) \in E(G)\}$ contains a triangle (v_1, v_2, v_3) which together with v_0 forms a complete graph on four vertices.

The following graph $G = (V, E)$ on seven vertices, taken from the paper of Linial and Rabinovich ([9]), does not contain a K^4, but every five vertices contain a triangle: $V(G) = \{0, 1, \ldots, 6\}$, $E(G) = \big\{(i, (i+1) \bmod 7), \quad 0 \leq i \leq 6\big\} \cup \big\{(i, (i+3) \bmod 7), \quad 0 \leq i \leq 6\big\}$. This example shows that $f_{3,4}(7) \leq 4$.

3 A lower bound for $f_{3,4}(n)$

Bollobás and Hind showed that $f_{r,s}(n) \geq n^{1/(s-r+1)}$. We can improve this bound slightly using a result of Ajtai, Erdős, Komlós and Szemerédi ([1]) that every K^s-free graph on n vertices with an average degree t contains a vertex independent set of size $c(n/t)\log(\log t/s)$ (instead of the value $n/(t+1)$ provided by Turán's theorem).

Theorem 1. *$f_{r,s}(n) \geq c_{r,s} n^{1/(s-r+1)} (\log\log n)^{1-1/(s-r+1)}$, where $c_{r,s}$ is a constant depending only on the values of r and s.*

Proof. As in the proof of Bollobás and Hind define a sequence of graphs

$$G = G_0, G_1, \ldots, G_{s-r}$$

by putting $G_{i+1} = G_i[\Gamma(v_i)]$ for $i = 0, 1, \ldots, s-r-1$, where v_i is a vertex of maximal degree in G_i. Obviously, every G_i does not contain a K^{s-i} for $i = 1, \ldots, s-r$. Denote $\alpha = 1/(s-r+1)$. If there exists an i such that

$$\Delta(G_i) < c'_{r,s} |G_i| n^{-\alpha} (\log\log n)^{\alpha},$$

take the first such i, denote it by i_0. According to the theorem of Ajtai, Erdős, Komlós and Szemerédi

$$\begin{aligned} ind(G_{i_0}) &> c''_{r,s} \frac{|G_{i_0}|}{|G_{i_0}| n^{-\alpha} (\log\log n)^{\alpha}} \log\left(\frac{\log |G_{i_0}| n^{-\alpha} (\log\log n)^{\alpha}}{s}\right) \\ &> c_{r,s} n^{\alpha} (\log\log n)^{1-\alpha}. \end{aligned}$$

In other case

$$\Delta(G_i) \geq c'''_{r,s}|G_i|n^{-\alpha}(\log\log n)^{\alpha}$$

for every $i = 0, 1, \ldots, s-r-1$, so

$$|G_{s-r}| \geq |G_0|(c'''_{r,s}n^{-\alpha}(\log\log n)^{\alpha})^{s-r},$$

and therefore

$$|G_{s-r}| \geq c_{r,s}n^{1/(s-r+1)}(\log\log n)^{1-1/(s-r+1)}.$$

Since G_{s-r} does not contain a K^r, the result follows. □

4 An upper bound for $f_{r,s}(n)$

To establish $f_{r,s}(n) \leq m$ we have to show that there exists a graph G of order n not containing a copy of K^s such that every set of m vertices of G contains a K^r. The existence of such G is shown by using the Lovasz Local Lemma ([6]; [2], Ch. 5), and Janson's inequality ([8]; [2], Ch. 8).

Consider a random graph $G(n,p)$ - a graph on n vertices in which all edges are chosen independently and with probability p, where the value of $p = p(n)$ will be chosen later. For a set S of s vertices let A_S be an event $G[S] \cong K^s$. Obviously, $Pr(A_S) = p^{\binom{s}{2}}$. For a set T of m vertices (where m will be an upper bound we wish to establish) let B_T an event that T does not contain a K^r.

Claim. $Pr(B_T) \leq c\exp\left\{-\binom{m}{r}p^{\binom{r}{2}} + m^{r+1}p^{\binom{r}{2}+r-1}\max\{m^{r-3}p^{(r-3)r/2}, 1\}\right\}$

Proof. We prove it by using Janson's inequality. Our notation will be consistent with that of [2].

For a set $X \subset T$ of size r let C_X be an event $G[X] \cong K^r$. Let $\epsilon = Pr[C_X] = p^{\binom{r}{2}}$. Our aim is to bound from above the probability $Pr[\bigwedge_{|X|=r}\overline{C}_X]$.

If the events C_X were mutually independent then the probability $Pr[\bigwedge_{|X|=r}\overline{C}_X]$ would be

$$M = \prod_{|X|=r} Pr(\overline{C}_X) = \left(1 - p^{\binom{r}{2}}\right)^{\binom{m}{r}} < \exp\left\{-\binom{m}{r}p^{\binom{r}{2}}\right\}.$$

Since in fact the events C_X and C'_X are mutually dependent if $|X \cap X'| \geq 2$, the probability $Pr[\bigwedge_{|X|=r}\overline{C}_X]$ may be greater than M. Janson's inequality asserts that this difference is in some sense not so large.

For every $2 \leq i \leq r-1$ let

$$h(i) = \binom{r}{i}\binom{m-r}{r-i}p^{\binom{r}{2}-\binom{i}{2}} \leq cm^{r-i}p^{\binom{r}{2}-\binom{i}{2}},$$

and let

$$\Delta^* = \sum_{i=2}^{r-1} h(i).$$

Let also

$$\mu = \sum_{|X|=r} Pr(C_X) = \binom{m}{r}p^{\binom{r}{2}}$$

and

$$\begin{aligned}\Delta &= \sum_{2\le|X\cap X'|\le r-1} Pr(C_X \wedge C'_X)\\ &= \sum_{|X|=r} Pr(C_X) \sum_{2\le|X\cap X'|\le r-1} Pr(C'_X/C_X)\\ &= \Delta^* \sum_{|X|=r} Pr(C_X) = \Delta^*\mu,\end{aligned}$$

In the expression for Δ^* the terms $h(2) \le cm^{r-2}p^{\binom{r}{2}-1}$ and $h(r-1) \le cmp^{r-1}$ are the only candidates for being the dominating term. If $mp^{r/2} \ll 1$, then $h(2) \ll h(r-1)$, otherwise $h(2) \gg h(r-1)$. So

$$\begin{aligned}\Delta = \Delta^*\mu &\le cm^r p^{\binom{r}{2}} \max\{m^{r-2}p^{\binom{r}{2}-1}, mp^{r-1}\}\\ &= cm^{r+1}p^{\binom{r}{2}+r-1}\max\{m^{r-3}p^{(r-3)r/2},1\}.\end{aligned}$$

Now Janson's inequality provides the following upper bound for $Pr[B_T] = Pr[\bigwedge_{|X|=r}\overline{C}_X]$:

$$\begin{aligned}Pr[\bigwedge_{|X|=r}\overline{C}_X] &\le M\exp\left\{\frac{1}{1-\epsilon}\frac{\Delta}{2}\right\}\\ &\le c\exp\left\{-\binom{m}{r}p^{\binom{r}{2}} + m^{r+1}p^{\binom{r}{2}+r-1}\max\{m^{r-3}p^{(r-3)r/2},1\}\right\}.\end{aligned}$$

$\Box(Claim)$

We shall now apply the Lovasz Local Lemma. For this purpose we define a dependency graph of the events. The events A_S and A'_S are independent unless S and S' have at least two vertices in common. The same is true for A_S and B_T or B_T and B'_T. Consider a graph whose vertices correspond to all A_S with S ranging over all s-sets of $V(G)$, and all B_T with T ranging over all m-sets of $V(G)$. Two vertices of the dependency graph are joined by an edge if the corresponding sets share at least two vertices. Each A_S is joined to $\sum_{i=2}^{s-1}\binom{s}{i}\binom{n-s}{s-i} \le cn^{s-2}$ events A'_S and to at most $\binom{n}{m}$ events B_T . Each B_T is joined to $\sum_{i=2}^{s}\binom{m}{i}\binom{n-m}{s-i} \le cm^2n^{s-2}$ events A_S (here we are assuming $m = o(n)$) and to at most $\binom{n}{m}$ events B'_T. Associate the same $0 < x \le 1$ with each vertex A_S and the same $0 < y \le 1$ with each vertex B_T. Now the Local Lemma asserts that if there exist p, m such that

$$x(1-x)^{cn^{s-2}}(1-y)^{\binom{n}{m}} \ge p^{\binom{s}{2}}$$

and

$$\begin{aligned}&y(1-x)^{cm^2n^{s-2}}(1-y)^{\binom{n}{m}}\\ &\qquad\ge c\exp\left\{-\binom{m}{r}p^{\binom{r}{2}} + m^{r+1}p^{\binom{r}{2}+r-1}\max\{m^{r-3}p^{(r-3)r/2},1\}\right\},\end{aligned}$$

then there exists a graph G on n vertices not containing a K^s, in which every induced graph on m vertices contains a K^r. Our aim is to find the minimal m for which there exists a solution of the above two inequalities. Elementary calculations give that the best choice is

$$
\begin{aligned}
m &= c_1 n^{(s-2)r/(s(s-1)-r)} (\log n)^{\left(\binom{s}{2}-\binom{r}{2}\right)/\left(\binom{s}{2}(r-1)-\binom{r}{2}\right)}, \\
p &= c_2 n^{-2(s-2)/(s(s-1)-r)} (\log n)^{1/\left(\binom{s}{2}(r-1)-\binom{r}{2}\right)}, \\
x &= c_3 p^{\binom{s}{2}}, \\
y &= c_4 \Big/ \binom{n}{m}.
\end{aligned}
$$

We have proved the following theorem:

Theorem 2. *$f_{r,s}(n) < c_{r,s} n^{(s-2)r/(s(s-1)-r)} (\log n)^{\left(\binom{s}{2}-\binom{r}{2}\right)/\left(\binom{s}{2}(r-1)-\binom{r}{2}\right)}$, where $c_{r,s}$ is a constant depending only on the values of r and s.*

Corollary 1. $f_{3,4}(n) \leq c n^{2/3} (\log n)^{1/3}$.

Corollary 2. $f_{s-1,s}(n) \leq c_s n^{(s-2)/(s-1)} (\log n)^{2/(s-1)(s-2)}$.

Compare now the bounds of Theorem 2 with the bounds obtained by Bollobás and Hind. While for the case $r = s - 1$ these bounds approximately match the bounds of [3], for the general case the bounds obtained here improve significantly previosly known bounds. For the case of $f_{3,4}(n)$ the bound of Theorem 2 is also better than the bound of Bollobás and Hind. Nevertheless, it is easy to see that the gap between the lower bound of Theorem 1 and the upper bound of Theorem 2 is still relatively large .

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