The Effect of Service Time Variability on Job Scheduling Fairness

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Abstract

Fairness is an inherent and fundamental factor of queue service disciplines in a large variety of queueing applications, ranging from computer systems, communications systems and call centers to airport and supermarket waiting lines. Service time variability across jobs is a major factor affecting both system performance and scheduling rules (for example, computer systems prioritize short jobs over long jobs). Service time variability and its effects on mean response times have been studied extensively. However, its effect on queue fairness has not been researched. This work studies the effect of service time variability on queue fairness. We use the RAQFM queue fairness measure, whose analysis for the case of the M/M/1 queue was provided in Raz et al. (2004b), and aim at studying it under a wide variety of service time distributions (rather than exponential only) with a large range of service time variability. For the LCFS-PR scheduling we provide a full analysis of the M/G/1 system. We find that for this system the fairness (when expressed as second moment of discrimination) depends on the first two moments of the service time and only on them. For other service disciplines (FCFS, LCFS-NPR, ROS-NPR, ROS-PR) we propose to use the common approach of mapping an arbitrary service time distribution into a Coxian distribution (via moment mapping) and to use a Markovian-type fairness analysis of RAQFM for deriving the fairness level of the single server system with Poisson arrivals. The analysis reveals that queue fairness is sensitive to service time variability and that the fairness ranking of common scheduling policies (e.g. FCFS, LCFS, ROS) depends on

this parameter.

1 Introduction

Queueing systems have been used in a wide variety of applications such as computer systems, call centers, Web services and communications networks, as well as waiting lines in airports, banks, public offices and others. Queueing Theory has been used for nearly a century to study the performance of such systems and how to operate them efficiently.

Service times and their distributions play an important role in affecting the performance of queueing systems, and the scheduling policies used. One can mention the Pollaczek-Khinchin formula (see queueing theory text books, e.g. Kleinrock (1975), Cooper (1981)) where the for the M/G/1 system the average delay is proportional to the second moment of the service time. Accounting for service times in scheduling policies has been Widely studied, mainly in the context of optimizing mean system delay or mean delay cost. A well known result in this context is the so-called μc rule. According to this rule in a single server system with K classes, if class j incurs cost c_j per time unit of wait and its service time has mean $1/\mu_j$, then the cost minimizing rule is to pick for service the customer with the largest value of $\mu_j c_j$. In other words, the larger the value of $\mu_j c_j$ the higher the priority of this customer (or its class) (see Cox and Smith (1961) Chapter 3.3 and Kleinrock (1976), Lippman (1975)).

Fairness has been recognized as a highly important performance aspect in queues. This recognition can be found in past studies such as Larson (1987) Rothkopf and Rech (1987) Palm (1953) Mann (1969) and Whitt (1984). Recent experimental studies of the reaction of humans to various queue situations (Rafaeli et al. (2002) and Rafaeli et al. (2003)) have shown that fairness in the queue is very important to humans, perhaps some times even more than the wait itself. In practice, fairness aspects seem to affect scheduling policies, in some cases not less than the wish to minimize mean waiting time or weighted mean wafting time). However, fairness considerations have rarely been expressed quantitatively, simply since queue fairness quantification was not available until quite recently.

The interest in computer job scheduling and in their fairness has recently raised interest in *quantitatively evaluating queue scheduling fairness*. Work in this area has been done in Avi-Itzhak and Levy (2004), Bender et al. (1998), Bansal and Harchol-Balter (2001), Wierman and Harchol-Balter (2003), Raz et al. (2004b), Raz et al. (2004a) and Avi-Itzhak et al. (2004).

Our goal in this work is studying the quantitative effect of service variability on the fairness to jobs in the system and on various common scheduling policies. At the intuitive level one can easily recognize that high variability may adversely affect queue fairness in a significant way, since it implies the processing of very long jobs and very short jobs at the same system. This brings up performance questions as well as operational questions, such as: 1) To what degree (quantitatively) service time variability affects job scheduling fairness, 2) How fair are common scheduling disciplines, as a function of the job size variability, and 3) Which scheduling disciplines achieve higher job fairness (as a function of the job variability). Since job fairness is one of the major concerns in choosing a scheduling disciplines, answers to these questions should be useful to system designers and operators.

To quantify queue fairness we must first select a measure (yarstick) of queue fairness. To this end, three different approaches have been proposed recently:

1. In Avi-Itzhak and Levy (2004) measures based on *order of service* have been devised.

2. The *slowdown* (a.k.a. stretch, normalized response time) was proposed as a metric of unfairness in several works. In Bender et al. (1998) the *max slowdown* is used as indication of unfairness. In Bansal and Harchol-Balter (2001) the *max mean slowdown* is used to evaluate the unfairness of the SRPT scheduling policy. In Wierman and Harchol-Balter (2003), the max mean slowdown is used as a criterion for evaluating whether a system is fair or unfair.

3. In Raz et al. (2004b) an analysis of the *resources allocated* by the system to the various customers forms the base for a fairness measure named Resource Allocation Queueing Fairness Measure (RAQFM).

As discussed in Avi-Itzhak et al. (2004) the first approach focuses on the *relative arrival times* of customers while the second approach focuses on their *relative service times*; as such both approaches have difficulties accounting for the *tradeoff* between relative seniority (the time spent in the system since arrival) and service requirement. The reader may recognize this tradeoff and its importance from his/her daily life experience where a very short job arrives to the queue just shortly after a very long job (e.g in a supermarket), bringing up the common dilemma of whom it is more fair to serve first.

The third approach (Raz et al. (2004b,a)) focuses on the *resources* of the system and their allocation, and thus allows to deal with the tradeoff between service requirement and seniority. The results derived in there show that the measure is indeed sensitive to both factors and reacts properly (intuitively) in a variety of cases of interest. We will therefore adopt the RAQFM measure as our fairness evaluation metrics in this study [1]

Having selected RAQFM, our first goal is to have an analysis of the fairness of common service disciplines, under the M/G/1 model. This is required in order to examine systems under various conditions of service variabilities. To this end, note that the work conducted on evaluating queue fairness using the RAQFM metric (Raz et al. (2004b,a)) has focused on *exponential service times (M/M/1 model)* and therefore, that work needs to be extended to deal with *general service time distribution.*

We start by presenting the model and reviewing the RAQFM measure (Section 2). We then (Section 3) turn to the analysis of the Processor Sharing (PS) and LCFS-Preemptive disciplines. First (Section 3.1), we recall from the literature that the unfairness of PS is 0 in all single server systems, including the M/G/1 model (thus it is the most fair policy), and regardless of service variability. Second (Section 3.2), we start our study by providing an exact analysis of queue unfairness (expressed as the second moment of discrimination) in the Last-Come-First-Served Preemptive-Resume (LCFS-PR) M/G/1 system. The analysis leads to a simple numerical recursion for evaluating the individual discriminations as well as the system's unfairness in this system. The results derived imply that system unfairness directly depends on the first two moments of the service times. That is, service variability is a major factor affecting queue fairness. Further, the results imply that the system unfairness *does not depend* on the third and higher moments of the service time.

We next (Section 4) turn to analyze the First-Come-First-Served (FCFS), LCFS non-preemptive (LCFS-NPR), Random-Order-of-Service Non-preemptive (ROS-NPR) and ROS-PR. We realize that the analysis of RAQFM for the M/G/1 model might be quite challenging. The performance measure of fairness (at least as used in RAQFM) is inherently more involved (mathematically) than the performance measure of waiting times. This is so since the latter involves the measures of individual jobs while the former involves a

[1]The reader may question whether the fairness measures developed in the analysis of Weighted Fair Queueing, like Absolute Fairness Bound and Relative Fairness Bound (see, e.g. Greenberg and Madras (1992), Keshav (1997), ch. 9 pp. 209-261, Golestani (1994), Zhou and Sethu (2002)) should be considered. Those measures seem to fit well streams of packets and less so individual jobs, on which our focus is in this work.

comparative measuring between different jobs.

To overcome this difficulty we turn to the common strategy of mapping a general service time distribution into a Coxian distribution, by matching the moments of the distributions, and analyzing the Markovian model of with the Coxian service time distribution. This is done in Section 4 where we first discuss the mapping procedure and then analyze the corresponding Markovian models. The analysis is carried out via a set of recursive equations, which can be solved numerically to yield the individual job discrimination as well as system unfairness.

To provide some insight into the behavior of the non-preemptive policies we provide in Section 5 an approximate analysis of discrimination in these systems, leading to some closed form approximate expressions. That analysis demonstrates that in non-preemptive systems, in the presence of highly variable service times, the positive discrimination experienced by the long jobs is the dominant factor in the system unfairness.

Lastly (Section 6.1) we turn to conduct a numerical evaluation of the models, examining their fairness sensitivity to service time variability. The major findings are:

1. Service variability significantly affects the fairness of scheduling policies, including their relative (fairness) ranking.

2. At high service time variability: At most load conditions the non-preemptive policies are the most unfair. At high load LCFS-PR is the most unfair. ROS-PR seems to be the most fair almost at all loads.

3. At low service time variability: At most ranges the policies maintain order of fairness: FCFS-NPR > ROS-NPR > ROS-PR > LCFS-NPR > LCFS-PR. At very high loads LCFS-NPR seems to become the most unfair.

2 Model, Notation and Review of RAQFM in a Single Server System

2.1 Model and Notation

Consider a single server queueing system. The system is subject to a stream of arriving customers, $C_1, C_2, \ldots$, arriving at this order. Let a_i and e_i denote the arrival and exit (departure) epochs of C_i respectively. Let S_i be a random variable denoting the service requirement (measured in time units) of C_i, where $S_1, S_2, ...$ are i.i.d as S.Let $s^{(1)} = E[S]$,

$s^{(2)} = E[S^2]$ $\sigma_S^2 = E[S^2] - (E[S])^2$ and $\gamma_S = \sigma_S/E[S]$, where γ_S is called the coefficient of variation. A specific series of values $\{a_i\}_{i=1}^L$ is called an *arrival pattern*. A specific series of values $\{a_i, s_i\}_{i=1}^L$ is called an *arrival and service pattern*.

At each epoch t the server grants service at rate $x_i(t) \geq 0$ to C_i. Let $N(t)$ denote the number of customers in the system at epoch t. The system is work-conserving, i.e. $\int_{a_i}^{e_i} x_i(t)dt = s_i$. The server has a service rate of one unit and is non-idling, i.e. $\forall t, N(t) > 0 \Rightarrow \sum_i x_i(t) = 1$.

2.2 Individual Customer Discrimination

The fundamental principle underlying RAQFM is the belief that at every epoch t, all customers present in the system deserve an equal share of the system's resources. This principle implies that the share of the server's resources a customer deserves at t is simply given by $1/N(t)$. We call this quantity the *momentary warranted service* of C_i at epoch t. Summing this for C_i yields $R_i \stackrel{def}{=} \int_{a_i}^{e_i} dt/N(t)$, the *warranted service* of C_i. The *(overall) discrimination* of C_i, denoted D_i is the difference between the warranted service and the granted service. Since the granted service is $S_i = \int_{a_i}^{e_i} x_i(t)dt$, then

$$D_i = S_i - R_i = S_i - \int_{a_i}^{e_i} dt/N(t). \tag{1}$$

A positive (negative) value of D_i means that a customer receives better (worse) treatment than it fairly deserves, and therefore it is *positively (negatively) discriminated*.

Since D_i consists of the difference between S_i and R_i, we may view S_i as the "positive discrimination" and denote it by $D_i^+ = S_i$, and R_i as the "negative discrimination" and denote it $D_i^- = -R_i$. Similarly define D^+ and D^- to be the steady state limiting values of D_i^+ and D_i^- respectively.

An alternative way to define D_i is to define the *momentary discrimination* of C_i at epoch t as

$$\delta_i(t) \stackrel{def}{=} x_i(t) - 1/N(t), \tag{2}$$

and then the overall discrimination of C_i is:

$$D_i = \int_{a_i}^{e_i} \delta_i(t)dt. \tag{3}$$

An important property of this measure is that it obeys, for every non-idling work-conserving system, and for every t: $\sum_i \delta_i(t) = 0$, that is, every positive discrimination is balanced by negative discrimination. This results from the fact that when the system is non-empty $\sum_i x_i(t) = 1$ (due to non-idling) and the overall momentarily warranted service at such epoch is 1 as well. An important outcome of this property is that if D is a random variable denoting the discrimination of an arbitrary customer when the system is in steady state, then $E[D] = 0$, namely the expected discrimination is zero. The proof is given in Raz et al. (2004a).

2.3 System Measure of Unfairness

To measure the unfairness of a system, using a particular policy, across all customers, that is, to measure the *system unfairness*, one would choose some summary statistics measure over the values D_i, or a function of the distribution of D, where D is a random variable denoting the discrimination of an arbitrary customer when the system is in steady state.

Since fairness inherently deals with differences in treatment of customers, a natural choice is the variance of customer discrimination. Since $E[D] = 0$, this equals the second moment and we denote this measure F_{D^2}. Other optional measures are the mean of distances $E[|D|]$ (denoted $F_{|D|}$) and the mean negative discrimination $-E[D|D < 0]$ (denoted $F_{D<0}$). Throughout this paper, the term "unfairness" refers to F_{D^2} since the paper focuses on this measure. In some instances we also mention $F_{|D|}$. $F_{D<0}$ seems to be less tractable and is not dealt with in this paper.

3 The Fairness of the PS and the LCFS-PR Scheduling Policies

3.1 The Fairness of the PS System

The Processor Sharing discipline was analyzed in Raz et al. (2004b) under general assumption of arrival and service times. Under such general processes, which include the M/G/1 case, it was shown that the discrimination of an individual job is identically 0 and so is the unfairness (expressed either as the second moment of discrimination or as the expected absolute value of discrimination). As such it is the "utmost fair" policy. The reader may verify these results by applying the definitions of discrimination and unfairness given in Section 2 to the PS discipline.

3.2 Analysis of Fairness in the LCFS-PR System

In this section we analyze the fairness and discriminations experienced in the Preemptive LCFS system. Consider a tagged customer C^* arriving at the LCFS-PR system. Let $k \geq 0$ be the number of customers it finds upon arrival. C^* enters service immediately, and these k customers will remain in the system until C^* leaves. Recall that S denotes a random variable representing the service time of C^*, with moments $s^{(1)}$ and $s^{(2)}$.

While C^* is served, customers arrive at the system at rate λ. Once such a customer arrives, it preempts C^*, starting a sub-busy-period, at the end of which (after all customers arriving during the busy period depart, including the preempting one) the service of C^* resumes. Let N be a random variable denoting the number of arrivals during S; this is exactly the number of times C^* will be preempted and a sub-busy-period will start. Since the arrival process is Poisson, we have:

$$E[N] = \lambda E[S]; \quad E[N^2] = \lambda^2 E[S^2] + \lambda E[S]. \tag{4}$$

Let $D|k$ be a random variable denoting the discrimination experienced by C^* conditioned on the number of customers (k) it finds in the system upon arrival. Let $D^+|k$ and $D^-|k$ be the conditional positive discrimination (granted service) and negative discrimination (warranted service). Let $D^{SE}|k$ and $D^Q|k$ be the conditional discriminations experienced by C^* while in service and while in the queue, respectively. Assuming that the value of S is s, we have:

$$D^{SE}|k = (1 - 1/k)s, \tag{5}$$

$$D^Q|k = \tilde{D}_1|k + \tilde{D}_2|k + ... + \tilde{D}_N|k \tag{6}$$

where $\tilde{D}_i|k$ is a random variable denoting the total discrimination experienced by C^* at the ith sub-busy period. It is important to note that, while N depends on S, the variables $\tilde{D}_i|k$ are i.i.d as $\tilde{D}|k$ (which denotes the discrimination experienced by C^* during an *arbitrary* sub-busy period). The following claim establishes a key relation between the variables $D|k$ and $\tilde{D}|k$:

Proposition 3.1. *For $k = 0, 1, ...$ the random variable $\tilde{D}|k$ is identical to the variable $D^-|k + 1$.*

Proof. $\tilde{D}|k$ is the discrimination C^* experiences from the moment a new customer, say C', arrives and until the sub-busy period of C' ends. Since during this sub-busy period both

C^* and C' are in the system their negative discrimination during this period is identical. Further, since C' sees exactly $k+1$ customers upon arrival, its discrimination is distributed as $D|k+1$ and its negative discrimination is distributed as $D^-|k+1$. □

Thus, the first and the second moments of $D|k$ are given by:

$$E[D|k, S = s] = E\left[s(1 - \frac{1}{k+1}) + Y(s,k)\right], \tag{7}$$

$$E[D^2|k, S = s] = E\left[s(1 - \frac{1}{k+1}) + Y(s,k)\right]^2 \tag{8}$$

where $Y(s,k) = \sum_{i=1}^{N(s)} \tilde{D}_i|k$ and $N(s)$ is the number of Poisson arrivals in an interval of length s. Hence

$$E[Y(s,k) = \lambda s E[\tilde{D}|k]; \quad E[(Y(s,k))^2] = \lambda s \sigma_{\tilde{D}|k} + (\lambda^2 s^2 + \lambda s)E[\tilde{D}|k]^2, \tag{9}$$

where $\sigma_{\tilde{D}|k}$ is the variance of $\tilde{D}|k$. Taking the expectations of in Eqs. 7 and 8 yields:

$$E[D|k, S = s] = s\left(1 - \frac{1}{k+1}\right) + \lambda s E[\tilde{D}|k], \tag{10}$$

and

$$E[D^2|k, S = s] = s^2\left(1 - \frac{1}{k+1}\right)^2 + \lambda E[\tilde{D}^2|k] + \lambda^2 s^2 E[\tilde{D}|k]^2 + 2\left(1 - \frac{1}{k+1}\right)\lambda s^2 E[\tilde{D}|k]. \tag{11}$$

Unconditioning on $S = s$ and using Proposition 3.1 we get

$$E[D|k] = s^{(1)}\left[(1 - \frac{1}{k+1}) + \lambda E[D^-|k+1\right] \tag{12}$$

$$E[D^2|k] = s^{(2)}\left(1 - \frac{1}{k+1} + \lambda E[D^-|k+1]\right)^2 + \lambda s^{(1)} E[(D^-)^2|k+1]. \tag{13}$$

Now, to derive the first two moments of $D^-|k$ one can repeat the analysis above only for the negative part of the discrimination, leading to equations that are similar to Equations 12 and 13:

$$E[D^-|k] = s^{(1)}\left[\frac{-1}{k+1} + \lambda E[D^-|k+1\right] \tag{14}$$

$$E[(D^-)^2|k] = E[S^2]\left(\frac{-1}{k+1} + \lambda E[D^-|k+1]\right)^2 + \lambda E[S]E[(D^-)^2|k+1]. \tag{15}$$

Equation 14 can be solved by successive substitution to yield:

$$E[D^-|0] = -s^{(1)}\sum_{i=1}^{\infty}\frac{\rho^{i-1}}{i} = -\frac{s^{(1)}}{\rho}ln(1-\rho). \tag{16}$$

$$E[D^-|k] = -\frac{s^{(1)}}{\rho^{k+1}}ln(1-\rho) + s^{(1)}\sum_{i=1}^{k}\frac{1}{i\rho^{k-i+1}}. \tag{17}$$

Finally, let g_k be the probability that C^* sees k customers upon arrival (which, due to PASTA, equals the steady state probability of having k customers in the system). Unconditioning Equation 13 on k we get:

$$E[D^2] = s^{(2)}\sum_{k=0}^{\infty} g_k\left(1 - \frac{1}{k+1} + \lambda E[D^-|k+1]\right)^2 + s^{(1)}\sum_{k=0}^{\infty}\lambda g_k E[(D^-)^2|k+1]. \tag{18}$$

The LCFS-PR system is a symmetric queue as defined by (Kelly, 1979, Section 3.3). Therefore[2]

$$g_k = (1-\rho)\rho^k, \quad k = 0, 1, 2, ... \tag{19}$$

where $\rho = \lambda E[S]$. Thus we have:

$$E[D^2] = s^{(2)}\sum_{k=0}^{\infty}(1-\rho)\rho^k\left(1 - \frac{1}{k+1} + \lambda E[D^-|k+1]\right)^2 + \rho\sum_{k=0}^{\infty}(1-\rho)\rho^k E[(D^-)^2|k+1]. \tag{20}$$

Equation 20 demonstrates a direct dependency of the second moment of discrimination on the first two moments of service time. Further, we may conclude the following important corollary:

Corollary 3.1. *The unfairness of the M/G/1 system with the LCFS-PR service regime, measured via the RAQFM measure (via the second moment of discrimination) depends on the first two moments of the service time S, and does not depend on higher moments of S.*

[2]Note that this form results also directly from the following simple argument: In the LCFS-PR iff a customer leaves behind him k customers he encounters k customers upon arrival. A custoem leaves behind $k+1$ customers iff he preempts a customer who Encountered k upon arrival. Since $E[N] = \lambda E[S] = \rho$ we have $g_{k+1} = \rho g_k$ implying $g_k = (1-\rho)\rho^k$, where g_k is the probability of encountering k upon arrival.

4 Analysis of Non-Preemptive Policies and ROS-Preemptive under Coxian distribution

In this section we analyze the FCFS, LCFS-NPR, ROS-PR and ROS-NPR policies. The analysis approach used is to take the (general) distribution of the service time, and approximate it by a Coxian distribution. This leads to a Markovian model for which the discrimination and fairness are then derived. For the lack of space we provide only the analysis of LCFS and ROS-PR. The analysis of the other models can be found in the appendix.

4.1 Approximating General Service Time distributions by Coxian Distributions

For the purpose of approximating a general service time distribution we use a second order moment matching technique (Adan and Resing, 2001, Chapter 2.5). In particular, we fit a phase-type distribution, either Coxian or Erlangian, on the mean, $s^{(1)}$, and the coefficient of variation, γ_S, of the given service time random variable S. We distinguish between two cases: (a) When $0 < \gamma_S < 1$ we seek an integer k such that $\frac{1}{k} \approx {\gamma_S}^2$, and fit a k-stage Erlang distribution, E_k, (see Section 4.2) with mean $s^{(1)}$. To match an arbitrary $0 < \gamma_S < 1$ it is possible to use a more sophisticated distribution such as the mixed Erlang distribution which selects between E_k and E_{k-1} distributions with some fixed probability. However, using the basic Erlang distribution leads to simpler recurrence equations and therefore we use it for the analysis. (b) When $\gamma_S > 1$ we use a Coxian-2 distribution (Adan and Resing, 2001, Chapter 2.4) which is composed of two exponential stages with mean lengths μ_i, $i = 1, 2$ where the move from the first stage to the second one is with probability p_1, and with probability $1 - p_1$ the service ends after the first stage. For the approximation we use the following parameters, suggested by Marie (1980):

$$\mu_1 = 2s^{(1)},\ \alpha = \frac{0.5}{\gamma_S},\ \mu_2 = \mu_1 \alpha$$

4.2 Conditional Discrimination in M/E$_r$/1

Consider the M/E$_r$/1 where the service time distribution is Erlang with r exponential stages. For this distribution the service is assumed to be composed of r phases (i.e., stages) arranged serially, where a new customer that enters service proceeds through stages

$1, \ldots, r$ one at a time; upon completion of the last stage the customer departs. The lengths of the stages are i.i.d exponentially with parameter $r\mu$.

In a work conserving non-idling M/E$_r$/1 system the time between the arrival of a customer and its departure is slotted by arrivals and stage completions. Let $T_i, i = 1, 2, \ldots$ be the duration of the i-th slot, then $T_i, i = 1, 2, \ldots$ are i.i.d. random variables exponentially distributed with parameter $\lambda + r\mu$; the first two moments of T_i are $t^{(1)} = \frac{1}{\lambda + r\mu}$ and $t^{(2)} = \frac{2}{(\lambda + r\mu)^2} = 2(t^{(1)})^2$. The probabilities that a slot ends with an arrival or with a stage completion are denoted by $\tilde{\lambda}$ and $\tilde{\mu}$ respectively.

$$\tilde{\lambda} = \frac{\lambda}{\lambda + r\mu} = \frac{\rho}{r + \rho} \qquad\qquad \tilde{\mu} = \frac{r\mu}{\lambda + r\mu} = \frac{r}{r + \rho}, \tag{21}$$

where $\rho = \lambda/\mu < 1$.

The system unfairness, given by $E[D^2]$, can be expressed as:

$$E[D^2] = P_0 E[D^2|0, 1] + \sum_{k=1}^{\infty} \sum_{j=1}^{r} P_{k,j} E[D^2|k, j], \tag{22}$$

where $P_{k,j},\ k \geq 1$ is the probability of finding k customers upon arrival and the served customer in stage j, $P_0 = 1 - \rho$ is the probability of finding an empty system, and $E[D^2|k, j]$ is the second moment of D for a customer who arrives to find k customers in the system and the served one in stage j. Note that $P_{k,j}$ can be derived numerically via standard techniques for solving steady state balance equations (see, for example, Kleinrock (1975)).

4.2.1 FCFS

For a tagged customer C residing in the system let a denote the number of customers ahead of C, let b denote the number of customers behind C, and let $j = 1, \ldots, r$ denote the stage of service in which the served customer is currently found. Such a customer is said to be in state $\mathcal{S}_{a,b,j}$. Due to the memoryless properties of the system, the state $\mathcal{S}_{a,b,j}$ captures all that is needed for predicting the future of C. The momentary discrimination at state $\mathcal{S}_{a,b,j}$ is independent of the current service stage. We denote it by $c(a, b)$, where j is omitted.

$$c(a, b) = \begin{cases} -\frac{1}{a+b+1} & a > 0 \\ 1 - \frac{1}{b+1} & a = 0 \end{cases} \cdots \tag{23}$$

Let $D(a,b,j)$ denote the accumulated discrimination of C during a walk starting at state $\mathcal{S}_{a,b,j}$ and ending at the departure of C, and let $d(a,b,j) \overset{def}{=} d^{(1)}(a,b,j)$ and $d^{(2)}(a,b,j)$ be the first and second moments of $D(a,b,j)$. Then

$$E[D|k,j] = \quad E[d(k,0,j)] \tag{24}$$

$$E[D^2|k,j] = \quad E[d^{(2)}(k,0,j)] \tag{25}$$

Assume C is in state $\mathcal{S}_{a,b,j}$. C will encounter one of the two following evens:

1. A new customer arrives into the system. The probability of this event is $\tilde{\lambda}$. Afterward C will move to state $\mathcal{S}_{a,b+1,j}$.
2. A customer completes its current stage. The probability of this event is $\tilde{\mu}$. If C is not being served ($a \neq 0$) it will move to $\mathcal{S}_{a,b,j+1}$ if $j \neq r$ or to $\mathcal{S}_{a-1,b,j}$ if $j = r$. If C is being served ($a = 0$) it will move to $\mathcal{S}_{0,b,j+1}$ if $j \neq r$ or will leave the system if $j = r$.

Thus for customers not in service

$$D(a,b,j) = \begin{cases} Tc(a,b) + D(a,b+1,j) & \text{w.p } \tilde{\lambda} \\ Tc(a,b) + D(a,b,j+1) & \text{w.p } \tilde{\mu},\ j \neq r \\ Tc(a,b) + \Delta(a>0)D(a-1,b,j) & \text{w.p } \tilde{\mu},\ j = r \end{cases} \tag{26}$$

where T is the duration of the current slot, and $\Delta()$ is an indicator function.

Taking expectation leads to the following recursive expression:

$$d(a,b,j) = \begin{cases} t^{(1)}c(a,b) + \tilde{\lambda}d(a,b+1,j) + \tilde{\mu}d(a,b,j+1) & j \neq r \\ t^{(1)}c(a,b) + \tilde{\lambda}d(a,b+1,j) + \Delta(a>0)\tilde{\mu}d(a-1,b,1) & j = r \end{cases} \tag{27}$$

From Eq.(26) for customers not in service

$$D(a,b,j)^2 = \begin{cases} (Tc(a,b) + D(a,b+1,j))^2 & \text{w.p } \tilde{\lambda} \\ (Tc(a,b) + D(a,b,j+1))^2 & \text{w.p } \tilde{\mu},\ j \neq r \\ (Tc(a,b) + \Delta(a>0)D(a-1,b,1))^2 & \text{w.p } \tilde{\mu},\ j = r \end{cases} \tag{28}$$

We now take its expectation, after expanding the quadratic terms, and get that

$$d^{(2)}(a,b,j) = t^{(2)}(c(a,b))^2 + \tilde{\lambda}d^{(2)}(a,b+1,j) +$$

$$2t^{(1)}c(a,b)\tilde{\lambda}d(a,b+1,j)+$$

$$\begin{cases} \tilde{\mu}d^{(2)}(a,b,j+1)+2t^{(1)}c(a,b)\tilde{\mu}d(a,b,j+1)) & j\neq r \\ \Delta(a>0)(\tilde{\mu}d^{(2)}(a-1,b,j)+2t^{(1)}c(a,b)\tilde{\mu}d(a-1,b,j))) & j=r \end{cases} \quad (29)$$

These recursive relations (equations (29), (27)) combined with equations (24), (23), (22), (21) can be used, via numerical computation, to derive the system unfairness measure, $V[D]$.

4.2.2 Preemptive ROS

Consider a preemptive ROS policy in which a preempted customer is immediately selectable for service, i.e., a preempted customer is allowed to immediately reenter service. Let $\overline{a} =< a_1,\ldots,a_r >$ be a vector of length r, where a_i is the number of customers other than C that need to complete $r-i+1$ stages of service. Let $a=\sum_i a_i$ denote the total number of customers in the system other than C. Let $c=1,\ldots,r$ be an integer variable such that $r-c+1$ is the number of stages that C needs to complete. Let s be a boolean variable which is 1 if C is in service and 0 if it is waiting. The state of C is denoted by $\mathcal{S}\overline{a},c,s,j$. In this state a_i,$i=1,\ldots,r$ customers need to complete $r-i+1$ stages, the tagged customer C needs to complete $r-c+1$ stages, the one in service is in its j-th stage, and it is C if $s=1$ and not C if $s=0$.

When C is in state $\mathcal{S}_{\overline{a},c,s,j}$ it will encounter one of the following possible events:

1. If $s=0$ the possible events are:

   (a) A customer arrives into the system and C is chosen to receive service next. The probability of this event is $\frac{\tilde{\lambda}}{a+2}$ and C will move to $\mathcal{S}_{<a_1+1,\ldots,a_r>,c,1,c}$, where $< a_1+1,\ldots,a_r >$ is the updated vector which includes the additional arriving customer that needs to complete r stages.

   (b) A customer arrives into the system and a waiting customer (other than C) which is left with $r-k+1$ stages is chosen to receive service next. The probability of this event is $\tilde{\lambda}\frac{\tilde{a_k}}{a+2}$ where $\tilde{a_k}=\begin{cases} a_k & k=2,\ldots,r \\ a_1+1 & k=1 \end{cases}$. Then C will move to $\mathcal{S}_{<a_1+1,\ldots,a_r>,c,0,k}$.

   (c) A customer completes its current stage j, where $j\neq r$. The probability of this event is $\tilde{\mu}$ and C moves to $S_{<\ldots,a_j-1,a_{j+1}+1,\ldots>,c,0,j+1}$, where $<\ldots,a_j-1,a_{j+1}+$

$1, \ldots >$ is the updated vector with a single decrease in the number of customers left with $r-j+1$ stages and a single increase in the number of customers left with $r-(j+1)+1$ stages.

(d) A customer completes service, leaves the system and C is chosen to receive service next. The probability of this event is $\tilde{\mu}/a$ and C will move to $\mathcal{S}_{<a_1,\ldots,a_r-1>,c,1,c}$, where $< a1, \ldots, a_r - 1 >$ is the updated vector which excludes the departing customer.

(e) A customer completes service, leaves the system and a waiting customer left with $r-k+1$ stages is chosen to receive service next. The probability of this event is $\tilde{\mu}\frac{a_k}{a}$ and C will move to $\mathcal{S}_{<a1,\ldots,a_r-1>,c,0,k}$.

2. If $s = 1$ the possible events are:

   (a) Same as (1a)

   (b) Same as (1b) but C moves to $\mathcal{S}_{<a_1+1,\ldots,a_r>,j,0,k}$

   (c) Same as (1c) but C moves to $S_{\overline{a},c+1,1,j+1}$

   (d) The customer in service completes its service. The probability of this event is $\tilde{\mu}$ and C leaves the system.

To simplify the recursive equations let $\overline{a_1+1} \stackrel{def}{=} < a_1+1, \ldots, a_r >$, $\overline{a_j-1, a_{j+1}+1} \stackrel{def}{=} < \ldots, a_j-1, a_{j+1}+1, \ldots >$, and $\overline{a_r-1} \stackrel{def}{=} < a1, \ldots, a_r - 1 >$.

For $s = 0$, $d(\overline{a}, c, s, j)$ and $d^{(2)}(\overline{a}, c, s, j)$, can be expressed as

$$
\begin{aligned}
d(\overline{a}, c, 0, j) = t^{(1)} c(\overline{a}, s) + \frac{\tilde{\lambda}}{a+2} d(\overline{a_1+1}, c, 1, c) + \\
\sum_{i:a_i>0} \tilde{\lambda} \frac{\tilde{a_i}}{a+2} d(\overline{a_1+1}, c, 0, i) + \\
\begin{cases} \tilde{\mu} d(\overline{a_j-1, a_{j+1}+1}, c, 0, j+1) & j \neq r \\ \frac{\tilde{\mu}}{a} d(\overline{a_r-1}, c, 1, c) + \sum_{i:a_i>0} \tilde{\mu}\frac{a_k}{a} d(\overline{a_r-1}, c, 0, i) & j = r \end{cases}
\end{aligned}
\tag{30}
$$

$$
d^{(2)}(\overline{a}, c, 0, j) = t^{(2)} c(\overline{a}, s)^2 + \frac{\tilde{\lambda}}{a+2} d^{(2)}(\overline{a_1+1}, c, 1, c) +
$$

$$2t^{(1)}c(\overline{a},s)\frac{\tilde{\lambda}}{a+2}d(\overline{a_1+1},c,1,c)+$$

$$\sum_{i:a_i>0}\tilde{\lambda}\frac{\tilde{a}_i}{a+2}d^{(2)}(\overline{a_1+1},c,0,i)+$$

$$2t^{(1)}c(\overline{a},s)\sum_{i:a_i>0}\tilde{\lambda}\frac{\tilde{a}_i}{a+2}d(\overline{a_1+1},c,0,i)+$$

$$\begin{cases}\tilde{\mu}d^{(2)}(\overline{a_j-1,a_{j+1}+1},c,0,j+1)+2t^{(1)}c(\overline{a},s)\tilde{\mu}d(\overline{a_j-1,a_{j+1}+1},c,0,j+1) & j\neq r\\ \frac{\tilde{\mu}}{a}d^{(2)}(\overline{a_r-1},c,1,c)+\sum_{i:a_i>0}\tilde{\mu}\frac{a_k}{a}d^{(2)}(\overline{a_r-1},c,0,i)+ & \\ 2t^{(1)}c(\overline{a},s)(\frac{\tilde{\mu}}{a}d(\overline{a_r-1},c,1,c)+\sum_{i:a_i>0}\tilde{\mu}\frac{a_k}{a}d(\overline{a_r-1},c,0,i)) & j=r\end{cases} \tag{31}$$

For $s=1$ they are given by

$$d(\overline{a},c,1,j)=t^{(1)}c(\overline{a},s)+\frac{\tilde{\lambda}}{a+2}d(\overline{a_1+1},c,1,c)+$$

$$\sum_{i:a_i>0}\tilde{\lambda}\frac{\tilde{a}_i}{a+2}d(\overline{a_1+1},j,0,i)+$$

$$\begin{cases}\tilde{\mu}d(\overline{a},c+1,1,j+1) & j\neq r\\ 0 & j=r\end{cases} \tag{32}$$

$$d^{(2)}(\overline{a},c,1,j)=t^{(2)}c(\overline{a},s)^2+\frac{\tilde{\lambda}}{a+2}d^{(2)}(\overline{a_1+1},c,1,c)+$$

$$2t^{(1)}c(\overline{a},s)\frac{\tilde{\lambda}}{a+2}d(\overline{a_1+1},c,1,c)+$$

$$\sum_{i:a_i>0}\tilde{\lambda}\frac{\tilde{a}_i}{a+2}d^{(2)}(\overline{a_1+1},j,0,i)+$$

$$2t^{(1)}c(\overline{a},s)\sum_{i:a_i>0}\tilde{\lambda}\frac{\tilde{a}_i}{a+2}d(\overline{a_1+1},j,0,i)+$$

$$\begin{cases}\tilde{\mu}d^{(2)}(\overline{a},c+1,1,j+1)+2t^{(1)}c(\overline{a},s)\tilde{\mu}d(\overline{a},c+1,1,j+1) & j\neq r\\ 0 & j=r\end{cases} \tag{33}$$

A customer arrives to the system either at state $\mathcal{S}_{\overline{0},1,1,1}$ when it is empty, where $\overline{0}$ is a zero vector of length r; or at state $\mathcal{S}_{\overline{a},1,0,j}$ when it serving customer at stage j and the number customers left with service stages is represented by $\overline{a}$. Then, for preemptive ROS

$$E[D^2] = P_0 d^{(2)}(\overline{0}, 1, 1, 1) + \sum_{k=1}^{\infty} \sum_{\overline{a}: a=k} \sum_{j=1}^{r} P_{\overline{a},j} d^{(2)}(\overline{a}, 1, 0, j] \tag{34}$$

where $P_{\overline{a},j}$, is the probability of finding a customers upon arrival such that the number of customers left with $r-i+1$ stages of service is a_i, and the served customer is in stage j.

4.2.3 Non-Preemptive LCFS

Here we give a short description of the state variable, $\mathcal{S}_{a,b,j}$, used to construct the recursive discrimination equations for this model. The full analysis is presented in Section A.1.1. Our approach is to preserve the notations of the FCFS model. At every slot let a denote the number of customers arrived earlier than C and thus to be served after C, and let b denote the number of customers arrived later than C and thus to be served before C. The state $S_{a,b,j}$ (where j is the stage of the customer in service) captures all that is needed for predicting the future of C.

4.2.4 Non Preemptive ROS

Similarly to the previous section we provide only a brief description of the state variable. Further detailed can be found in Section A.1.2. For a tagged customer C, denote by a the number of customers in the system other than C, and by s a boolean variable which is 1 if C is in service and 0 if it is waiting. Then, in state $\mathcal{S}a, s, j$ there are a customers in addition to C, the one in service is in its j-th stage, and it is C if $s = 1$ and not C if $s = 0$.

4.3 Conditional Discrimination in M/Cox$_2$/1

For the analysis of M/Cox$_2$/1 model we preserve the notations of the the M/E$_r$/1 model and use the same state variables; this leads to simpler recursive equations than the M/E$_r$/1 equations. The full description of the M/Cox$_2$/1 model and its analysis is given in Section B of the appendix.

4.4 Complexity perspective

Observe that the complexity of the discrimination computation (i.e., the recursive equations complexity) is dependent upon the number of stages used to approximate the service time distribution. This implies that M/Cox$_2$/1 equations have lower complexity than M/E$_r$/1 (r states versus 2). Nonetheless, for the M/E$_r$/1 one may avoid high complexity analysis due to the convergence of the discrimination equations (see Section 6.1). Thus, in practice, the computational complexity is based only on a small number of stages.

5 Analysis of Fairness and its Properties

In this section we are interested in understanding the behavior of the discrimination function in the presence of high variability service times. We do this by focusing on non-preemptive systems and studying the discrimination during the service of long customers. To this end, it will be convenient to break the discrimination of C_i to several components. Let D_i^{SE} and D_i^{Q} be the discriminations experienced by C_i while in service and while (waiting) in queue, respectively. We may further break D_i^{SE} to the positive discrimination observed in service, denoted and obeying $D_i^{SE+} = S_i$, and to the corresponding negative discrimination D_i^{SE-} obeying $D_i^{SE-} + D_i^{Q} = -R_i$ (which is the overall warranted service). Recall also the notations $D_i^{+} = S_i$ and $D_i^{-} = -R_i$ (see Section 2.2). For the corresponding steady state variables we use the same notation where the index i is omitted.

5.1 Expected Positive and Negative Discriminations

Recalling that under RAQFM the expected discrimination obeys $E[D] = 0$, it immediately follows that:

Observation 5.1. *Under RAQFM, for any single server system and any work conserving policy, the expected values of the positive discrimination and of the negative discrimination are equal to each other:*

$$E[D^{+}] = -E[D^{-}]. \tag{35}$$

5.2 Non Preemptive systems: The effect of a Customer with Long Service on the Discrimination of Other Customers

Consider a tagged customer, C^{*}, who resides in the system, and who encounters, during her waiting time, the service of a very long job (denote the customer with the long job by C_L).

To achieve some insight, we use a simplistic model and assume that customers are of two types whose service time are exponentially distributed with means $1/\mu_1$ and $1/\mu_2$ where the second type corresponds to the very large jobs and thus $\mu_2 << \mu_1$. Then the service time of C_L is exponentially distributed with mean $1/\mu_2$. The arrival rate (Poisson) into the system is assumed to be λ

Note that λ/μ_2 is not necessarily smaller than 1 (for stability). In fact, we are interested in cases where $\lambda/\mu_2 >> 1$. We also assume that service is non-preemptive, thus, once the service of the long job started it will be carried out to completion.

Let K be the number of customers present at the system when the service of C_L starts, or when C^* arrives, whichever is later; obviously $K \geq 2$ since both C_L and C^* reside in the system. Let $\bar{t}$ be the expected duration until the next event (service completion of C_L or arrival of a new customer), then $\bar{t} = 1/(\lambda + \mu_2)$. Let $D^-_{(k)}$ be the negative discrimination experienced by C^* during the service of C_L given that $K = k$ and let $d^{(1)-}_{(k)} = E[D^-_{(k)}]$ and $d^{(2)-}_{(k)} = E[(D^-_{(k)})^2]$.

Proposition 5.1. $d^{(1)-}_{(k)}$ *, $k = 2, , 3, ...$ is monotonically decreasing, i.e.* $d^{(1)-}_{(k+1)} < d^{(1)-}_{(k)}$.

Proof. The proof is carried out by examining two systems, one that starts with k customers and one that starts with $k + 1$ customers. If the systems are subject to exactly the same arrival and departure processes (until the departure of C^*), then at every epoch of arrival or departure the first system will have one less customer. Since the temporal negative discrimination at t is given by $1/N(t)$ it follows that the discrimination in the first system is larger than that in the second system for *every sample path.* This directly implies the monotonicity of the expected values as stated in the proposition. □

To bound the value of $d^{(1)-}_{(k)}$, it is now sufficient to bound the value of $d^{(1)-}_{(2)}$, which we do next. We look at the events occurring while C_L is served, these can be either an arrival or the service completion of C_L. Since both occur at exponential rates the expected duration until the next event is given by $\bar{t} = 1/(\lambda + \mu_2)$, the probability that the next event is an arrival is given by $p = \lambda/(\lambda + \mu_2)$ and the probability that the next event is a service completion is given by $1 - p$. Also, the negative momentary discrimination at the first interval (time until first event) is -1/2, at the second interval is given by -1/3 and so on. Thus, the over all expected negative discrimination accumulated by C^* during the service of C_L is given by:

$$d_{(2)}^{(1)-} = -\bar{t}\sum_{i=0}^{\infty}\frac{1}{2+i}p^i. \tag{36}$$

To obtain a closed form expression we rewrite this expression as:

$$d_{(2)}^{(1)-} = -\frac{\bar{t}}{p^2}\int dp \sum_{i=2}^{\infty}\frac{d}{dp}\frac{p^i}{i} = \frac{t}{p^2}\int\frac{p}{1-p}dp, \tag{37}$$

which yields

$$d_{(2)}^{(1)-} = -\frac{\bar{t}}{p^2}[-p - ln(1-p)]. \tag{38}$$

For relatively large values of $1/\mu_2$ where p is close to 1, we have:

$$d_{(2)}^{(1)-} \approx -\frac{1}{\lambda+\mu_2}ln(\frac{1}{1-p}) \approx -\frac{1}{\lambda}ln(\frac{1}{1-p}), \tag{39}$$

$$d_{(2)}^{(1)-} \approx \frac{-1}{\lambda}ln(\frac{1}{\mu_2}). \tag{40}$$

5.3 Non Preemptive systems: The effect of Long Services on the Discrimination of the served Customer

We now repeat the analysis performed in the previous section, but now we focus on the discrimination experienced by C_L while being served. Let K be the number of customers present at the system when the service of C_L starts. Assume that $K \geq 2$ (that is C_L is not alone). Let $D_{(k)}$ be the negative discrimination experienced by C_L during its service given that $K = k$ and let $d_{(k)}^{(1)} = E[D_{(k)}]$. Similarly to the previous section one can show that $d_{(k)}^{(1)}$ is monotonically non decreasing in k. Also, similarly to Equation 36 we get:

$$d_{(2)}^{(1)} = \bar{t}\sum_{i=0}^{\infty}\left(1-\frac{1}{2+i}\right)p^i = \bar{t}\left(\frac{1}{1-p} - \sum_{i=0}^{\infty}\frac{1}{2+i}p^i\right). \tag{41}$$

Following the analysis of the previous section we get, when p approaches 1:

$$d_{(2)}^{(1)} \approx \frac{1}{\lambda+\mu_2}\left(\frac{1}{1-p} - ln(\frac{1}{1-p})\right), \tag{42}$$

$$d_{(2)}^{(1)} \approx \frac{1}{\lambda}\left(\frac{1}{\mu_2} - ln(\frac{1}{\mu_2})\right). \tag{43}$$

Observation 5.2. *Comparison of Equations 43 and 40 shows that while the discrimination of the served customer (C_L) is proportional to $\frac{1}{\mu_2}$, which is the expected duration of the service time, the negative discrimination of a waiting customer (C^*) is proportional only to the log of $\frac{1}{\mu_2}$.*

6 Numerical results and Observations

6.1 Numerical Results

In this section we numerically evaluate the systems studied, aiming at examining their unfairness as function of the system load, service time variability and Scheduling policy. We conduct an evaluation for all the scheduling policies studied under a wide range of service time variability. For low service time variability we use the M/E_r/1 model and for the high variability the M/COX_2/1 model. The results for the non-preemptive systems are calculated numerically and verified via a simulation, while the results of the preemptive policies are derived via simulation.

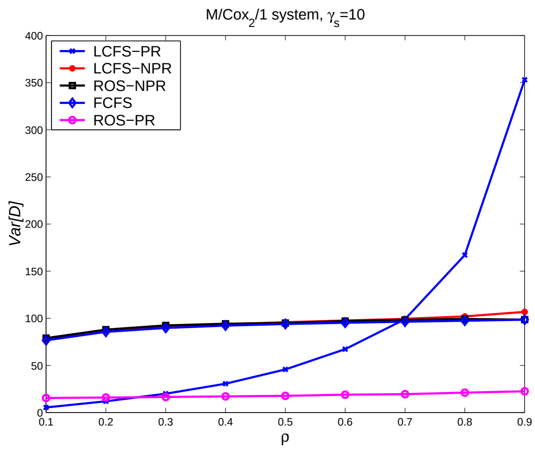

Figure 1: Unfairness for high variability system, M/COX_2/1 (Coefficient of variation $\gamma_S = 10$)

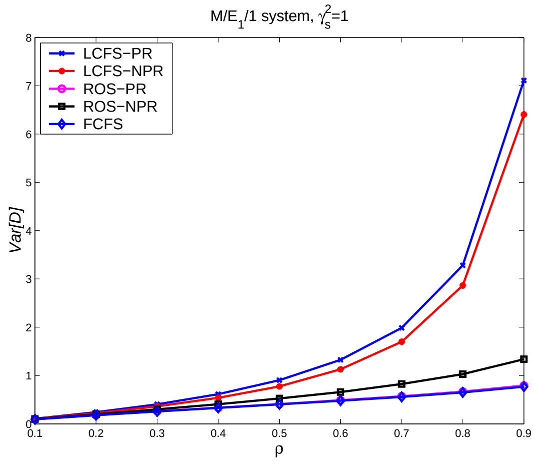

Figure 2: Unfairness for medium variability system, M/E_1/1 (Coefficient of variation $\gamma_S = 1$)

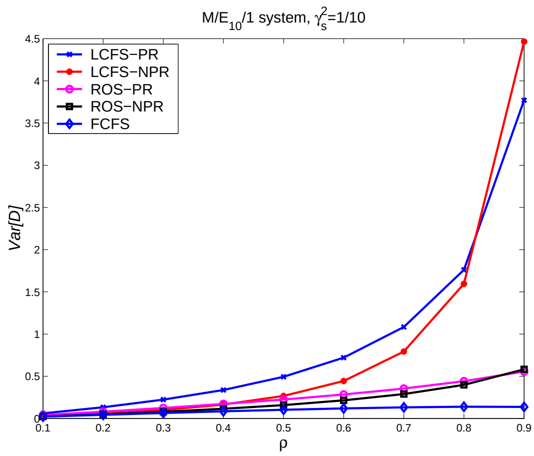

Figure 3: Unfairness for low variability system, M/E_{10}/1 (Coefficient of variation $\gamma_S = 1/\sqrt{10}$)

In all cases examined we set the mean service time to $E[S] = 1$ and use the arrival rate λ to control the system load. We also vary the service time coefficient of variation γ_S. The examination is carried out for loads ranging between 0.1 and 0.9 and γ_S varies through the values $1/(\sqrt{20}), 1/(\sqrt{15}), 1/(\sqrt{10}), 1/(\sqrt{5}), 1, 5, 10, 15, 20$. For compactness we do not present here all the results.

Figure 1 depicts the unfairness in the system under high service time variability $\gamma_S = 10$. The behavior for $\gamma_S = 5, 15, 20$ is quite similar and thus is not presented. Figure

2 depicts the unfairness in the system under medium service time variability, $\gamma_S = 1$ (the M/M/1 case). Figure 3 depicts the unfairness in the system under low service time variability, $\gamma_S = 1/(10^{0.5})$ (the M/E_{10}/1 system).

6.2 Observations

One may observe the following properties:

1. The variability of service time significantly affects the fairness experienced in the various disciplines. In fact, service time variability affects also the relative (fairness) ranking of the scheduling policies. For example, at $\rho = 0.6$, the relative fairness ranking for $\gamma_S = 10$ (Figure 1) is ROS-PR $>$ LCFS-PR $>$ FCFS-NPR $\approx$ ROS-NPR $\approx$ LCFS-NPR (where $>$ should read as "more fair" and $\approx$ as "approximately identically fair"). In contrast, for $\gamma_S = 1$ (Figure 2) it is NPR-FCFS $\approx$ PR-ROS $>$ ROS-NPR $>$ LCFS-NPR $>$ LCFS-PR.

2. For service times with high variability we observe the following properties (demonstrated in Figure 1, and observed in all the high variability cases we examined):

   (a) The unfairness of all non-preemptive policies is about the same. The reason for this is that the dominant discrimination, in this case, is the positive discrimination of the long jobs (see the results derived in Section 5), which becomes dominant since unfairness is taken as the second moment of discrimination. Thus, the particular order of service has negligible effect on the over-all discrimination, in these policies.

   (b) For low to medium loads the non-preemptive policies are the most unfair while the ROS-Preemptive is the most fair.

   (c) For high loads the LCFS-PR becomes the most unfair (while ROS-PR maintains its highest fairness rank).

   (d) The unfairness of the Non-preemptive policies is roughly proportional to the second moment of service time (at all loads).

3. For service times with low variability we observe:

   (a) The unfairness values for the M/E_r/1 model are affected by the values of r but seem to converge once r reaches the values between 5 and 10.

(b) At most ranges of load the fairness relative ranking obeys FCFS > ROS > LCFS and $NPR > PR$. Both of these relations agree with the common sense intuition and with prior results indicating that in the case of deterministic service (zero service time variability) serving jobs in the order they arrive is the most fair order among non-preemptive policies.

(c) At very high load ($\rho = 0.9$ and above) the Non-preemptive policies seem to become more unfair than their preemptive counter part. This property, which repeats in some of the other cases studied does not have yet a good explanation and requires more study to validate it as well as to explain it.

Remark 6.1. It is interesting to view the ROS-PR and the LCFS-PR policies. While the former seems to be very fair at most cases (the most fair in many cases), the latter is very unfair (most unfair) in most cases. This suggests that the *preemption* factor drives LCFS-PR towards fairness (discriminating against long jobs) while the LCFS factor (discrimination against senior jobs) drives it towards unfairness. The latter factor does not exists in ROS-PR, which is the reason it is very fair. This observation should be put in the context of previous research, e.g., Wierman and Harchol-Balter (2003) that found LCFS-PR to be very fair (using the slow-down approach for fairness).

7 Concluding Remarks

This work aimed at understanding how service time variability affects fairness in queueing time. We analyzed basic common service disciplines via models (either an M/G/1 system (LCFS-PR) or a mapping of the M/G/1 system to a system with Coxian distribution) that accounted for the service time variability. We showed that the system unfairness is significantly affected by the first two moments of the service time; for the LCFS-PR we showed that higher moments of service time do not affect the unfairness. We demonstrated that fairness is very sensitive to service time variability including affecting the relative (fairness) ranking of the various scheduling policies.

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A Appendix

A.1 Conditional Discrimination in M/E$_r$/1

A.1.1 Non-Preemptive LCFS

Let C denote the tagged customer. For consistency we assume that the queue is ordered in order of arrival and customers are admitted into service from the tail of the queue. At every slot let a denote the number of customers arrived earlier than C and thus to be served after C. Let b denote the number of customers arrived later than C and thus to be served before C. The state $S_{a,b,j}$ (where j is the stage of the customer in service) captures all that is needed for predicting the future of C. Note that using this description a customer is served in $\mathcal{S}_{a,0,j}$, i.e., when $b = 0$.

An arriving customer starts either in state $\mathcal{S}_{0,0,1}$ when the system is empty, or in state $\mathcal{S}_{k-1,1,j}$ when the system is serving a customer at stage j. Thus

$$E[D|k,j] = \begin{cases} E[d(k-1,1,j)] & k > 0 \\ E[d(0,0,0)] & k = j \end{cases} \tag{44}$$

$$E[D^2|k,j] = \begin{cases} E[d^{(2)}(k-1,1,j)] & k > 0 \\ E[d^{(2)}(0,0,1)] & k = j \end{cases} \tag{45}$$

Using the same notations as in Section **??** we have:

$$c(a,b) = \begin{cases} -\frac{1}{a+b+1} & b > 0 \\ 1 - \frac{1}{a+1} & b = 0 \end{cases} \tag{46}$$

Here too, when C is in state $\mathcal{S}_{a,b,j}$ it will encounter one of two possible events:

1. A customer arrives into the system. The probability of this event is $\tilde{\lambda}$. If C was in service ($b = 0$) it will move to $\mathcal{S}_{a+1,0,j}$, otherwise to $\mathcal{S}_{a,b+1,j}$
2. A customer completes its current stage. The probability of this event is $\tilde{\mu}$. If C was in service and $j = r$ it leaves the system. If C wasn't in service it moves to $\mathcal{S}_{a,b,j+1}$ if $j \neq r$ or to $\mathcal{S}_{a,b-1,1}$ if $j = r$.

This leads to the following recursive expressions for $D(a,b,j)$. For $b > 0$

$$D(a,b,j) = \begin{cases} Tc(a,b) + D(a,b+1,j) & \text{w.p } \tilde{\lambda} \\ Tc(a,b) + D(a,b,j+1) & \text{w.p } \tilde{\mu},\ j \neq r, \\ Tc(a,b) + D(a,b-1,1) & \text{w.p } \tilde{\mu},\ j = r. \end{cases} \tag{47}$$

and for $b = 0$

$$D(a,0,j) = \begin{cases} Tc(a,0) + D(a+1,0,j) & \text{w.p } \tilde{\lambda} \\ Tc(a,0) + D(a,0,j+1) & \text{w.p } \tilde{\mu},\ j \neq r, \\ Tc(a,0) & \text{w.p } \tilde{\mu},\ j = r. \end{cases} \tag{48}$$

From these, as in Section 4.2.1, we derive the recursive equations for expressing $d(a,b,j)$ and $d^{(2)}(a,b,j)$.

For $b > 0$

$$d(a,b,j) = \begin{cases} t^{(1)}c(a,b) + \tilde{\lambda}d(a,b+1,j) + \tilde{\mu}d(a,b,j+1) & j \neq r \\ t^{(1)}c(a,b) + \tilde{\lambda}d(a,b+1,j) + \tilde{\mu}d(a,b-1,1) & j = r \end{cases} \tag{49}$$

$$\begin{aligned} d^{(2)}(a,b,j) = {} & t^{(2)}(c(a,b))^2 + \tilde{\lambda}d^{(2)}(a,b+1,j) \\ & + 2t^{(1)}c(a,b)\tilde{\lambda}d(a,b+1,j)) + \\ & \begin{cases} \tilde{\mu}d^{(2)}(a,b,j+1) + 2t^{(1)}c(a,b)\tilde{\mu}d(a,b,j+1) & j \neq r \\ \tilde{\mu}d^{(2)}(a,b-1,j) + 2t^{(1)}c(a,b)\tilde{\mu}d(a,b-1,1) & j = r \end{cases} \end{aligned} \tag{50}$$

and for $b = 0$

$$d(a,0,j) = \begin{cases} t^{(1)}c(a,0) + \tilde{\lambda}d(a+1,0,j) + \tilde{\mu}d(a,0,j+1) & j \neq r \\ t^{(1)}c(a,0) + \tilde{\lambda}d(a+1,0,j) & j = r \end{cases} \tag{51}$$

$$\begin{aligned} d^{(2)}(a,0,j) = {} & t^{(2)}(c(a,0))^2 + \tilde{\lambda}d^{(2)}(a+1,0,j) + \\ & 2t^{(1)}c(a,0)\tilde{\lambda}d(a+1,0,j)) + \\ & \begin{cases} \tilde{\mu}d^{(2)}(a,0,j+1) + 2t^{(1)}c(a,0)\tilde{\mu}d(a,0,j+1) & j \neq r \\ 0 & j = r \end{cases} \end{aligned} \tag{52}$$

A.1.2 Non-Preemptive ROS

For a tagged customer C, let a denote the number of customers in the system other than C. Consider a boolean variable s which is 1 if C is in service and 0 if it is waiting. The notation used in this section remains unchanged, except that b is replaced by s. In state $\mathcal{S}a, s, j$ there are a customers in addition to C, the one in service is in its j-th stage, and it is C if $s = 1$ and not C if $s = 0$.

The momentary discrimination at this state is $c(a, s)$,

$$c(a,s) = \begin{cases} -\frac{1}{a+1} & s = 0 \\ 1 - \frac{1}{a+1} & s = 1 \end{cases} \tag{53}$$

A customer arrives to the system either at state $\mathcal{S}_{0,1,1}$ when it is empty, or at state $\mathcal{S}_{k,0,j}$ when it serving customer at stage j. Then

$$E[D|k,j] = \begin{cases} E[d(k,0,j)] & k > 0 \\ E[d(0,1,1)] & k = 0 \end{cases} \tag{54}$$

$$E[D^2|k,j] = \begin{cases} E[d^{(2)}(k,0,j)] & k > 0 \\ E[d^{(2)}(0,1,1)] & k = 0 \end{cases} \tag{55}$$

When C is in state $\mathcal{S}_{a,s,j}$ it will encounter one of the following possible events:

1. If $s = 0$ the possible events are:

   (a) A customer arrives into the system. The probability of this event is $\tilde{\lambda}$ and C will move to $\mathcal{S}_{a+1,0,j}$.

   (b) A customer completes its current stage j, where $j \neq r$. The probability of this event is $\tilde{\mu}$ and C moves to $S_(a, 0, j + 1)$.

   (c) A customer completes service, leaves the system and C is chosen to receive service next. The probability of this event is $\tilde{\mu}/a$. C will move to $\mathcal{S}_{a-1,1,1}$.

   (d) A customer completes service, leaves the system and C is not chosen to receive service next. The probability of this event is $\tilde{\mu}(a-1)/a$. C will move to $\mathcal{S}_{a-1,0,1}$.

2. If $s = 1$ the possible events are:

   (a) Same as (1a) but C will move to $\mathcal{S}_{a+1,1,j}$.

   (b) Same as (1b) but C will move to $S_{a,1,j+1}$.

   (c) The customer in service completes its service. The probability of this event is $\tilde{\mu}$ and C leaves the system.

Using the same method as in the previous two sections this leads to the following recursive expressions:

$$d(a,s,j) = t^{(1)}c(a,s) + \tilde{\lambda}d(a+1,s,j) +$$
$$\begin{cases} \tilde{\mu}d(a,s,j+1) & j \neq r \\ \frac{\tilde{\mu}}{a}d(a-1,1,1) + \tilde{\mu}\frac{a-1}{a}d(a-1,0,1) & j = r, s = 0 \\ 0 & j = r, s = 1 \end{cases} \tag{56}$$

$$d^{(2)}(a,s,j) = t^{(2)}(c(a,s))^2 + \tilde{\lambda}d^{(2)}(a+1,s,j) +$$
$$2t^{(1)}c(a,s)\tilde{\lambda}d(a+1,s,j) +$$
$$\begin{cases} \tilde{\mu}d^{(2)}(a,s,j+1) + 2t^{(1)}c(a,b)\tilde{\mu}d(a,s,j+1)) & j \neq r \\ \frac{\tilde{\mu}}{a}d^{(2)}(a-1,1,1) + \tilde{\mu}\frac{a-1}{a}d^{(2)}(a-1,0,1) + & \\ 2t^{(1)}c(a,s)(\frac{\tilde{\mu}}{a}d(a-1,1,1) + \tilde{\mu}\frac{a-1}{a}d(a-1,0,1)) & j = r, s = 0 \\ 0 & j = r, s = 1 \end{cases} \tag{57}$$

B Conditional Discrimination in M/Cox$_2$/1

Consider the M/Cox$_2$/1 where the service time distribution is a two-stage Coxian. For this distribution the service is assumed to be composed of two serially arranged stages, where a new customer that enters service starts with stage 1 and after its completion enters stage 2 with probability p_1. The mean length of stage i is μ_i, $i = 1, 2$.

Similarly to M/E$_r$/1 the time between the arrival of a customer and its departure is slotted by arrivals and stage completions, however, for Coxian service the slot duration is dependant upon the stage of service in which the served customer is currently found. Let $T_{i,j}$, $j = 1, 2$, $i = 1, 2, \ldots$ be the duration of the i-th slot, where j is the stage of the served customer. Then, $T_{i,j}$ is a random variable exponentially distributed with parameter $\lambda + \mu_j$; the first two moments of $T_{i,j}$ are $t_j^{(1)} = \frac{1}{\lambda+\mu_j}$ and $t_j^{(2)} = \frac{2}{(\lambda+\mu_j)^2} = 2(t_j^{(1)})^2$. The probabilities that a slot ends with an arrival or with a stage completion are denoted by $\tilde{\lambda}_j$ and $\tilde{\mu}_j$ respectively.

$$\tilde{\lambda}_j = \frac{\lambda}{\lambda + \mu_j} \qquad \tilde{\mu}_j = \frac{\mu_j}{\lambda + \mu_j} \tag{58}$$

. Similarly to Eq.(22) the system unfairness is expressed as:

$$E[D^2] = P_0 E[D^2|0,1] + \sum_{k=1}^{\infty}\sum_{j=1}^{2} P_{k,j} E[D^2|k,j], \tag{59}$$

B.0.3 FCFS

We preserve the notations of Section 4.2.1 and denote by $\mathcal{S}_{a,b,j}$ the state of a tagged customer C, where a is the number of customers ahead of C, b is the number of customers behind C, and j is the stage of customer in service.

The momentary discrimination at state $\mathcal{S}_{a,b,j}$ is given by

$$c(a,b) = \begin{cases} -\frac{1}{a+b+1} & b > 0 \\ 1 - \frac{1}{b+1} & b = 0 \end{cases} \tag{60}$$

Similarly to (24)

$$E[D|k,j] = \quad E[d(k,0,j)] \tag{61}$$

$$E[D^2|k,j] = \quad E[d^{(2)}(k,0,j)] \tag{62}$$

Using a similar method of analysis as in Section **??** we have that

$$d(a,b,j) = t_j^{(1)} c(a,b) + \tilde{\lambda}_j d(a,b+1,j) + \\ \begin{cases} p_1 \tilde{\mu}_j d(a,b,j+1) + \Delta(a>0)(1-p_1)\tilde{\mu}_j d(a-1,b,1) & j=1 \\ \Delta(a>0)\tilde{\mu}_j d(a-1,b,1) & j=2 \end{cases} \tag{63}$$

where T_j is the duration of the current slot and the served customer is located at stage j during this slot. The second moment is given by

$$d^{(2)}(a,b,j) = t_j^{(2)}(c(a,b))^2 + \tilde{\lambda} d^{(2)}(a,b+1,j) + \\ 2t_j^{(1)} c(a,b)\tilde{\lambda} d(a,b+1,j) + \\ \begin{cases} p_1\tilde{\mu}_j d^{(2)}(a,b,j+1) + \Delta(a>0)(1-p_1)\tilde{\mu}_j d^{(2)}(a-1,b,1) + \\ 2t_j^{(1)} c(a,b)(p_1\tilde{\mu}_j d^{(2)}(a,b,j+1) + \Delta(a>0)(1-p_1)\tilde{\mu}_j d^{(2)}(a-1,b,1)) & j=1 \\ \Delta(a>0)(\tilde{\mu}_j d^{(2)}(a-1,b,j) + 2t_j^{(1)} c(a,b)\tilde{\mu}_j d(a-1,b,j))) & j=2 \end{cases} \tag{64}$$

B.0.4 Non-Preemptive LCFS

We preserve the notations of Section A.1.1 and denote by $\mathcal{S}_{a,b,j}$ the state of C, where a is the number of customers arrived earlier than C and thus to be served after C, b is the number of customers arrived later than C and thus to be served before C, and j is the stage of the served customer. The conditional discrimination (Eq.(44), Eq.(45)) and the momentary discrimination (Eq.(46)) remains the same as in Section A.1.1.

By examining the possible events that C encounter we get that for $b > 0$

$$d(a,b,j) = t_j c(a,b) + \tilde{\lambda}_j d(a,b+1,j) + \begin{cases} p_1\tilde{\mu}_j d(a,b,j+1) + (1-p_1)\tilde{\mu}_j d(a,b-1,1) & j=1 \\ \tilde{\mu}_j d(a,b-1,1) & j=2 \end{cases} \tag{65}$$

$$d^{(2)}(a,b,j) = t_j^{(2)}(c(a,b))^2 + \tilde{\lambda}_j d^{(2)}(a,b+1,j) + 2t_j^{(1)}c(a,b)d(a,b+1,j) \begin{cases} p_1\tilde{\mu}_j d^{(2)}(a,b,j+1) + (1-p_1)\tilde{\mu}_j d^{(2)}(a,b-1,1) + \\ 2t_j^{(1)}c(a,b)(p_1\tilde{\mu}_j d(a,b,j+1) + (1-p_1)\tilde{\mu}_j d(a,b-1,1)) & j=1 \\ \tilde{\mu}_j d^{(2)}(a,b-1,1) + 2t_j^{(1)}c(a,b)\tilde{\mu}_j d(a,b-1,1) & j=2 \end{cases} \tag{66}$$

For $b = 0$

$$d(a,0,j) = t_j c(a,0) + \tilde{\lambda}_j d(a+1,0,j) + \begin{cases} p_1\tilde{\mu}_j d(a,0,j+1) & j=1 \\ 0 & j=2 \end{cases} \tag{67}$$

$$d^{(2)}(a,0,j) = t_j^{(2)}(c(a,0))^2 + \tilde{\lambda}_j d^{(2)}(a+1,0,j) + 2t_j^{(1)}c(a,0)d(a+1,0,j) \begin{cases} p_1\tilde{\mu}_j d^{(2)}(a,0,j+1) + 2t_j^{(1)}c(a,0)p_1\tilde{\mu}_j d(a,0,j+1) & j=1 \\ 0 & j=2 \end{cases} \tag{68}$$

B.0.5 Non Preemptive ROS

We preserve the notations of Section A.1.2 and denote by $\mathcal{S}a, s, j$ the state of C, where a is the number of customers in the system other than C, s is 1 if C is in service and 0 if it is waiting, and j is the state of the served customer. The conditional discrimination (Eq.(54), Eq.(54)) and the momentary discrimination (Eq.(53)) remains the same as in Section A.1.2.

By examining the possible events that C encounter we get that

$$d(a,s,j) = t_j^{(1)} c(a,s) + \tilde{\lambda}_j d(a+1,s,j) +$$

$$\begin{cases} p_1 \tilde{\mu}_j d(a,s,j+1) + & \\ \Delta(a>0)(1-p_1)\tilde{\mu}_j(\frac{1}{a} d(a-1,1,1) + \frac{a-1}{a} d(a-1,0,1)) & j=1 \\ \Delta(a>0)\tilde{\mu}_j(\frac{1}{a} d(a-1,1,1) + \frac{a-1}{a} d(a-1,0,1)) & j=2 \end{cases} \quad (69)$$

$$d^{(2)}(a,b,j) = t_j^{(2)} (c(a,s))^2 + \tilde{\lambda}_j d^{(2)}(a+1,s,j) +$$

$$2t_j^{(1)} c(a,s) \tilde{\lambda}_j d(a+1,s,j) +$$

$$\begin{cases} p_1 \tilde{\mu}_j (d^{(2)}(a,s,j+1) + 2t_j^{(1)} c(a,s) d(a,s,j+1)) + & \\ \Delta(a>0)(1-p_1)\tilde{\mu}_j(\frac{1}{a} d^{(2)}(a-1,1,1) + \frac{a-1}{a} d^{(2)}(a-1,0,1) + & \\ 2t_j^{(1)} c(a,s)(\frac{1}{a} d(a-1,1,1) + \frac{a-1}{a} d(a-1,0,1)) & j=1 \\ \Delta(a>0)\tilde{\mu}_j(\frac{1}{a} d^{(2)}(a-1,1,1) + \frac{a-1}{a} d^{(2)}(a-1,0,1) + & \\ 2t_j^{(1)} c(a,s)(\frac{1}{a} d^{(2)}(a-1,1,1) + \frac{a-1}{a} d^{(2)}(a-1,0,1)) & j=2 \end{cases} \quad (70)$$