Foundation of Cryptography (0368-4162-01), Lecture 1

One Way Functions

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Section 1

Notation

Notation I

- For $t \in \mathbb{N}$, let $[t] := \{1, \ldots, t\}$.
- Given a string $x \in \{0,1\}^*$ and $0 \le i < j \le |x|$, let $x_{i,\ldots,j}$ stands for the substring induced by taking the $i, \ldots, j$ bit of x (i.e., $x[i] \ldots, x[j]$).
- Given a function f defined over a set $\mathcal{U}$, and a set $\mathcal{S} \subseteq \mathcal{U}$, let $f(\mathcal{S}) := \{f(x) \colon x \in \mathcal{S}\}$, and for $y \in f(\mathcal{U})$ let $f^{-1}(y) := \{x \in \mathcal{U} \colon f(x) = y\}$.
- poly stands for the set of all polynomials.
- The worst-case running-time of a *polynomial-time algorithm* on input x, is bounded by $p(|x|)$ for some $p \in$ poly.
- A function is *polynomial-time computable*, if there exists a polynomial-time algorithm to compute it.

Notation II

- PPT stands for probabilistic polynomial-time algorithms.
- A function $\mu \colon \mathbb{N} \mapsto [0,1]$ is negligible, denoted $\mu(n) = \text{neg}(n)$, if for any $p \in \text{poly}$ there exists $n' \in \mathbb{N}$ with $\mu(n) \leq 1/p(n)$ for any $n > n'$.

Distribution and random variables I

- The support of a distribution P over a finite set $\mathcal{U}$, denoted $\mathrm{Supp}(P)$, is defined as $\{u \in \mathcal{U} : P(u) > 0\}$.
- Given a distribution P and en event E with $\Pr_P[E] > 0$, we let $(P \mid E)$ denote the conditional distribution P given E (i.e., $(P \mid E)(x) = \frac{D(x) \wedge E}{\Pr_P[E]}$).
- For $t \in \mathbb{N}$, let let U_t denote a random variable uniformly distributed over $\{0,1\}^t$.
- Given a random variable X, we let $x \leftarrow X$ denote that x is distributed according to X (e.g., $\Pr_{x \leftarrow X}[x = 7]$).
- Given a final set $\mathcal{S}$, we let $x \leftarrow \mathcal{S}$ denote that x is uniformly distributed in $\mathcal{S}$.

Distribution and random variables II

- We use the convention that when a random variable appears twice in the same expression, it refers to a *single* instance of this random variable. For instance, $\Pr[X = X] = 1$ (regardless of the definition of X).
- Given distribution P over $\mathcal{U}$ and $t \in \mathbb{N}$, we let P^t over $\mathcal{U}^t$ be defined by $D^t(x_1, \ldots, x_t) = \Pi_{i \in [t]} D(x_i)$.
- Similarly, given a random variable X, we let X^t denote the random variable induced by t independent samples from X.

Section 2

One Way Functions

One-Way Functions

Definition 1 (One-Way Functions (OWFs))

A polynomial-time computable function $f : \{0,1\}^* \mapsto f : \{0,1\}^*$ is one-way, if for any PPT A

$$\Pr_{y \leftarrow f(U_n)}[A(1^n, y) \in f^{-1}(y)] = \text{neg}(n)$$

U_n: a random variable uniformly distributed over $\{0,1\}^n$

polynomial-time computable: there exists a polynomial-time algorithm F , such that $F(x) = f(x)$ for every $x \in \{0,1\}^*$

PPT : probabilistic polynomial-time algorithm

neg: a function $\mu\colon \mathbb{N} \mapsto [0,1]$ is a *negligible* function of n, denoted $\mu(n) = \text{neg}(n)$, if for any $p \in \text{poly}$ there exists $n' \in \mathbb{N}$ such that $g(n) < 1/p(n)$ for all $n > n'$

We will typically omit 1^n from the parameter list of A

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    - Asymptotic
    - Efficiently computable
    - On the average
    - Only against PPT's

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4. Where do we find them

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5. Non uniform OWFs

Definition 2 (Non-uniform OWF))

A polynomial-time computable function $f : \{0,1\}^* \mapsto \{0,1\}^*$ is one-way, if for any polynomial-size family of circuits $\{C_n\}_{n \in \mathbb{N}}$

$$\Pr_{y \leftarrow f(U_n)}[C_n(y) \in f^{-1}(y)] = \text{neg}(n)$$

Length preserving functions

Definition 3 (length preserving functions)

A function $f : \{0,1\}^* \mapsto f : \{0,1\}^*$ is length preserving, if $|f(x)| = |x|$ for any $x \in \{0,1\}^*$

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Theorem 4

Assume that OWFs exit, then there exist length-preserving OWFs

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Theorem 4

Assume that OWFs exit, then there exist length-preserving OWFs

Proof idea: use the assumed OWF to create a length preserving one

Partial domain functions

Definition 5 (Partial domain functions)

For $m, \ell \colon \mathbb{N} \mapsto \mathbb{N}$, let $h \colon \{0,1\}^{m(n)} \mapsto \{0,1\}^{\ell(n)}$ denote a function defined over input lengths in $\{m(n)\}_{n \in \mathbb{N}}$, and maps strings of length $m(n)$ to strings of length $\ell(n)$.

The definition of one-wayness naturally extends to such functions.

OWFs imply Length Preserving OWFs cont.

Let $f : \{0,1\}^* \mapsto \{0,1\}^*$ be a OWF, let $p \in \text{poly}$ be a bound on its computing-time and assume wlg. that p is monotony increasing (can we?).

Construction 6 (the length preserving function)

Define $g \colon \{0,1\}^{p(n)} \mapsto \{0,1\}^{p(n)}$ as

$$g(x) = f(x_{1,\ldots,n}), 0^{p(n)-\left|f(x_{1,\ldots,n})\right|}$$

Note that g is length preserving and efficient (why?).

OWFs imply Length Preserving OWFs cont.

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Note that g is length preserving and efficient (why?).

Claim 7

g is one-way.

OWFs imply Length Preserving OWFs cont.

Let $f : \{0,1\}^* \mapsto \{0,1\}^*$ be a OWF, let $p \in$ poly be a bound on its computing-time and assume wlg. that p is monotony increasing (can we?).

Construction 6 (the length preserving function)

Define $g \colon \{0,1\}^{p(n)} \mapsto \{0,1\}^{p(n)}$ as

$$g(x) = f(x_{1,\ldots,n}), 0^{p(n) - \left|f(x_{1,\ldots,n})\right|}$$

Note that g is length preserving and efficient (why?).

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g is one-way.

How can we prove that g is one-way?

OWFs imply Length Preserving OWFs cont.

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Note that g is length preserving and efficient (why?).

Claim 7

g is one-way.

How can we prove that g is one-way?
Answer: using reduction

Proving that g is one-way

Proof:
Assume that g is not one-way. Namely, there exists PPT A a $q \in$ poly and an infinite $\mathcal{I} \subseteq \{p(n) \colon n \in \mathbb{N}\}$, with

$$\Pr_{y \leftarrow g(U_n)}[\mathrm{A}(y) \in g^{-1}(y)] > 1/q(n) \qquad (1)$$

for any $n \in \mathcal{I}$.

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$$\Pr_{y \leftarrow g(U_n)}[\mathsf{A}(y) \in g^{-1}(y)] > 1/q(n) \tag{1}$$

for any $n \in \mathcal{I}$.
We would like to use A for inverting f.

Algorithm 8 (The inverter B)

Input: 1^n and $y \in \{0,1\}^*$.

1. Let $x = \mathsf{A}(1^{p(n)}, y, 0^{p(n)-|y|})$.
2. Return $x_{1,\ldots,n}$.

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2. Return $x_{1,\ldots,n}$.

Claim 9

Let $\mathcal{I}' := \{n \in \mathbb{N} \colon p(n) \in \mathcal{I}\}$. Then

1. $\mathcal{I}'$ is infinite
2. For any $n \in \mathcal{I}'$, it holds that $\Pr_{y \leftarrow g(U_n)}[\mathsf{B}(y) \in f^{-1}(y)] > 1/q(p(n))$.

in contradiction to the assumed one-wayness of f. $\square$

Conclusion

Remark 10

- We directly related the hardness of f to that of g
- The reduction is not “security preserving"

From partial domain functions to all-length functions

Construction 11

Given a function $f\colon \{0,1\}^{m(n)} \mapsto \{0,1\}^{\ell(n)}$, $f_{all} : \{0,1\}^* \mapsto \{0,1\}^*$ as

$$f_{all}(x) = f(x_{1,\ldots,k(n)}), 0^{n-k(n)}$$

where $n = |x|$ and $k(n) := \max\{m(n') \leq n\colon n' \in \mathbb{N}\}$.

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Claim 12

Assume that f is a one-way function and that m is monotone, polynomial-time commutable an satisfies $\frac{m(n+1)}{m(n)} \leq p(n)$ for some $p \in$ poly, then f_{all} is a one-way function. Further, if f is length preserving, then so is f_{all}.

Proof: ?

Weak One Way Functions

Definition 13 (weak one-way functions)

A polynomial-time computable function $f : \{0,1\}^* \mapsto f : \{0,1\}^*$ is α-one-way, if

$$\Pr_{y \leftarrow f(U_n)}[\mathsf{A}(1^n, y) \in f^{-1}(y)] \leq \alpha(n)$$

for any PPT A and large enough $n \in \mathbb{N}$.

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2. Examples
3. Can we “amplify" weak OWF to strong ones?

Strong to weak OWFs

Claim 14

Assume there exists OWFs, then there exist functions that are $\frac{2}{3}$-one-way, but not (strong) one-way

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Proof: let f be a OWF. Define $g(x) = (1, f(x))$ if $x_1 = 1$, and 0 otherwise.

Weak to Strong OWFs

Theorem 15

Assume there exists $(1-\alpha)$*-weak OWFs with* $\alpha(n) > 1/p(n)$ *for some* $p \in \text{poly}$*, then there exists (strong) one-way functions.*

Weak to Strong OWFs

Theorem 15

Assume there exists $(1-\alpha)$-weak OWFs with $\alpha(n) > 1/p(n)$ for some $p \in \text{poly}$, then there exists (strong) one-way functions.

Proof: we assume wlg that f is length preserving (can we do so?)

Construction 16 (g – the strong one-way function)

Let $t\colon \mathbb{N} \mapsto \mathbb{N}$ be a polynomial-time computable function satisfying $t(n) \in \omega(\log n/\alpha(n))$. Define $g\colon (\{0,1\}^n)^{t(n)} \mapsto (\{0,1\}^n)^{t(n)}$ as

$$g(x_1, \ldots, x_t) = f(x_1), \ldots, f(x_t)$$

Weak to Strong OWFs

Theorem 15

Assume there exists $(1-\alpha)$-weak OWFs with $\alpha(n) > 1/p(n)$ for some $p \in \text{poly}$, then there exists (strong) one-way functions.

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Claim 17

g is one-way.

Proving that g is one-way – the naive approach

Let A be a potential inverter for g, and assume that A tries to attacks each of the t outputs of g *independently*. Then

$$\Pr_{y \leftarrow g(U_n^{t(n)})}[\mathsf{A}(y) \in g^{-1}(y)] \leq (1-\alpha(n))^{t(n)} \leq e^{-\omega(\log n)} = \mathrm{neg}(n)$$

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A less naive approach would be to assume that A goes over output sequentially.
Unfortunately, we can assume none of the above.

Failing Sets

Failing Sets

Definition 18 (failing set)

A function $f : \{0,1\}^n \mapsto \{0,1\}^{\ell(n)}$ has a $(\delta(n), \varepsilon(n))$-failing set for A, if for large enough n, exists set $\mathcal{S}(n) \subseteq \{0,1\}^{\ell(n)}$ with

1. $\Pr[f(U_n) \in \mathcal{S}(n)] \geq \delta(n)$, and
2. $\Pr[\mathrm{A}(y) \in f^{-1}(y)] < \varepsilon(n)$, for every $y \in \mathcal{S}(n)$

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Claim 19

Let f be a $(1-\alpha)$-OWF. Then f has $(\alpha(n)/2, 1/p(n))$-failing set for any PPT A and $p \in$ poly.

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Let f be a $(1-\alpha)$-OWF. Then f has $(\alpha(n)/2, 1/p(n))$-failing set for any PPT A and $p \in$ poly.

Proof: Assume $\exists$ PPT A, a $p \in$ poly and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that for every $n \in \mathcal{I}$, $\exists \mathcal{L}(n) \subseteq \{0,1\}^n$ with

1. $\Pr[f(U_n) \in \mathcal{L}(n)] \geq 1 - \alpha(n)/2$, and
2. $\Pr[\mathsf{A}(y) \in f^{-1}(y)] \geq 1/p(n)$, for every $y \in \mathcal{L}(n)$

We'll use A to contradict the hardness of f.

Using A to invert f

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Algorithm 20 (The inverter B)

Input: $y \in \{0,1\}^n$.
Do (with fresh randomness) for $np(n)$ times:
If $x = \mathsf{A}(y) \in f^{-1}(y)$, return x

Clearly, B is a PPT

Using A to invert f

Algorithm 20 (The inverter B)

Input: $y \in \{0,1\}^n$.
Do (with fresh randomness) for $np(n)$ times:
If $x = \mathrm{A}(y) \in f^{-1}(y)$, return x

Clearly, B is a PPT

Claim 21

For every $n \in \mathcal{I}$, it holds that
$\Pr_{y \leftarrow f(U_n)}[\mathrm{B}(y) \in f^{-1}(y)] > 1 - \alpha(n)$

Hence, f is not $(1 - \alpha(n))$-one-way $\square$

Proof of Claim 21(all probabilities below are also over $y \leftarrow f(U_n)$):

$$\Pr[\mathrm{B}(y) \in f^{-1}(y)]$$

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&\quad \geq (1 - \alpha(n)/2) \cdot (1 - (1 - 1/p(n))^{np(n)})
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&\quad \geq (1 - \alpha(n)/2) \cdot (1 - (1 - 1/p(n))^{np(n)}) \\
&\quad \geq (1 - \alpha(n)/2) \cdot (1 - 2^{-n}) > 1 - \alpha(n). \square
\end{aligned}
$$

Proving that g is one-way

We show that if g is not OWF, then f has no flailing-set of the “right" type.

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Claim 22

Assume $\exists$ PPT A, $p \in$ poly and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ s.t.

$$\Pr_{z \leftarrow g(U_n^{t(n)})}[\mathsf{A}(z) \in g^{-1}(z)] \geq 1/p(n) \tag{2}$$

for every $n \in \mathcal{I}$. Then $\exists$ PPT B and $q \in$ poly s.t.

$$\Pr_{y \leftarrow \mathcal{S}}[\mathsf{B}(y) \in f^{-1}(y)] \geq 1/q(n) \tag{3}$$

for every $n \in \mathcal{I}$ and $\mathcal{S} \subseteq \{0,1\}^n$ with $\Pr_{y \leftarrow f(U_n)}[\mathcal{S}] \geq \alpha(n)/2$.

Namely, f does not have a $(\alpha(n)/2, 1/q(n))$-failing set.

Algorithm B

Algorithm 23 (No failing-set algorithm B)

Input: $y \in \{0,1\}^n$.

1. Choose $z = (z_1, \ldots, z_t) \leftarrow g(U_n^t)$ and $i \leftarrow [t]$
2. Set $z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t)$
3. Return $\mathsf{A}(z')_i$

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3. Return $\mathsf{A}(z')_i$

Fix $n \in \mathcal{I}$ and a set $\mathcal{S} \subseteq \{0,1\}^n$ of the right probability. We analyze B's success probability using the following (inefficient) algorithm B^*:

Algorithm B*

Definition 24 (Bad)

For $z \in Im(g)$ (the image of g), we set $Bad(z) = 1$ iff $\nexists i \in [t]$ with $z_i \in \mathcal{S}$.

B* differ from B in the way it chooses z': in case Bad$(z) = 1$, it sets $z' = z$. Otherwise, it sets i to an arbitrary index $j \in [t]$ with $z_j \in \mathcal{S}$, and sets z' as B does with respect to this i.

Algorithm B^*

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For $z \in Im(g)$ (the image of g), we set $Bad(z) = 1$ iff $\nexists i \in [t]$ with $z_i \in \mathcal{S}$.

B^* differ from B in the way it chooses z': in case $\text{Bad}(z) = 1$, it sets $z' = z$. Otherwise, it sets i to an arbitrary index $j \in [t]$ with $z_j \in \mathcal{S}$, and sets z' as B does with respect to this i.

Claim 25

$\Pr_{y \leftarrow \mathcal{S}}[B^*(y) \in f^{-1}(y)] \geq \frac{1}{p(n)} - \text{neg}(n)$,

and therefore $\Pr_{y \leftarrow \mathcal{S}}[B(y) \in f^{-1}(y)] \geq \frac{1}{t(n)p(n)} - \text{neg}(n)$.$\square$

Claim 25 follows from the following two claims,

Claim 26

$\Pr_{z \leftarrow g(U_n^t)}[\mathrm{Bad}(z)] = \mathrm{neg}(n)$

Claim 27

Let $Z = g(U_n^t)$ and let Z' be the value of z' induced by a random execution of B^* on $y \leftarrow (f(U_n) \mid f(U_n) \in \mathcal{S})$. Then Z and Z' are identically distributed.

The claims imply Claim 25.

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$$\Pr_{y \leftarrow \mathcal{S}}[\mathsf{B}^*(y) \in f^{-1}(y)] \geq \Pr_{z \leftarrow g(U_n^t)}[\mathsf{A}(z) \in g^{-1}(z) \wedge \neg \mathsf{Bad}(z)] \tag{4}$$

The claims imply Claim 25.

$$\Pr_{y \leftarrow \mathcal{S}}[\mathrm{B}^*(y) \in f^{-1}(y)] \geq \Pr_{z \leftarrow g(U_n^t)}[\mathrm{A}(z) \in g^{-1}(z) \wedge \neg \mathrm{Bad}(z)] \tag{4}$$

$$\begin{aligned} &\Pr_{z \leftarrow g(U_n^t)}[\mathrm{A}(z) \in g^{-1}(z)] \\ &\leq \Pr[\mathrm{A}(z) \in g^{-1}(Z) \wedge \neg \mathrm{Bad}(z)] + \Pr[\mathrm{Bad}(z)] \end{aligned} \tag{5}$$

The claims imply Claim 25.

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$$\begin{aligned} &\Pr_{z \leftarrow g(U_n^t)}[\mathsf{A}(z) \in g^{-1}(z)] \\ &\leq \Pr[\mathsf{A}(z) \in g^{-1}(Z) \wedge \neg \mathsf{Bad}(z)] + \Pr[\mathsf{Bad}(z)] \end{aligned} \tag{5}$$

It follows that

$$\begin{aligned} \Pr_{y \leftarrow \mathcal{S}}[\mathsf{B}^*(y) \in f^{-1}(y)] &\geq \Pr_{z \leftarrow g(U_n^t)}[\mathsf{A}(z) \in g^{-1}(z)] - \mathsf{neg}(n) \\ &\geq \frac{1}{p(n)} - \mathsf{neg}(n). \square \end{aligned}$$

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Proof of Claim 27: Consider the following process for sampling Z_i:

1. Let $\beta = \Pr_{y \leftarrow f(U_n)}[\mathcal{S}]$. Set $\ell_i = 1$ wp β and $\ell_i = 0$ otherwise.
2. If $\ell_i = 1$, let $y \leftarrow (f(U_n) \mid f(U_n) \in \mathcal{S})$. Otherwise, set $y \leftarrow (f(U_n) \mid f(U_n) \notin \mathcal{S})$.

It is easy to see that the above process is correct (samples Z correctly).

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It is easy to see that the above process is correct (samples Z correctly).
Now all that B* does, is repeating Step 2 for one of the i's with $\ell_i = 1$ (if such exists) □

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  What properties of the weak OWF have we used in the proof?