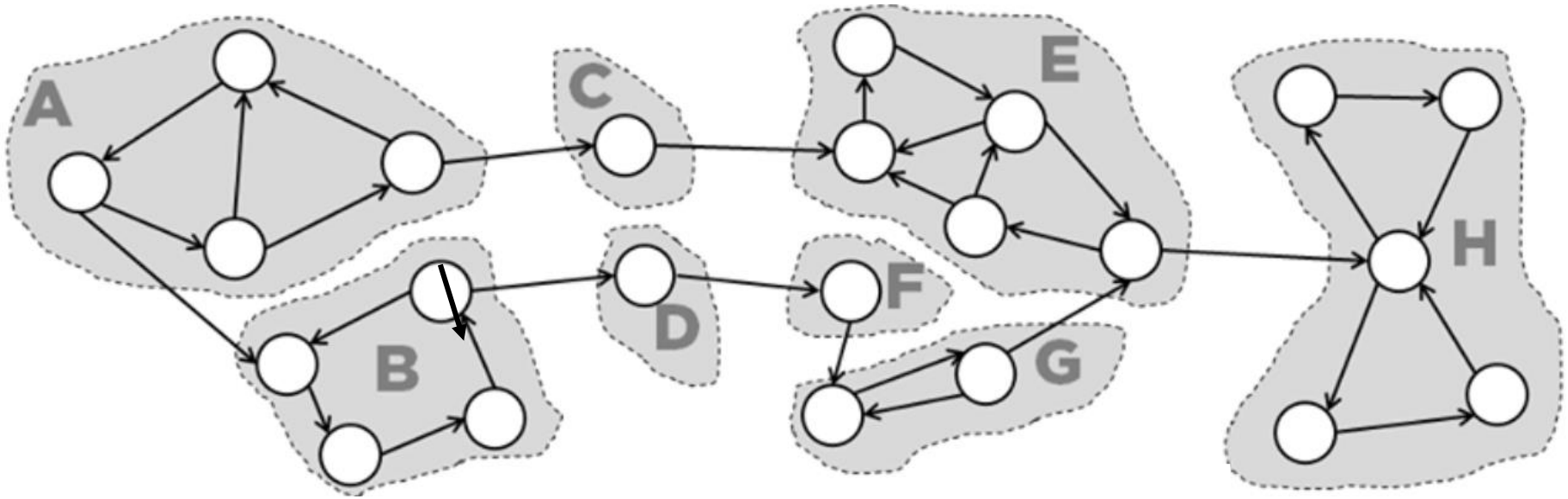
Strongly Connected Components with DFS

- Tarjan 1972 (wikipedia)
- Performs a single DFS with an additional **S**tack

Strongly connected components (SCC)

Def: Equivalence classes of the relation (u, v) iff there is a path from u to v and a path from v to u



Goal: Compute for each vertex the number of its component

Strongly connected components (SCC)

Obs: Let u and v be in a SC C. Let P be a path from u to v then every vertex on P is also in C

```
DFS(G)
For each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
For each vertex u ∈ G.V
    if u.color == WHITE
        DFS-Visit(G,u)
```

```
DFS-Visit(G,u)
time = time + 1
u.d = time
u.color = GRAY
For each vertex v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-Visit(G,v)
u.color = BLACK
time = time + 1
u.f = time
```

Thm 2: $[v.d, v.f] \subset [u.d, u.f]$ iff v is a descendant of u in the DFS-forest, $[v.d, v.f] \cap [u.d, u.f] = \emptyset$ iff u and v are unrelated in the DFS-forest

```
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    if v.color == WHITE
        v.π = u
        DFS-Visit(G,v)
u.color = BLACK
time = time + 1
u.f = time
```

Thm 3: v is a descendant of u in the DFS-forest iff when we invoke DFS-Visit(u) there is a white path from u to v

Edge classification

- Tree edges: (u,v), DFS-Visit(v) was called from DFS-visit(u)
- Back edges: (u,v) such that v is an ancestor of u in the DFS-forest
- Forward edges: nontree edges (u,v) such that v is a descendant of u
- Cross edges: (u,v) such that $v.f < u.d$

Observe, among non-tree edges:

1. (Only) backward edges go to a vertex with a later finish time
2. (Only) forward edges go to a vertex with later discovery time

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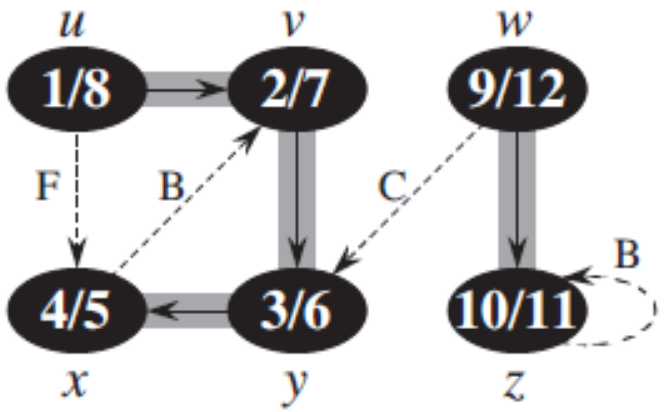
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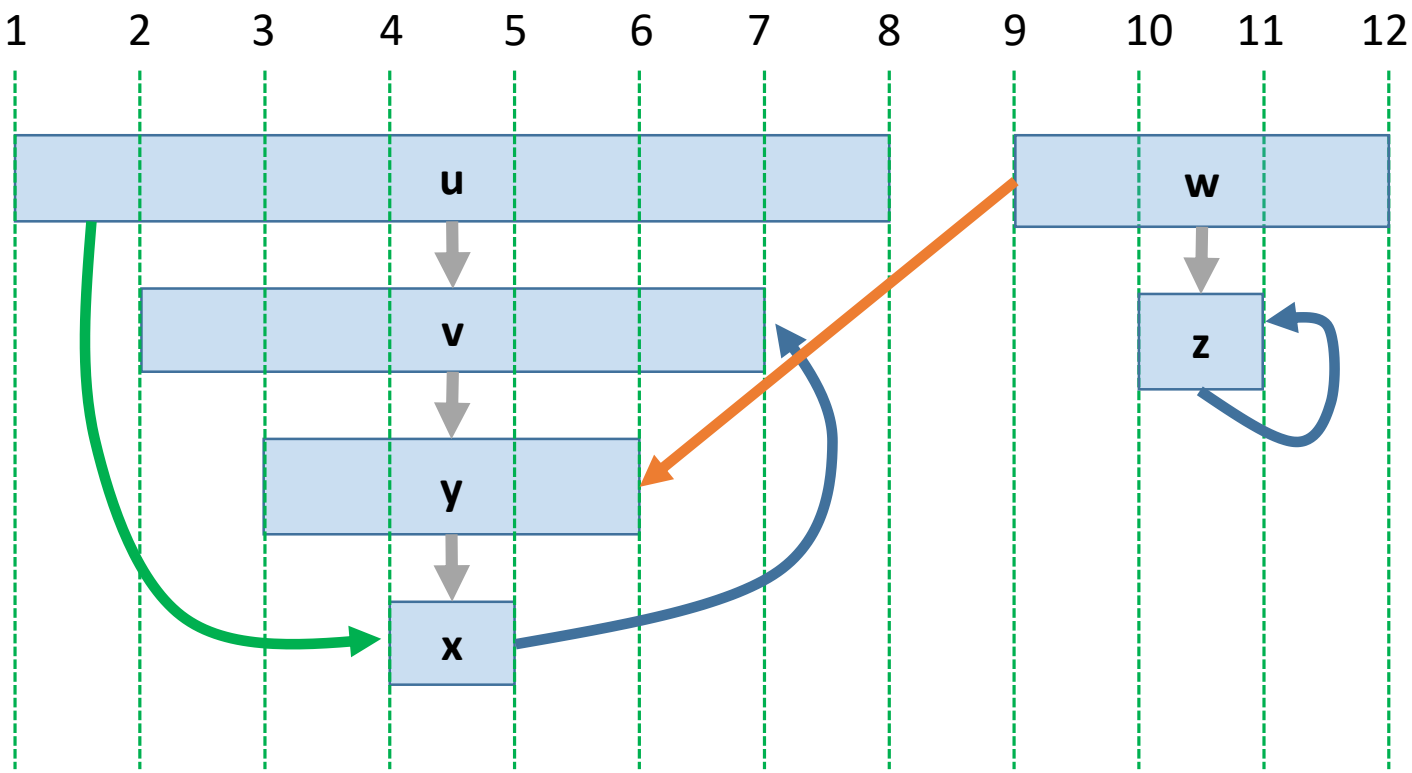
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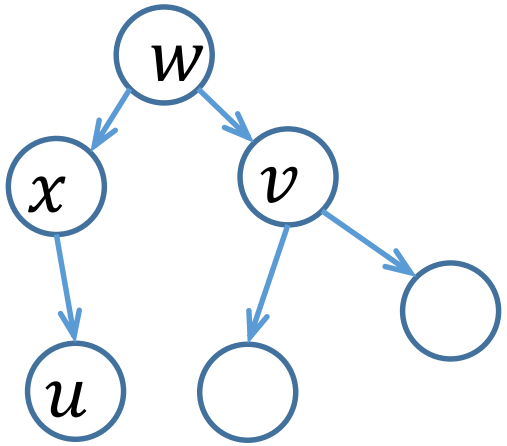
Tarjan's algorithm

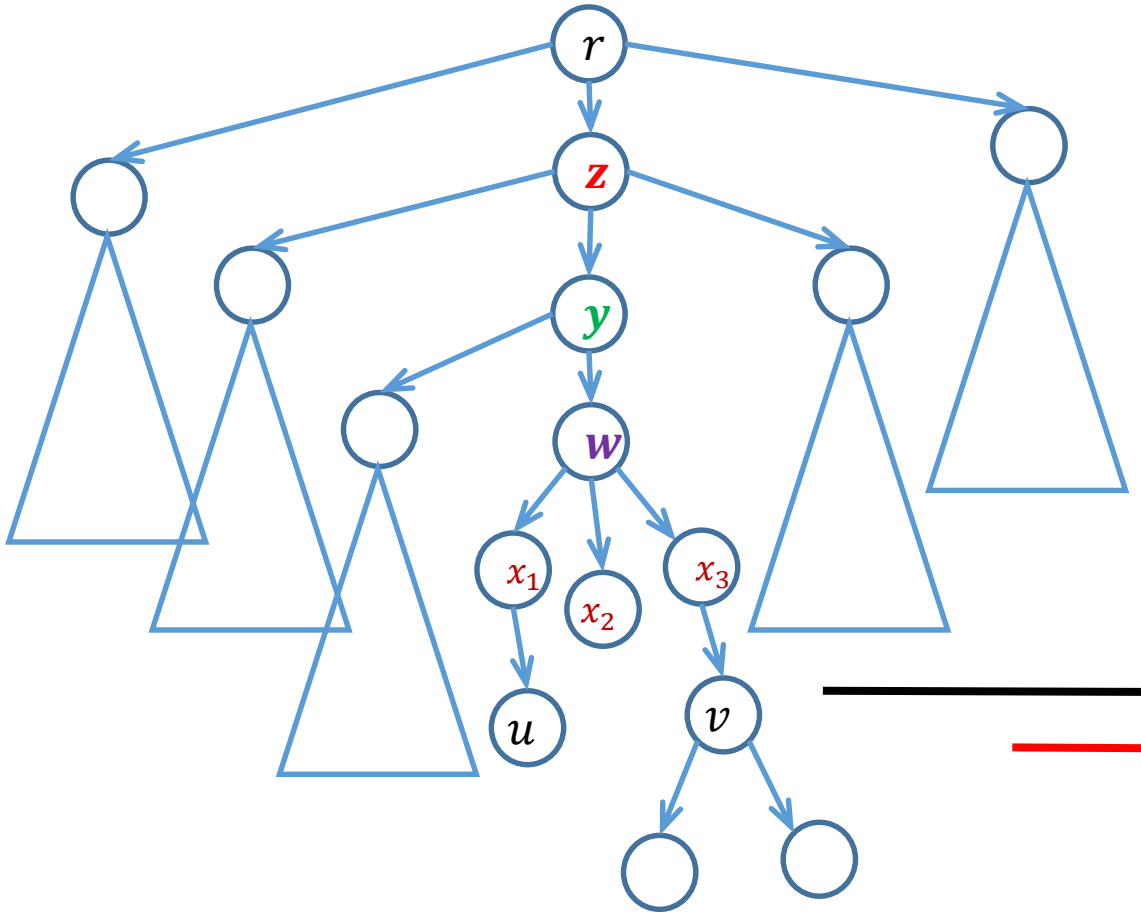
- We will find SCCs using only one DFS

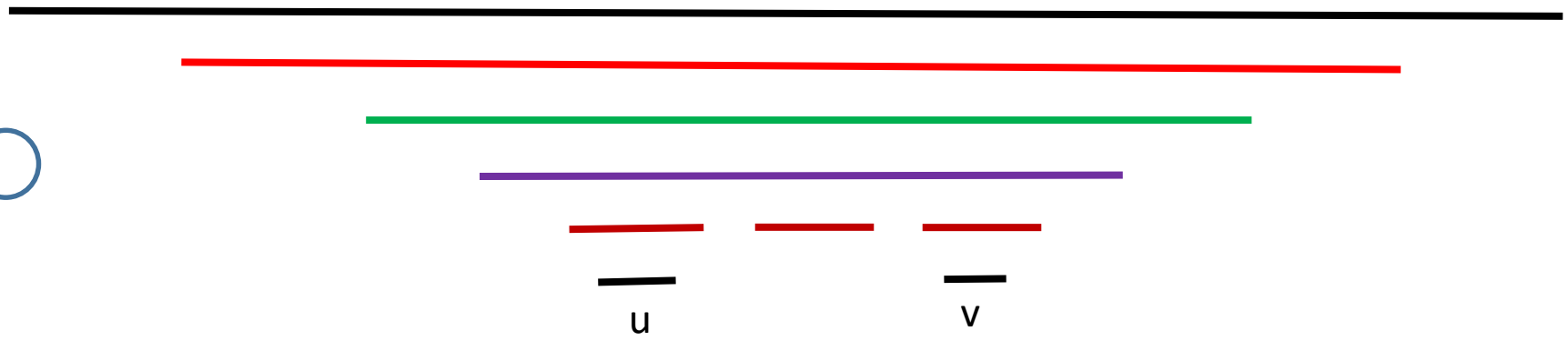
Thm 9: If u and v are in the same SC then they are in same tree of the DFS-forest and $LCA(u, v)$ is also in this component



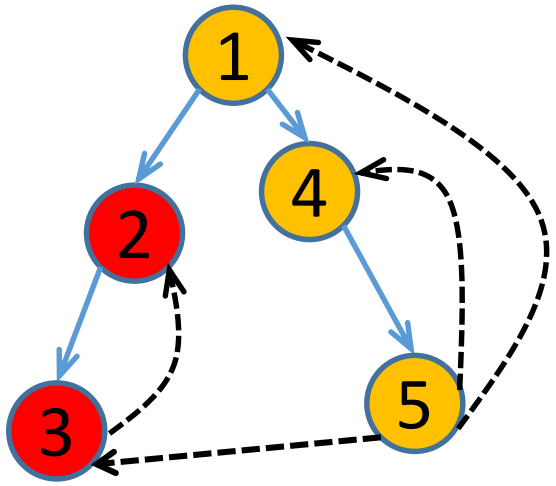
Corr: Each SC form a **connected** subtree of the DFS-forest

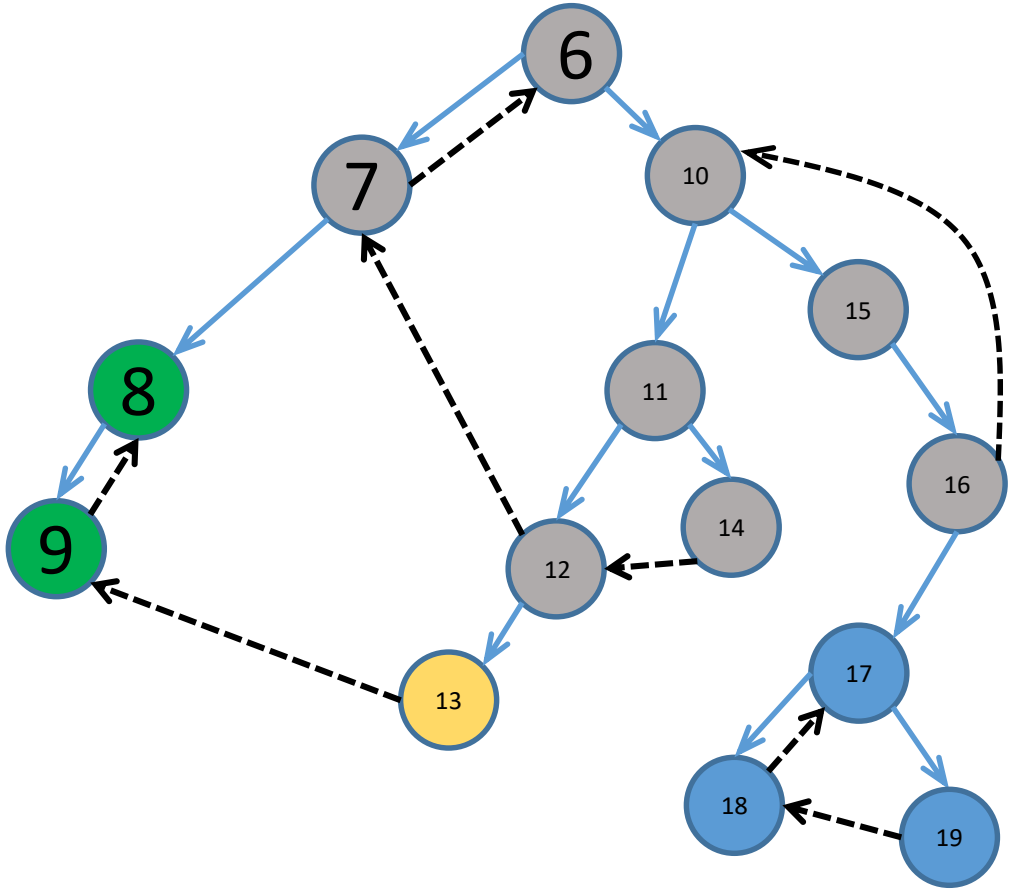
Proof: “by picture”, order the children of every node from left to right by their discovery time



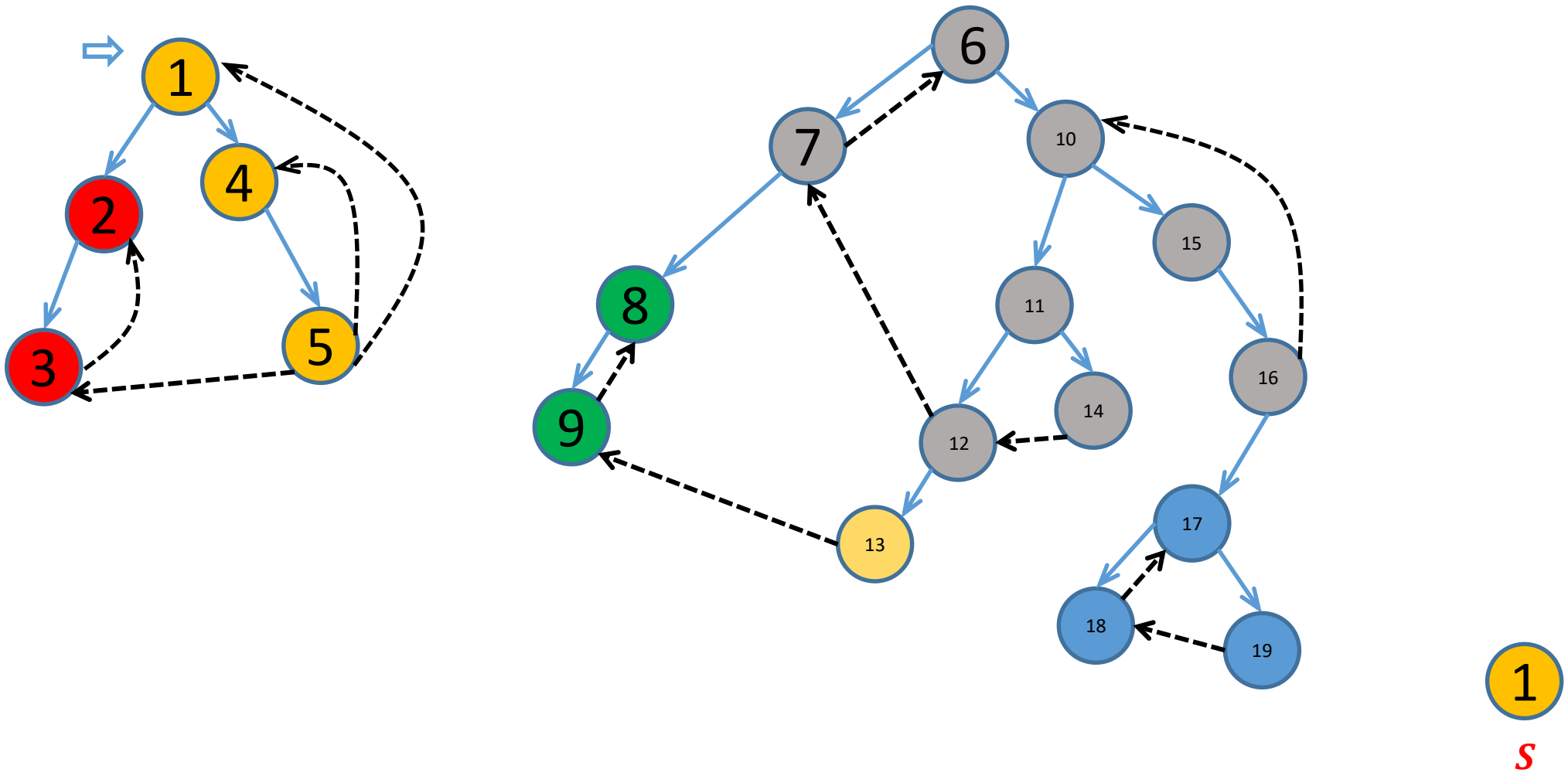


Identify the roots of these subtrees





Use an additional stack (S)



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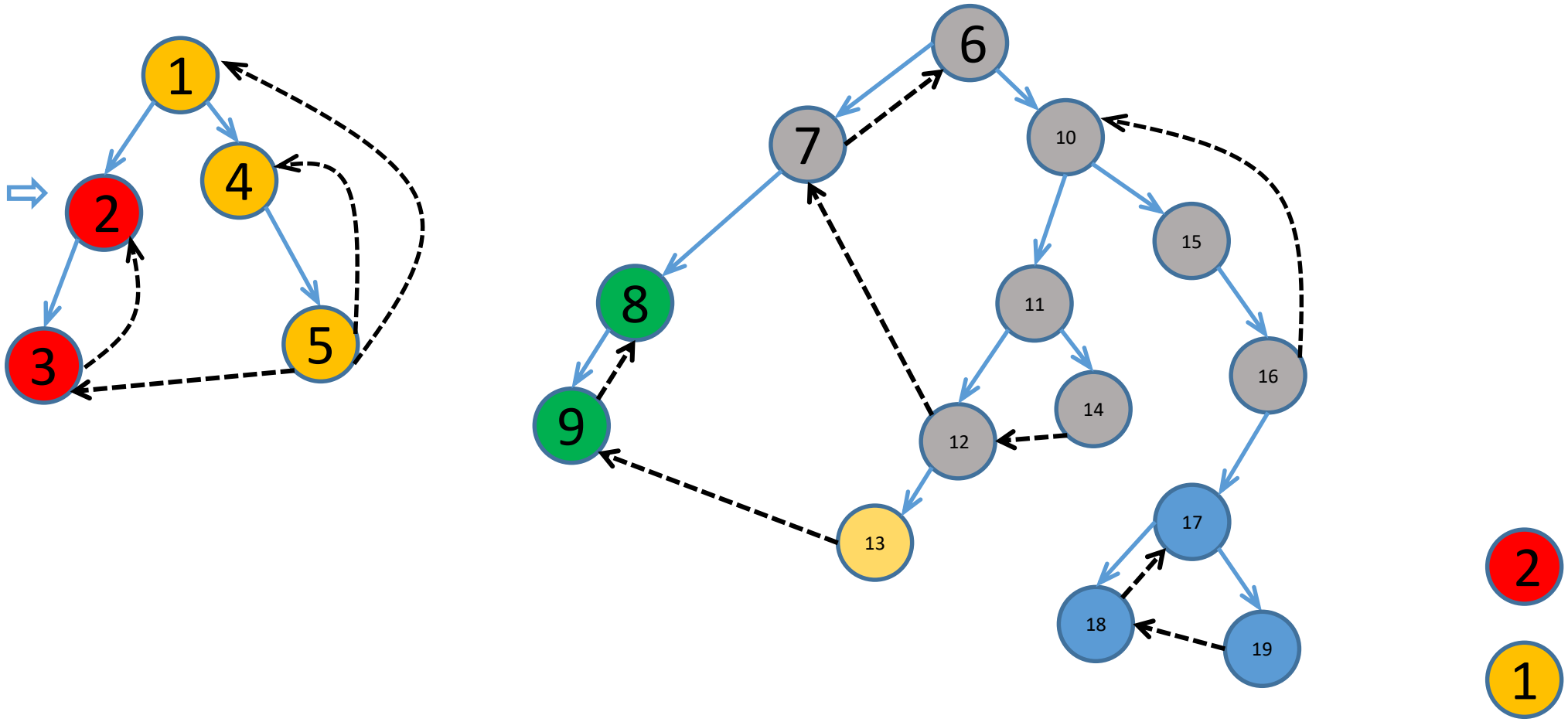

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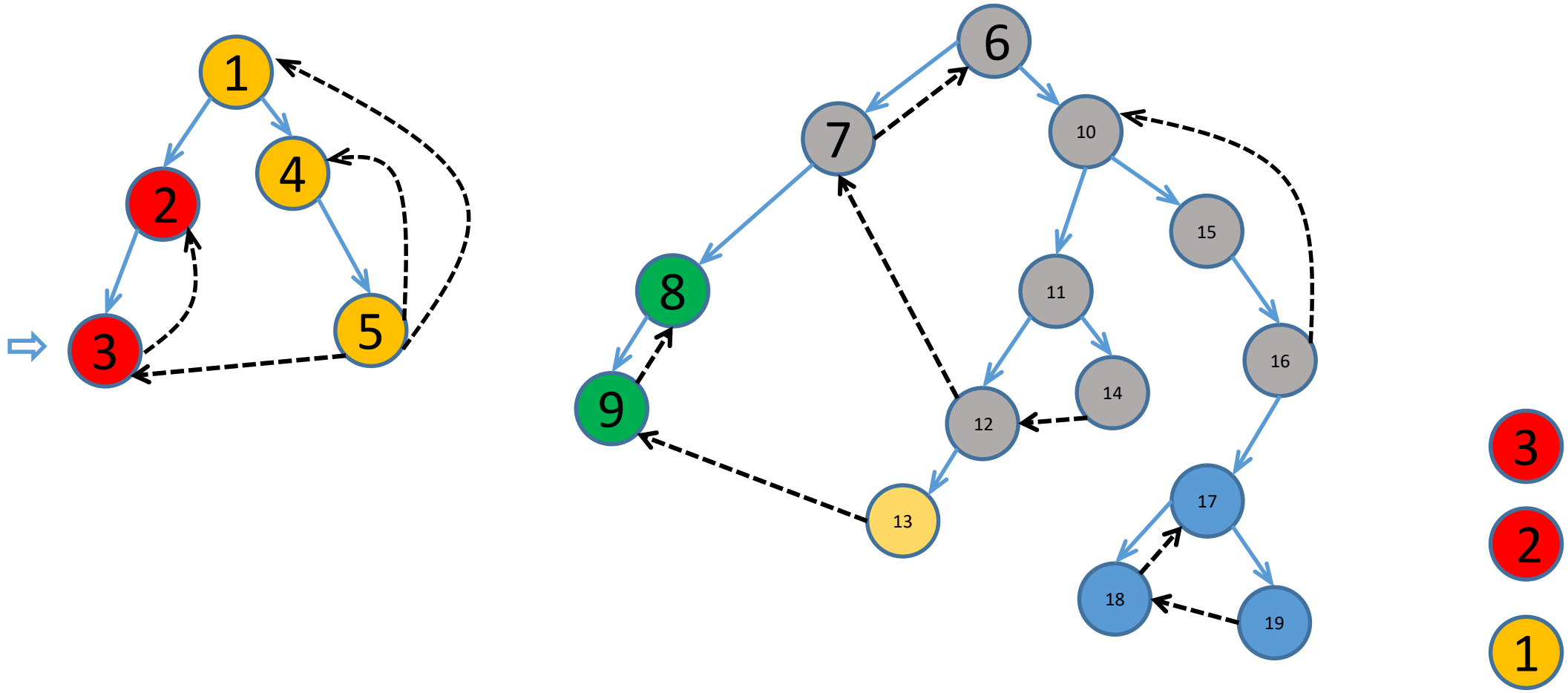

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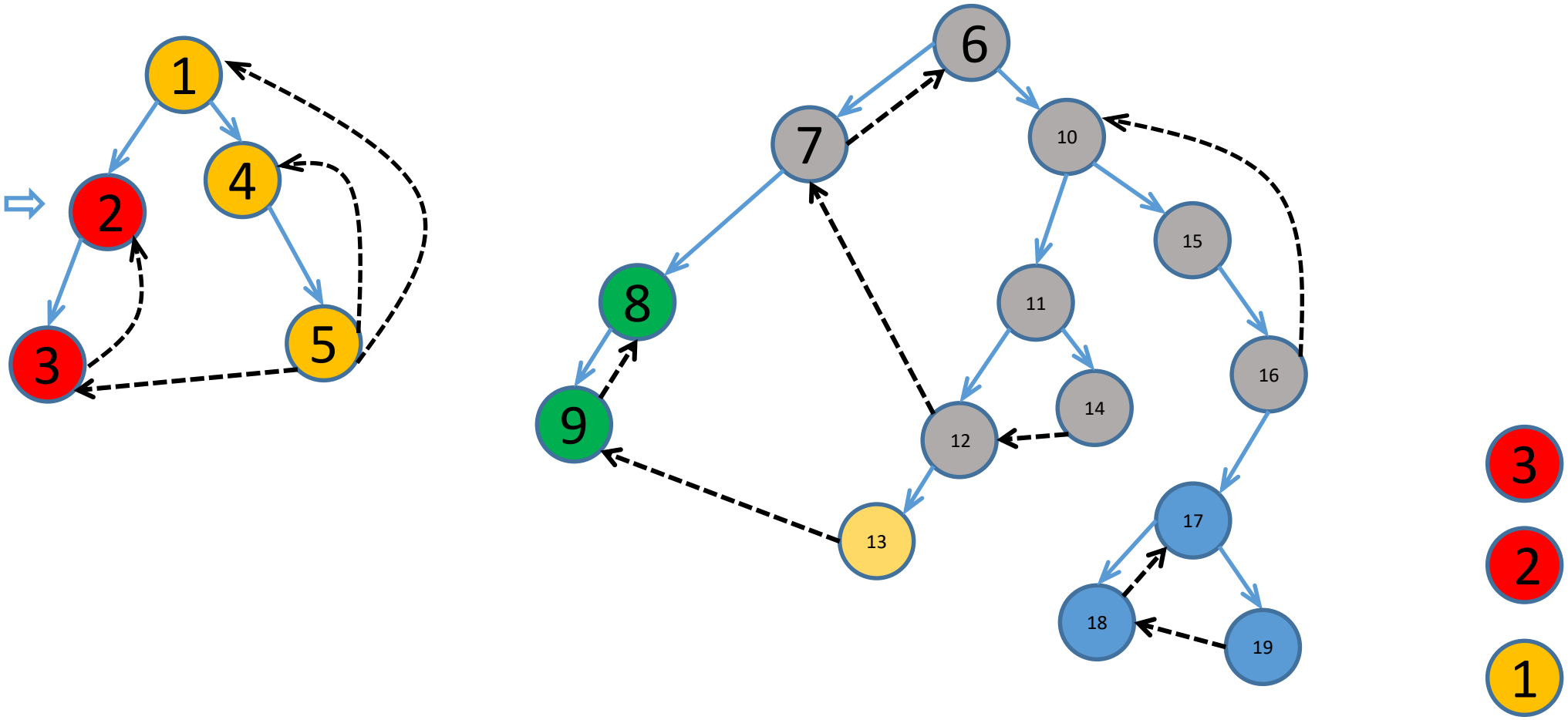

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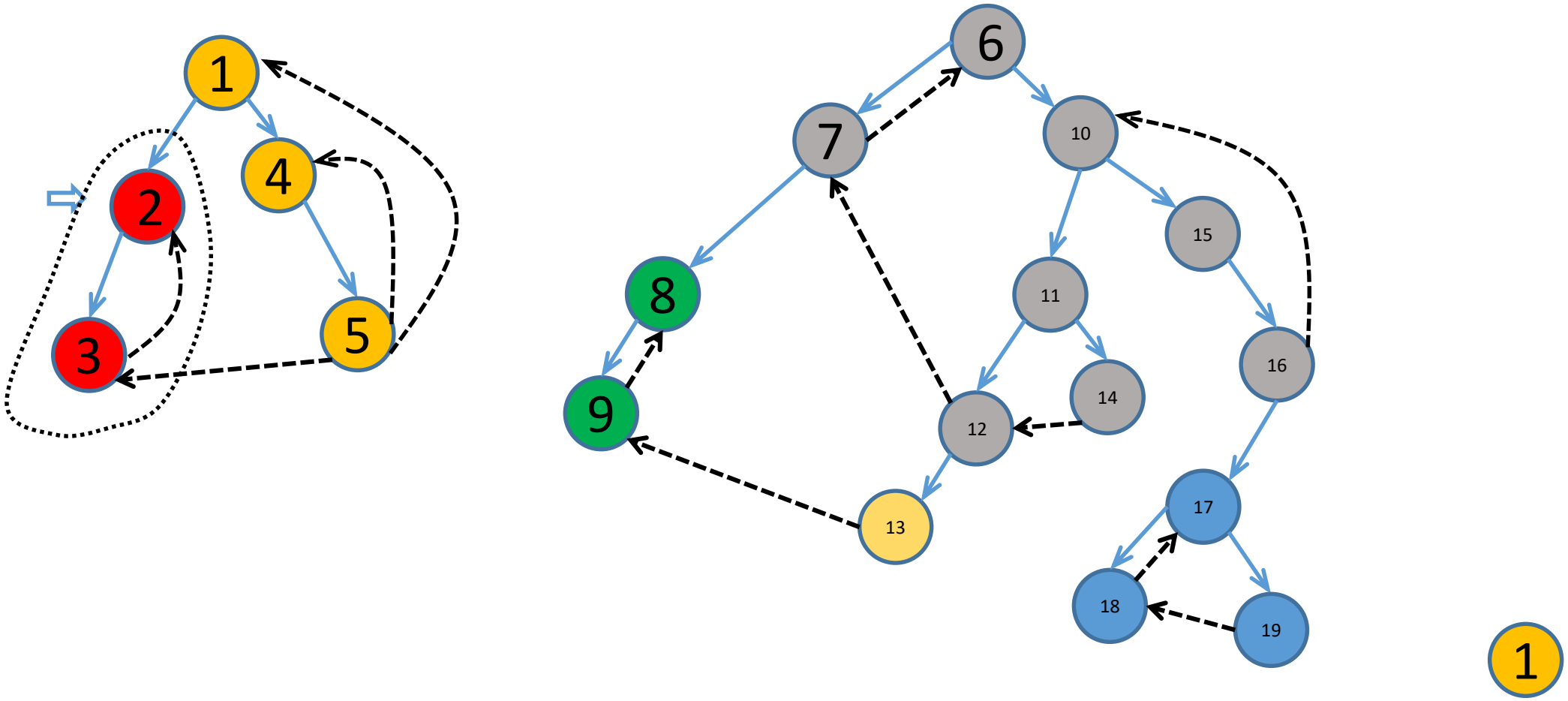

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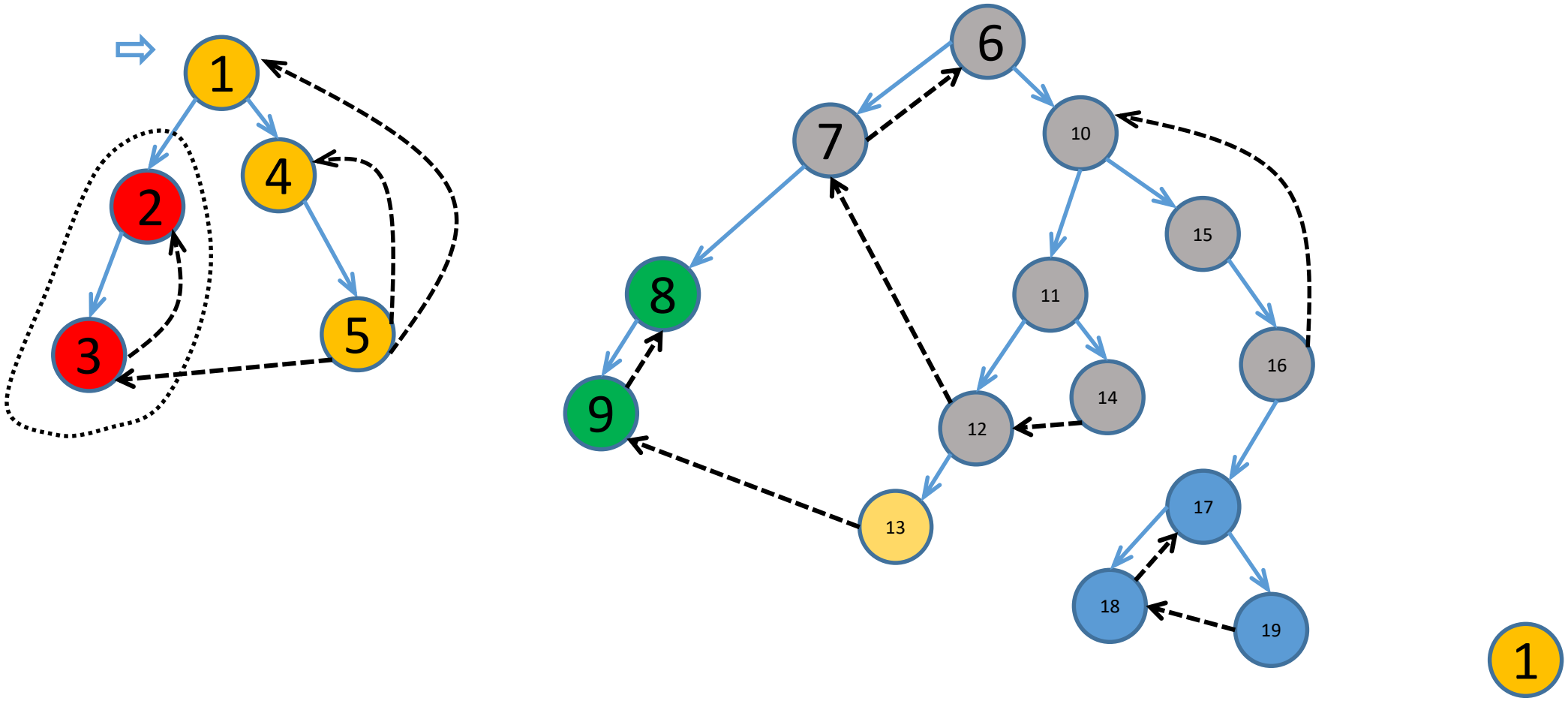

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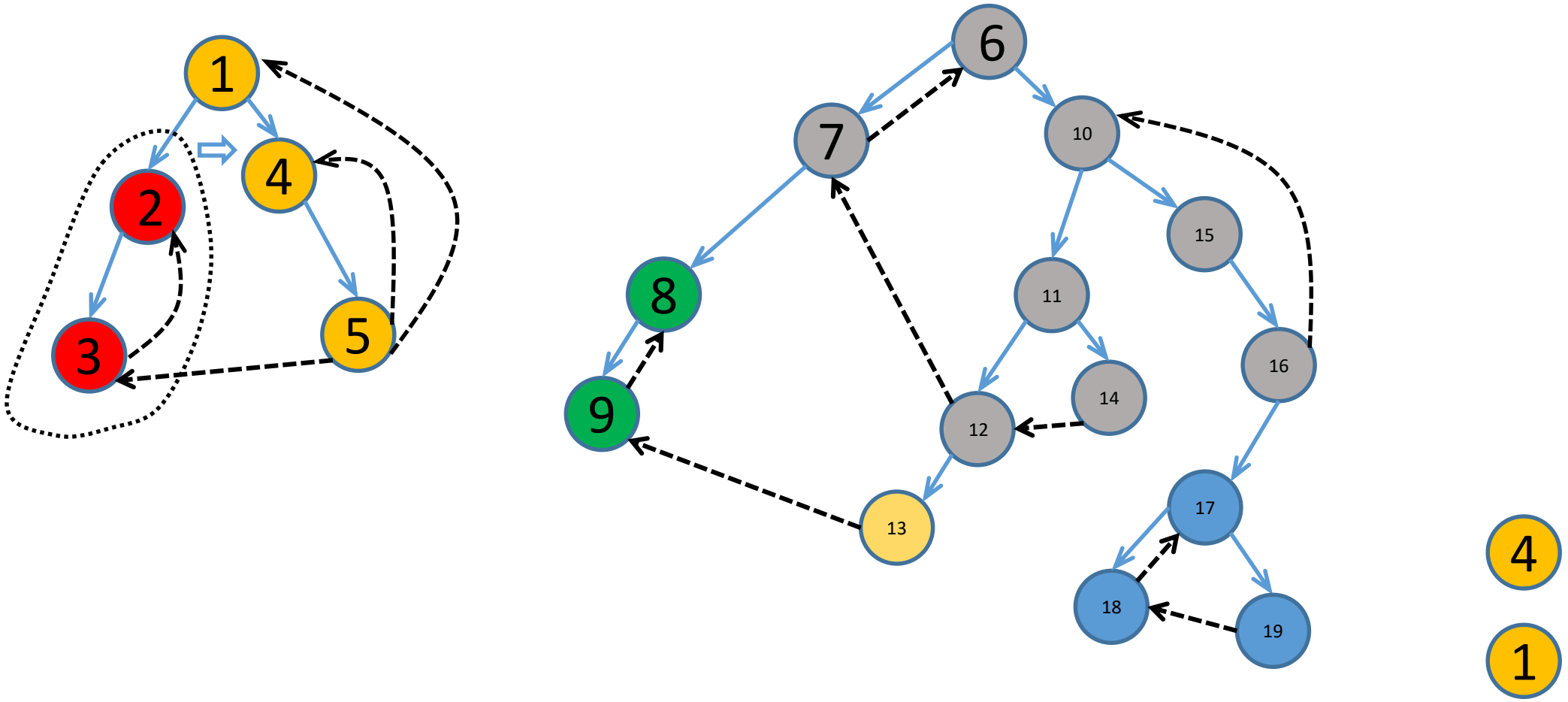

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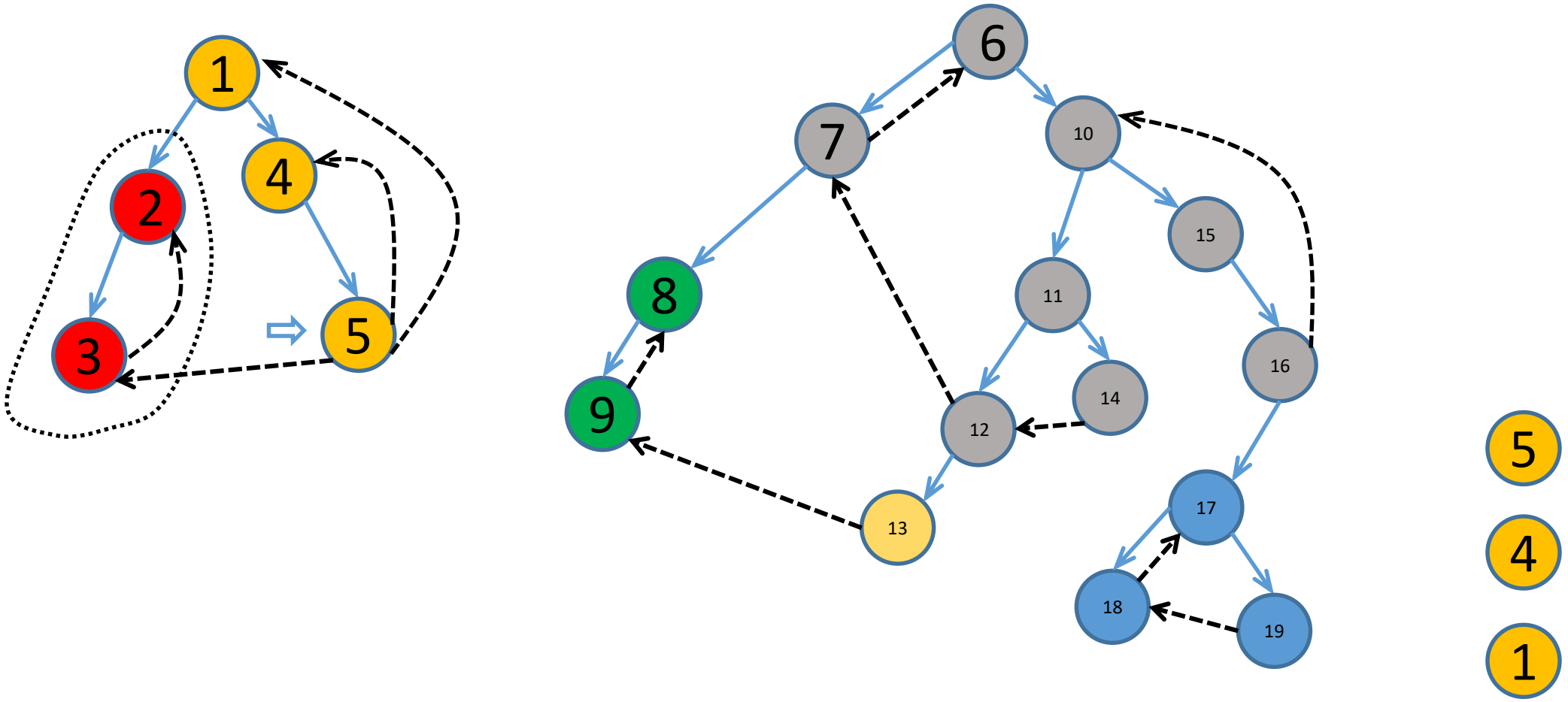

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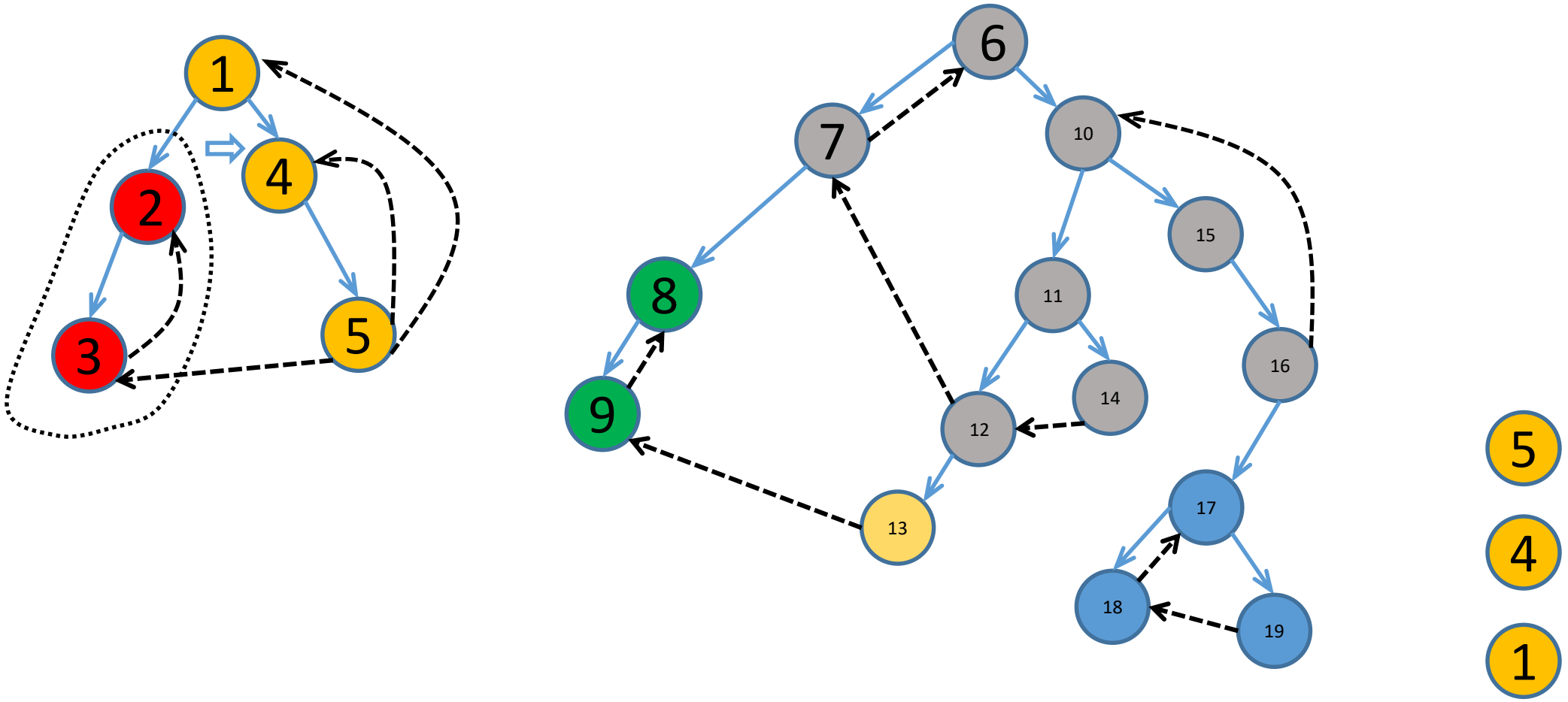

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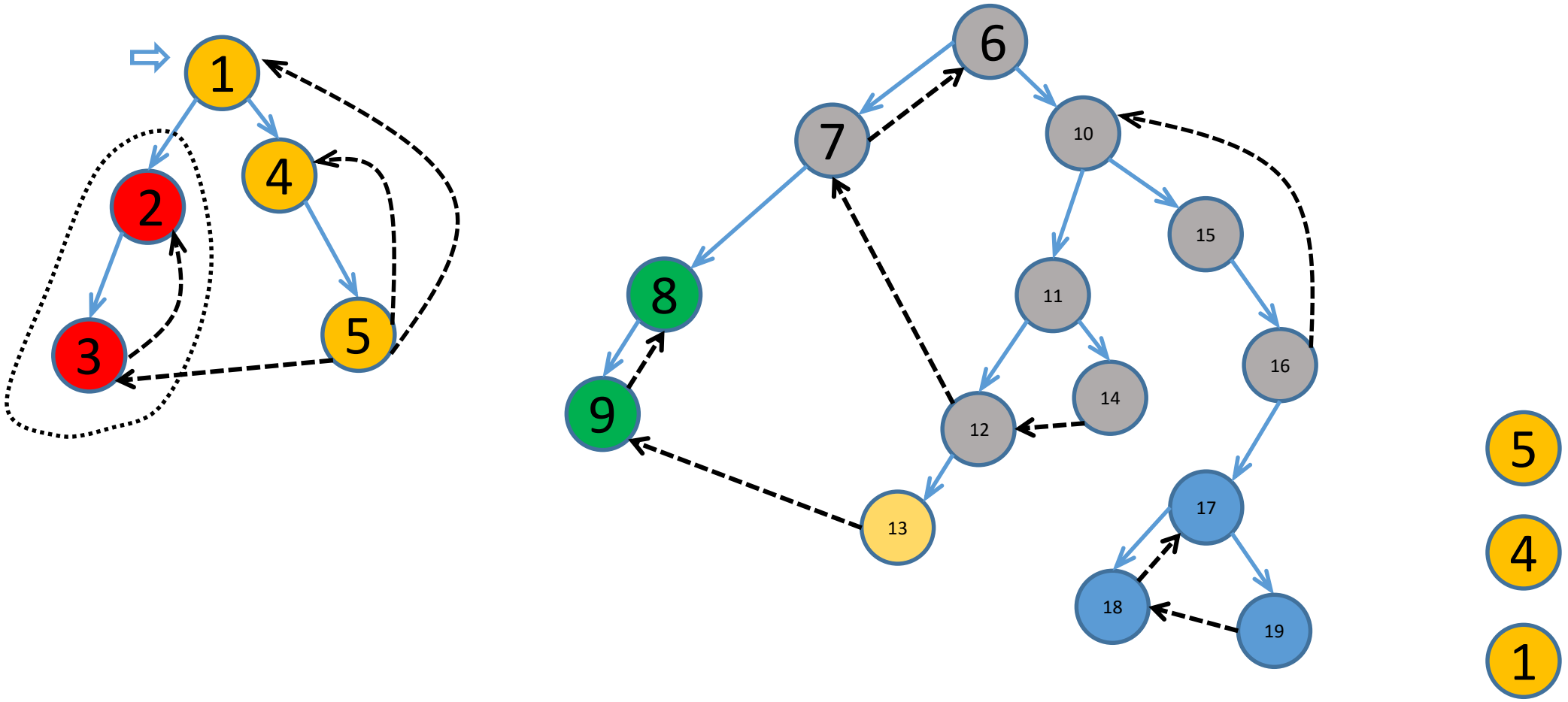

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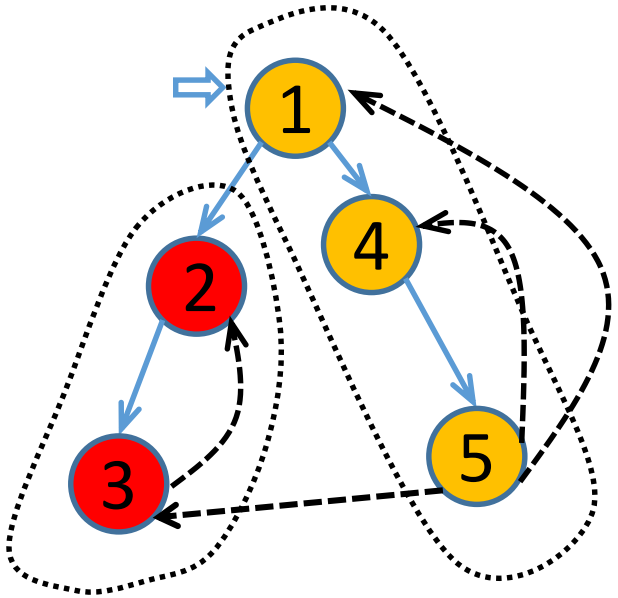

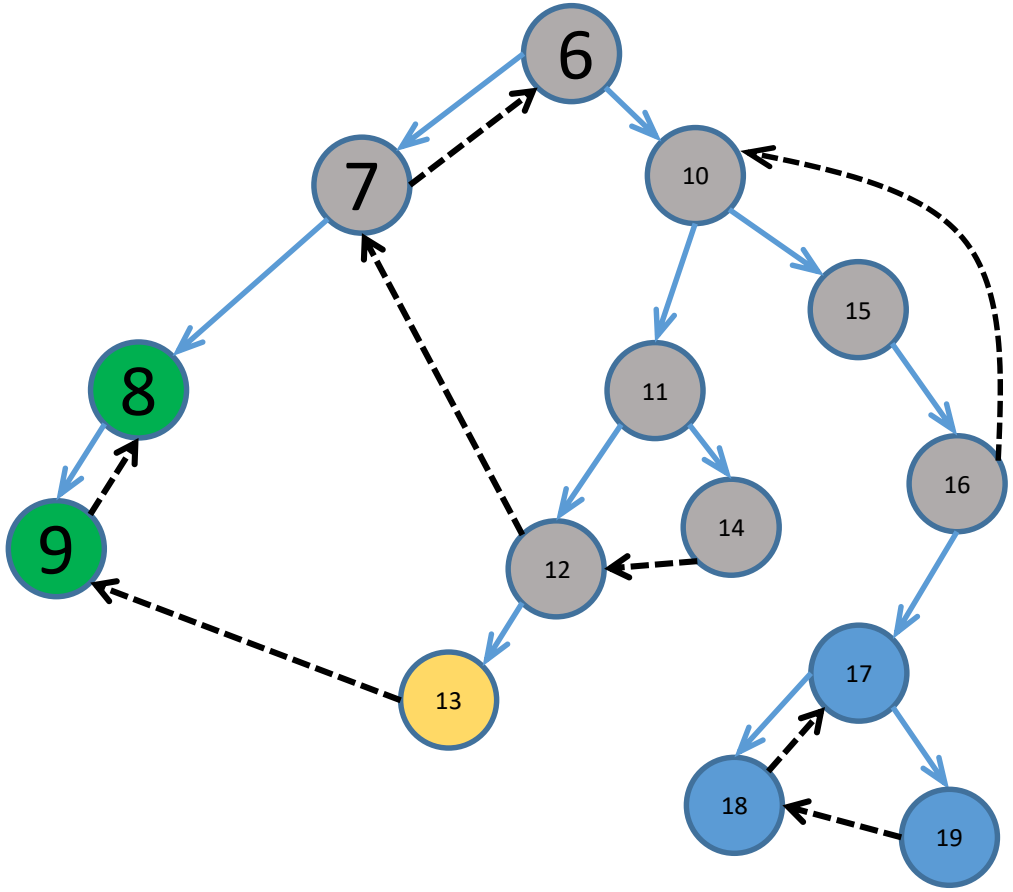

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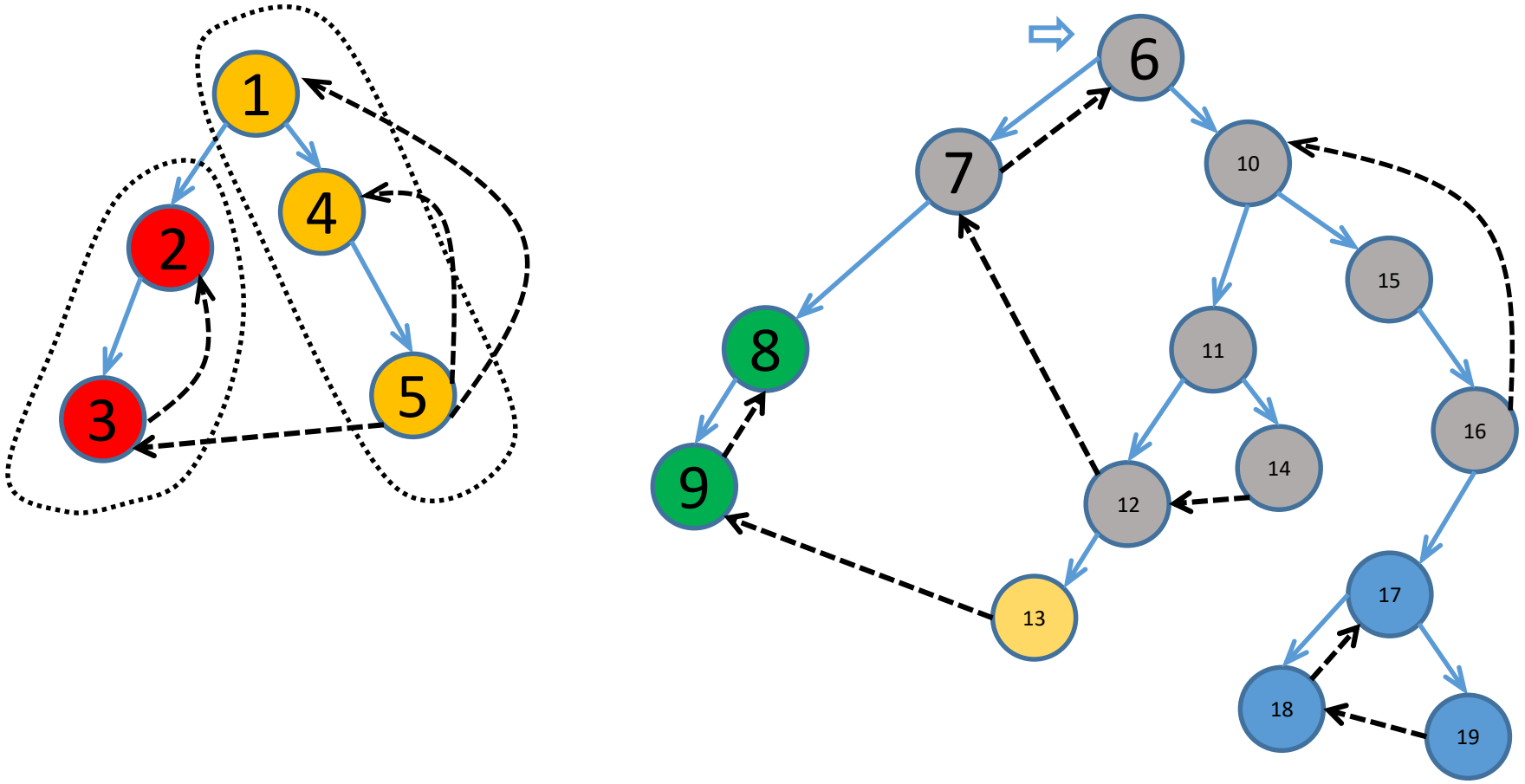

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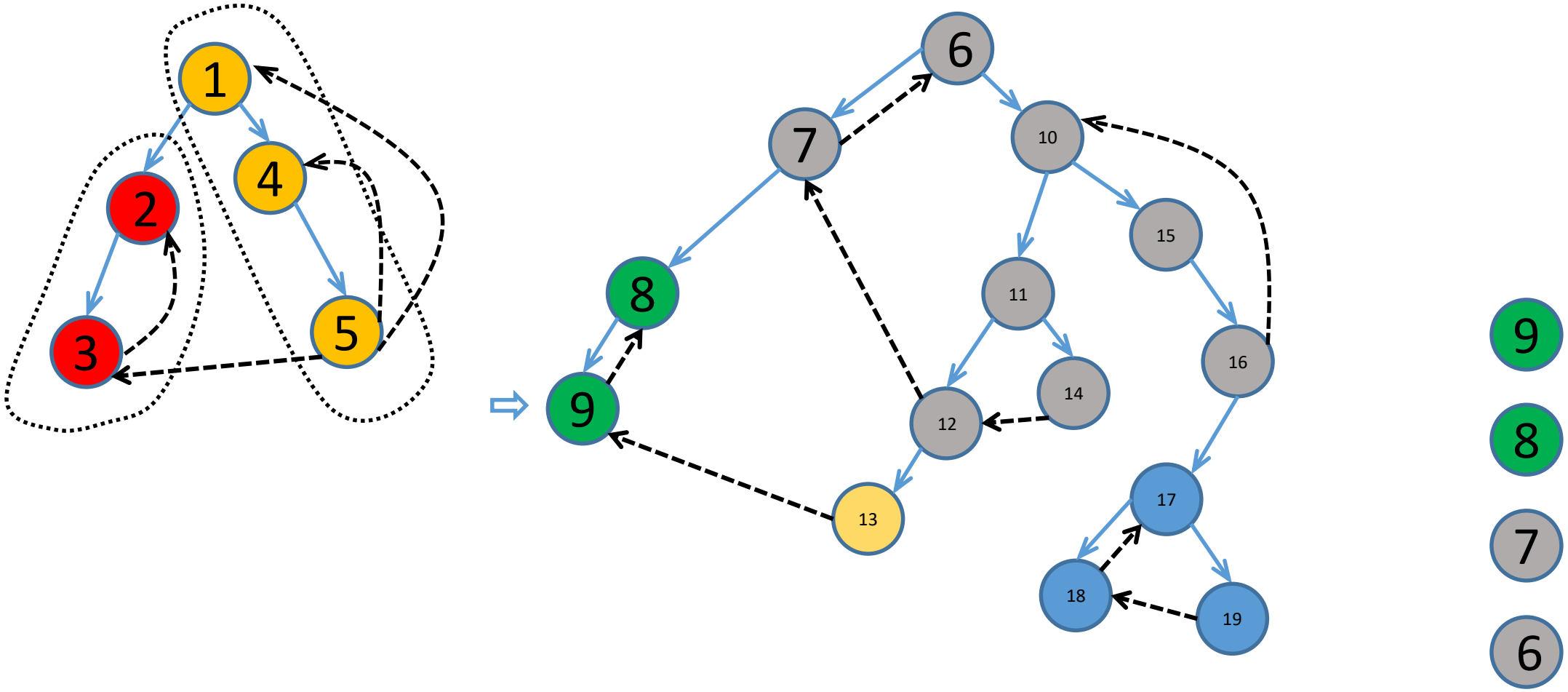

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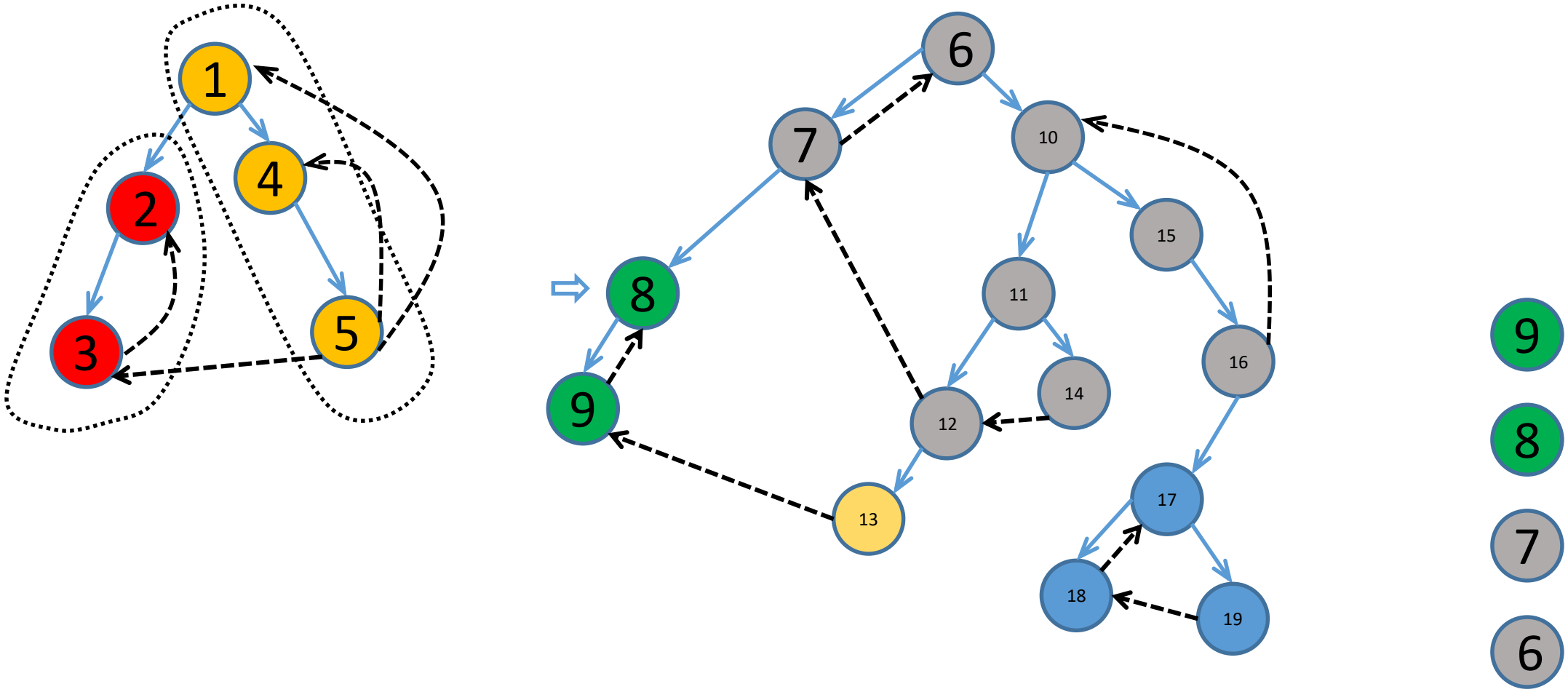

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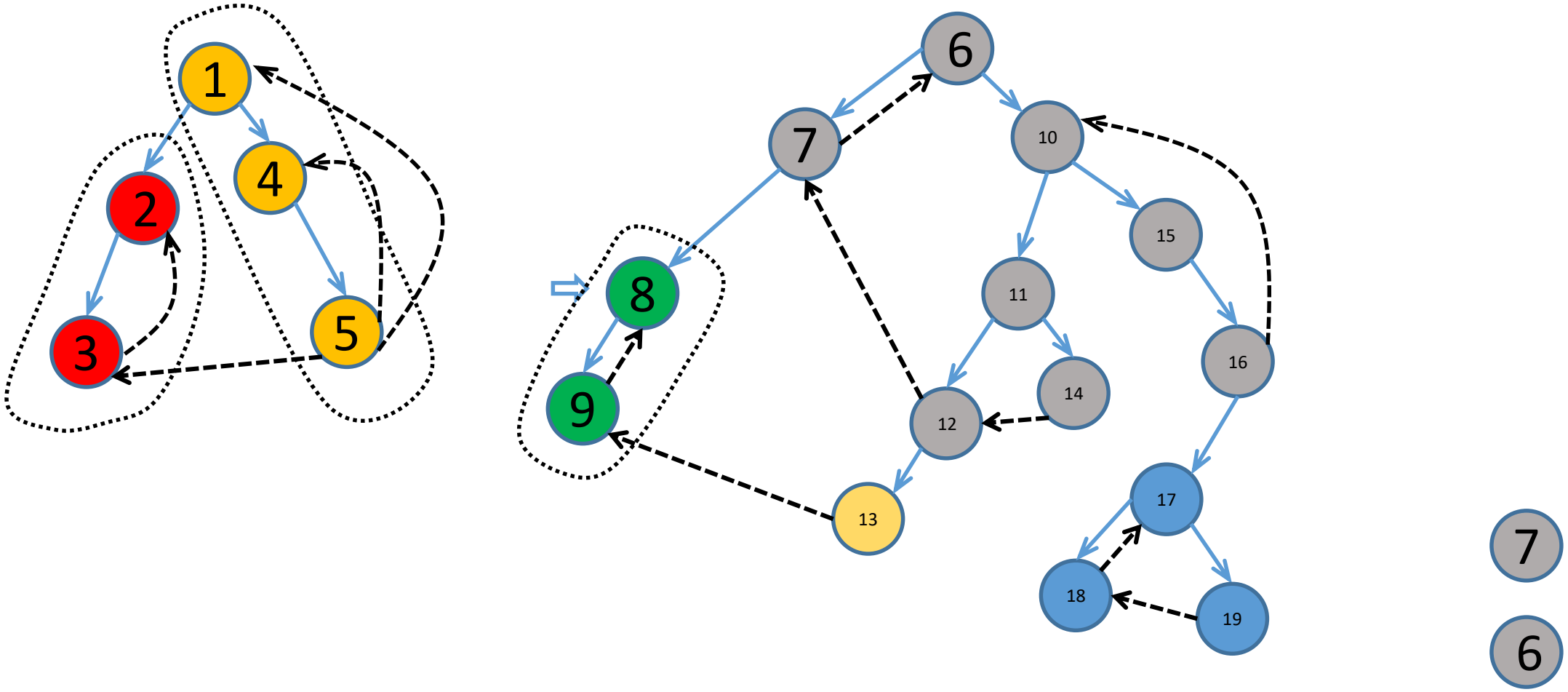

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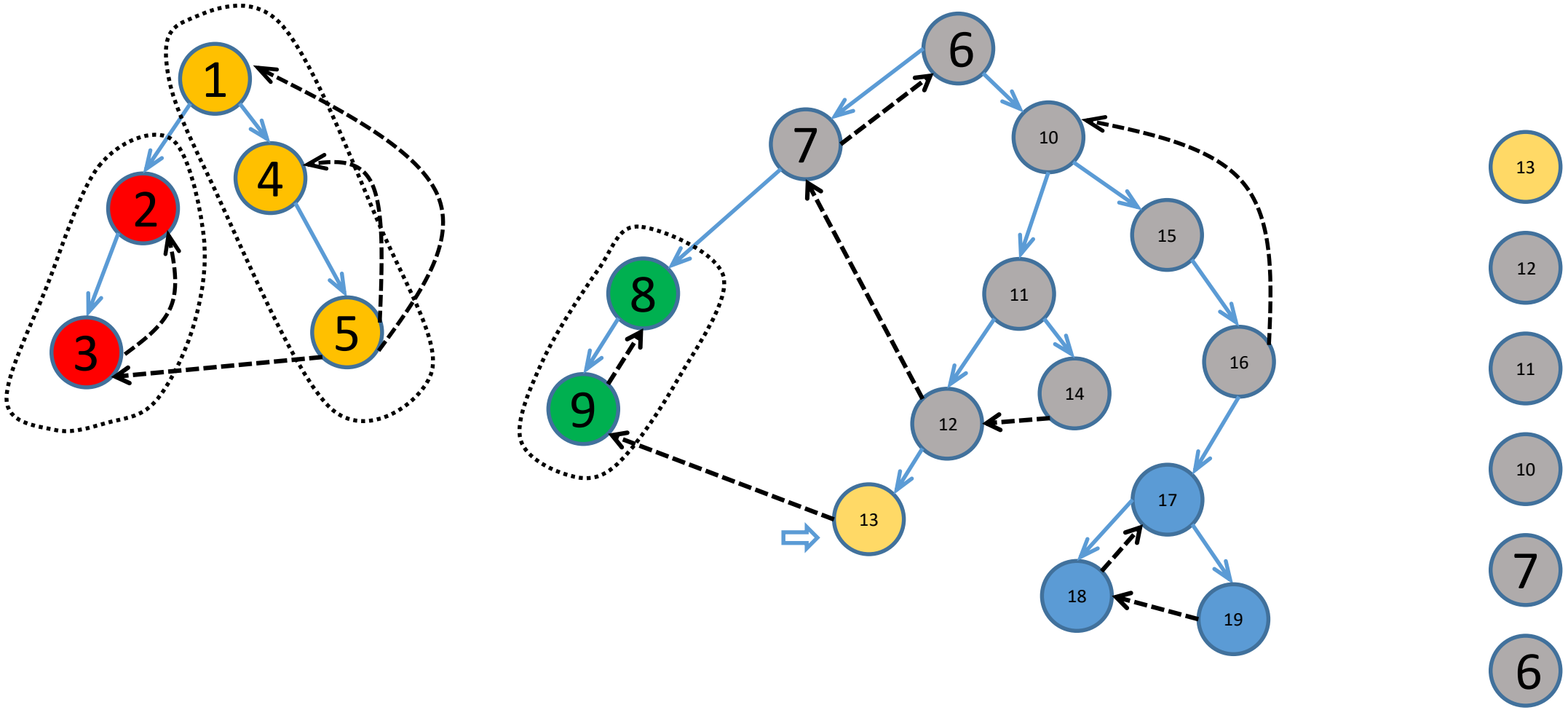

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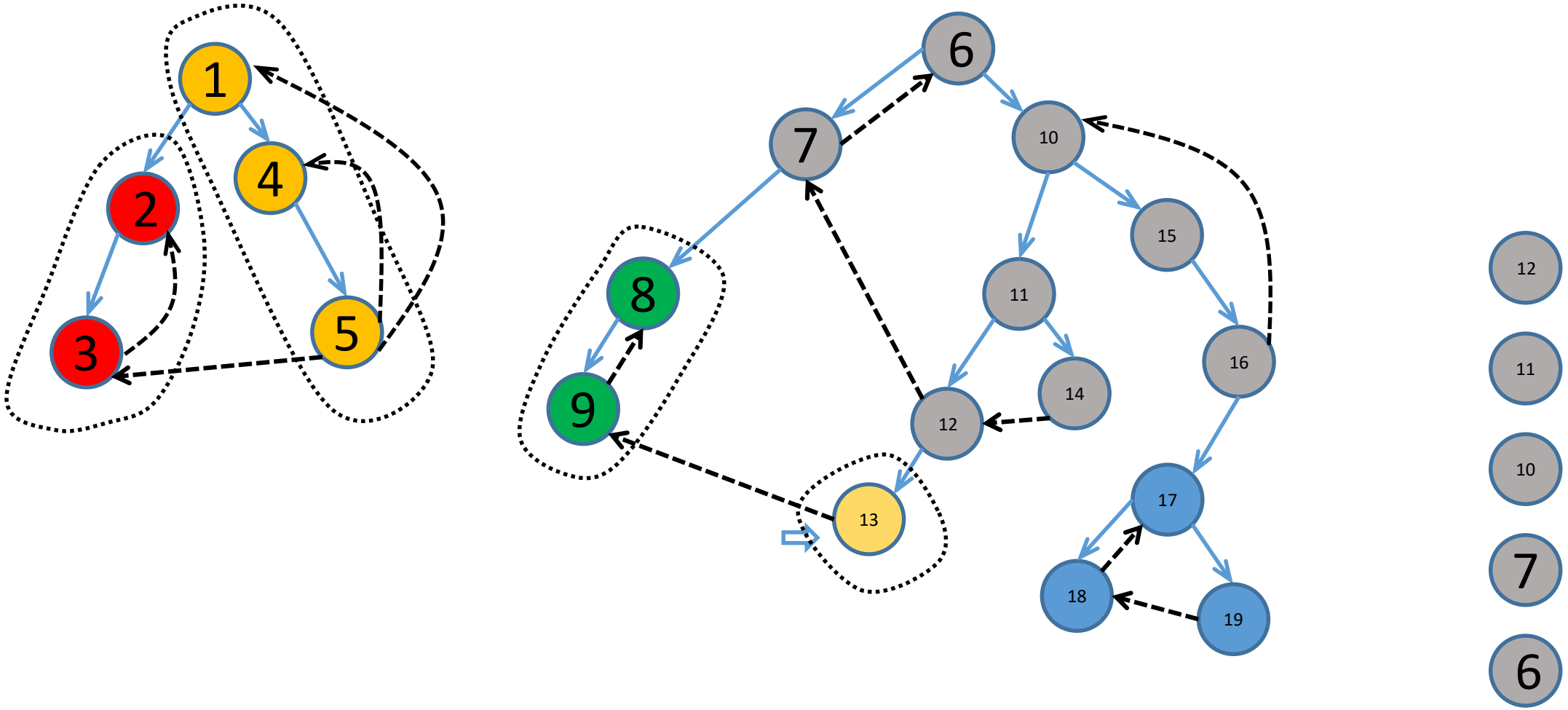

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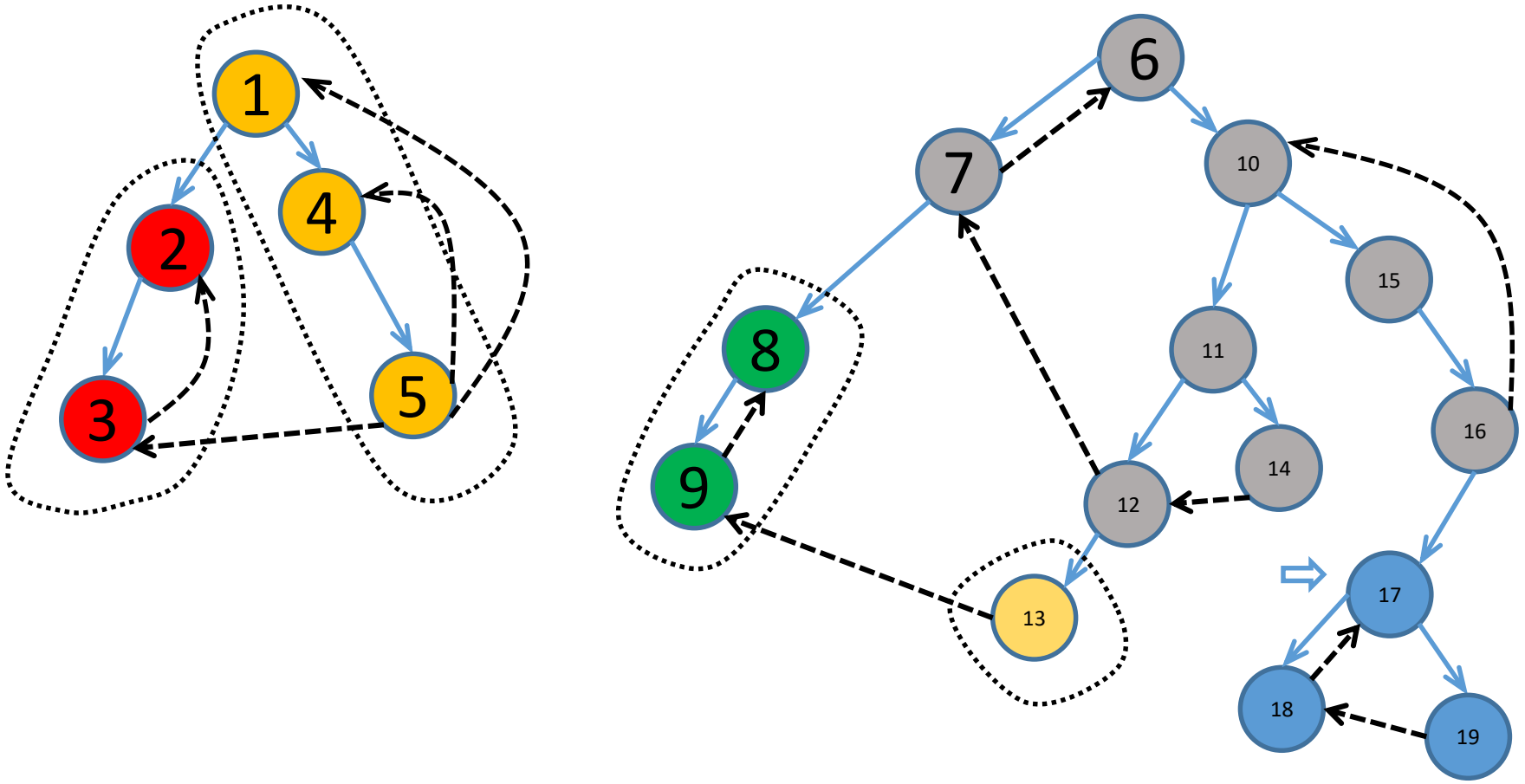

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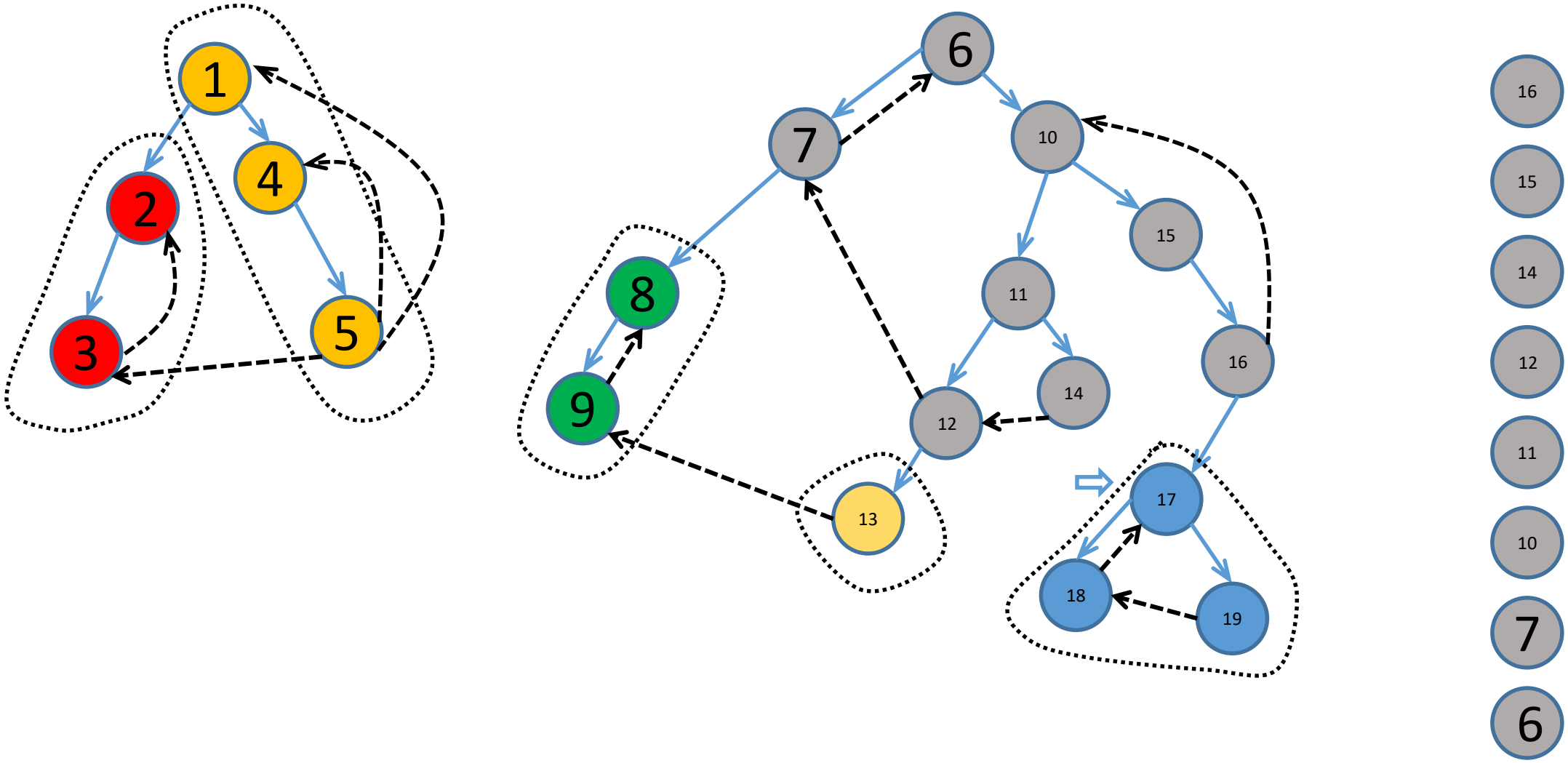

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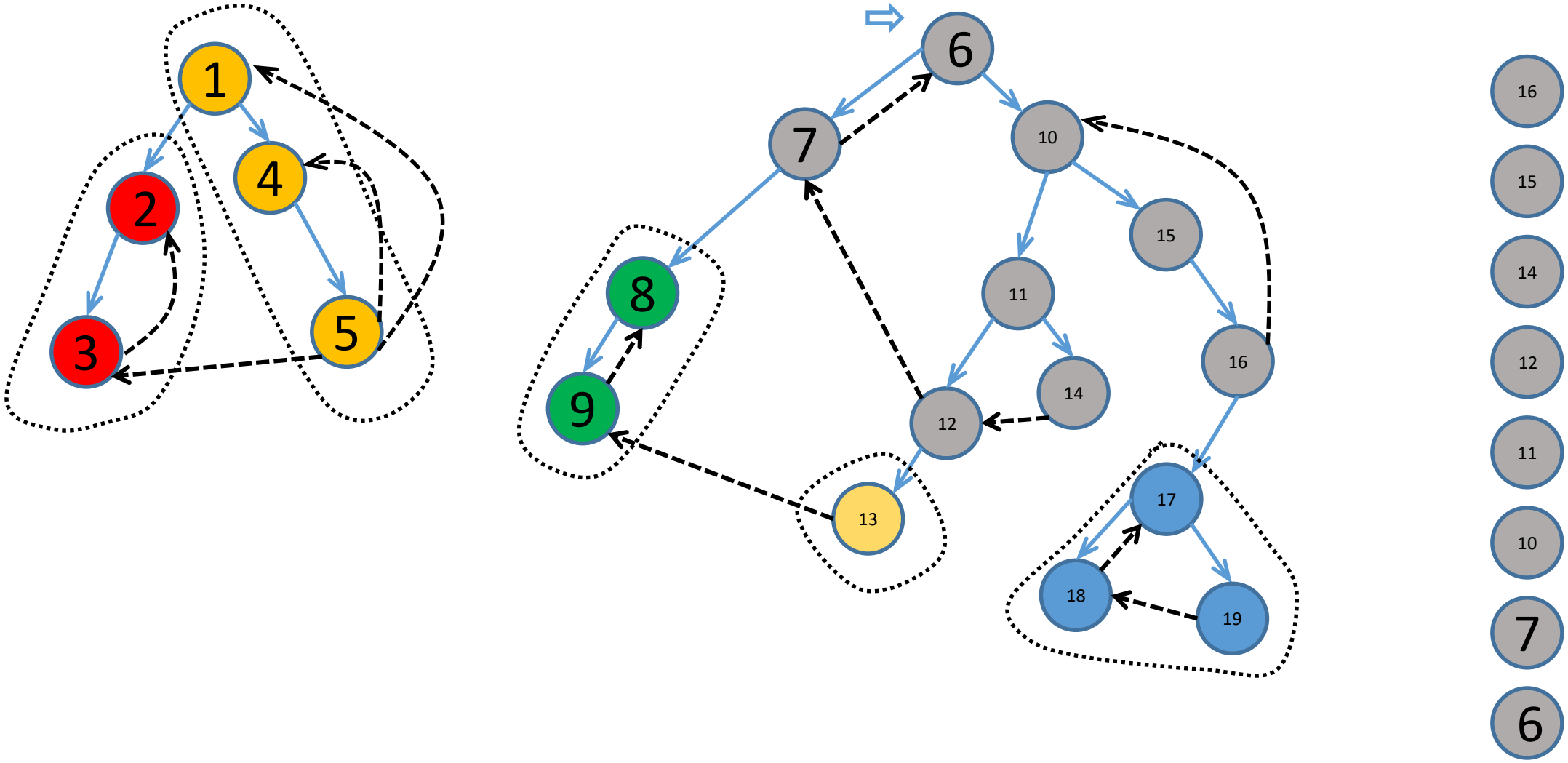

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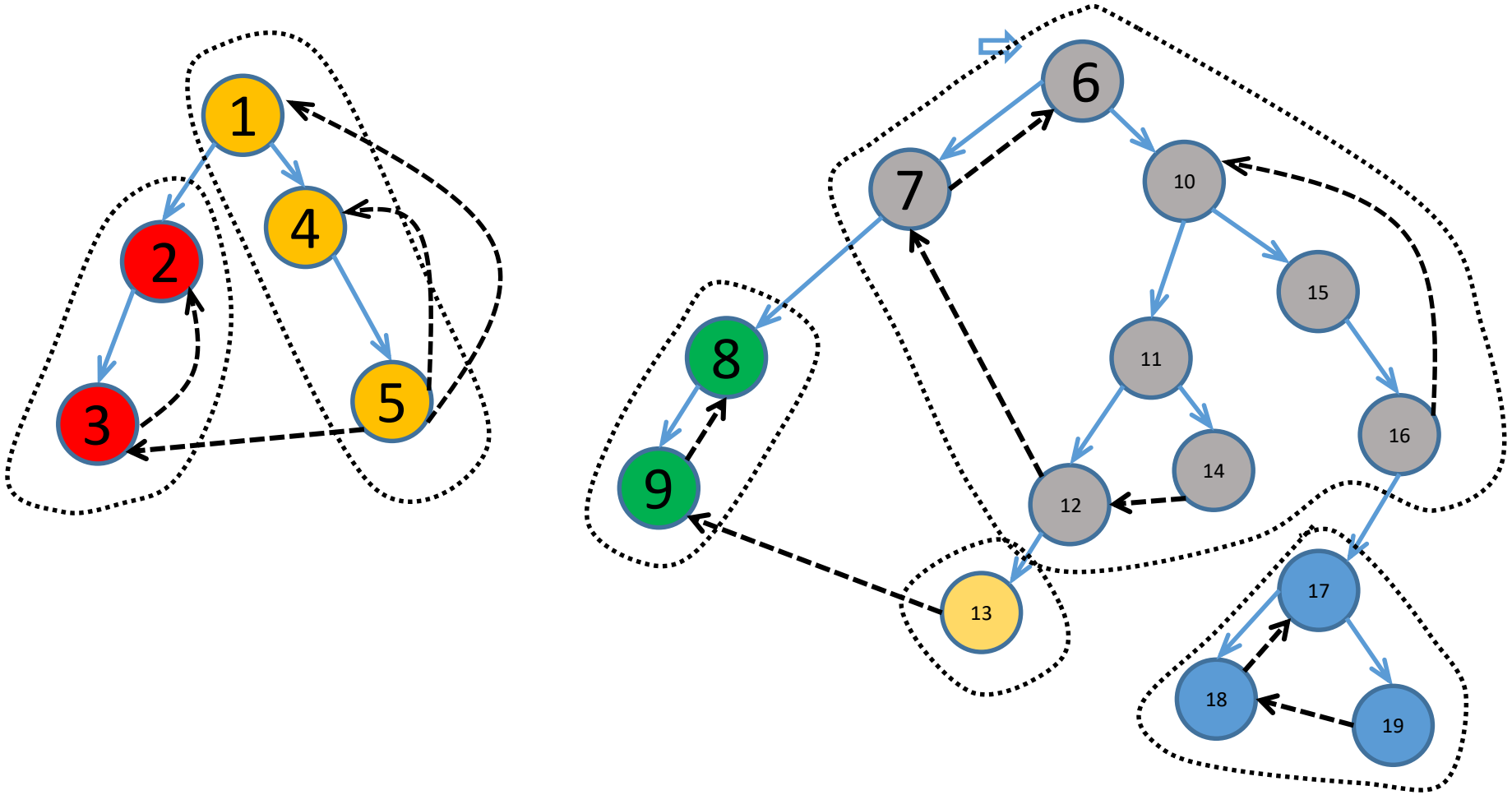


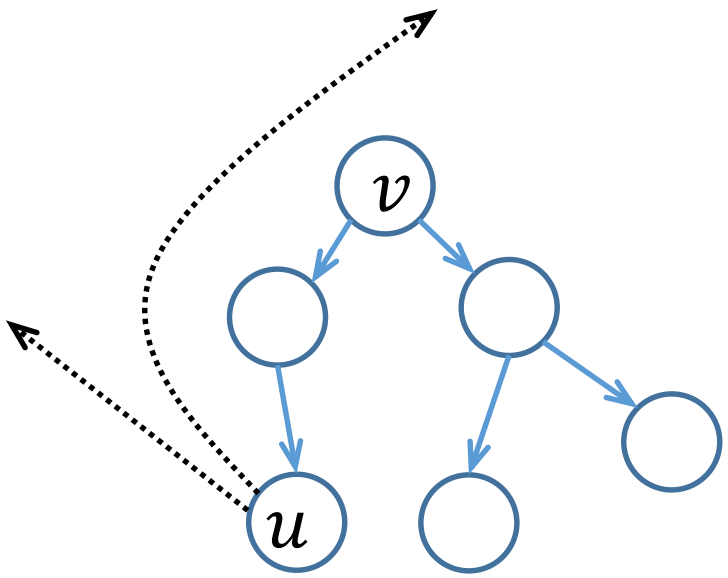
The Stack Invariant

- Let $C_1, C_2, \dots, C_i$ be the SC that intersect the path of the GRAY DFS vertices ordered top down.
- Then S contains the vertices of each C_j that we have already visited
- The vertices of C_1 are at the bottom of S and $\forall j > 1$ the vertices of C_j are consecutive and on top of the vertices of C_{j-1}
- $\forall j$ the root of C_j is the bottommost vertex among all the vertices of C_j

- Proof by induction on the steps of the algorithm

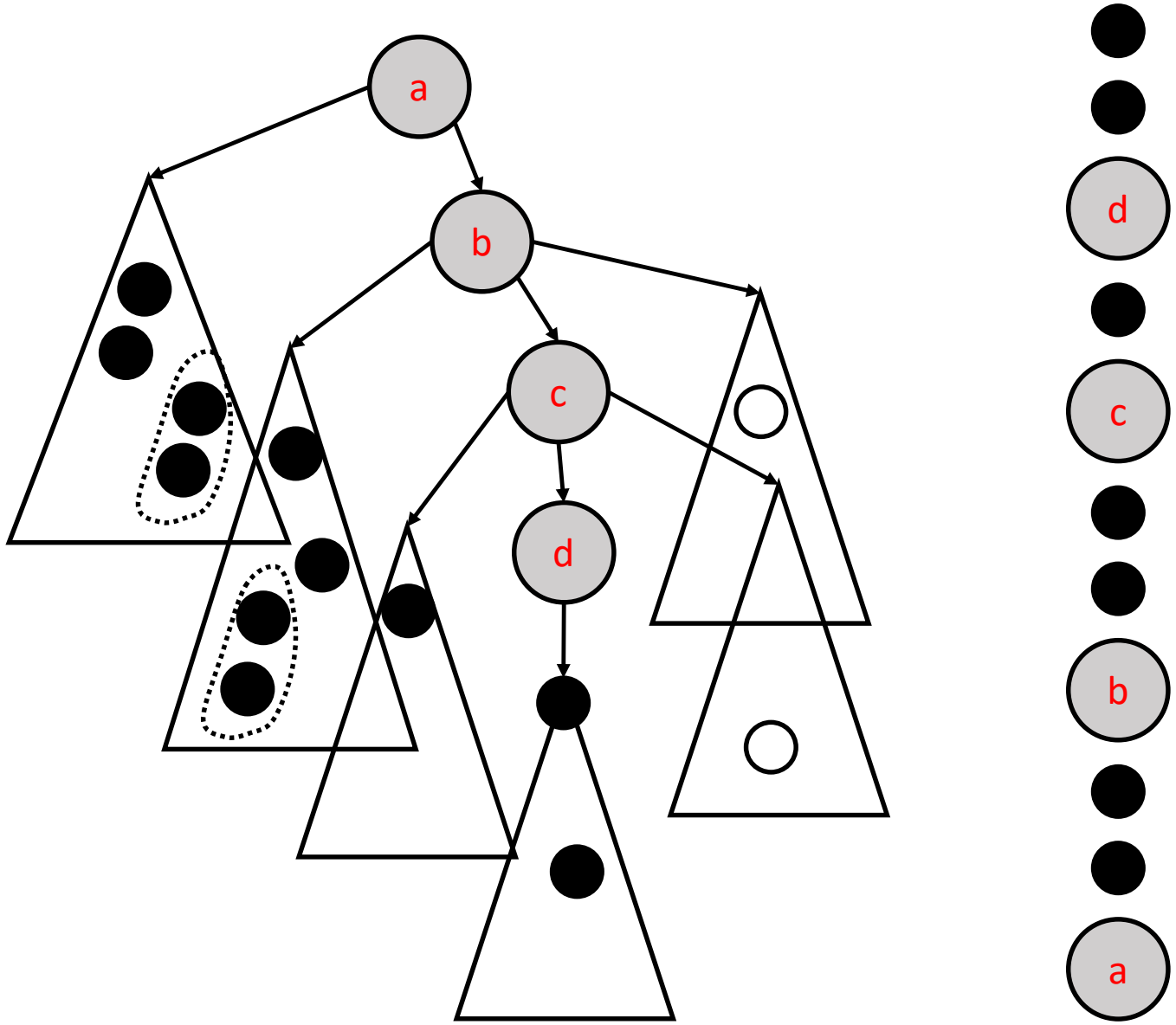
Characterize these roots ?

Thm 10: v is a **root** of a component **iff** when the DFS backtracks from v no back-edges or cross-edges going out of its subtree (in the DFS-forest) to vertices still in the stack.



Proof: Assume we apply this rule then the following Invariants hold:

- The components we have already identified are correct
- **If we backtracked from v and v is not a root then v is in the same component as its lowest GRAY ancestor**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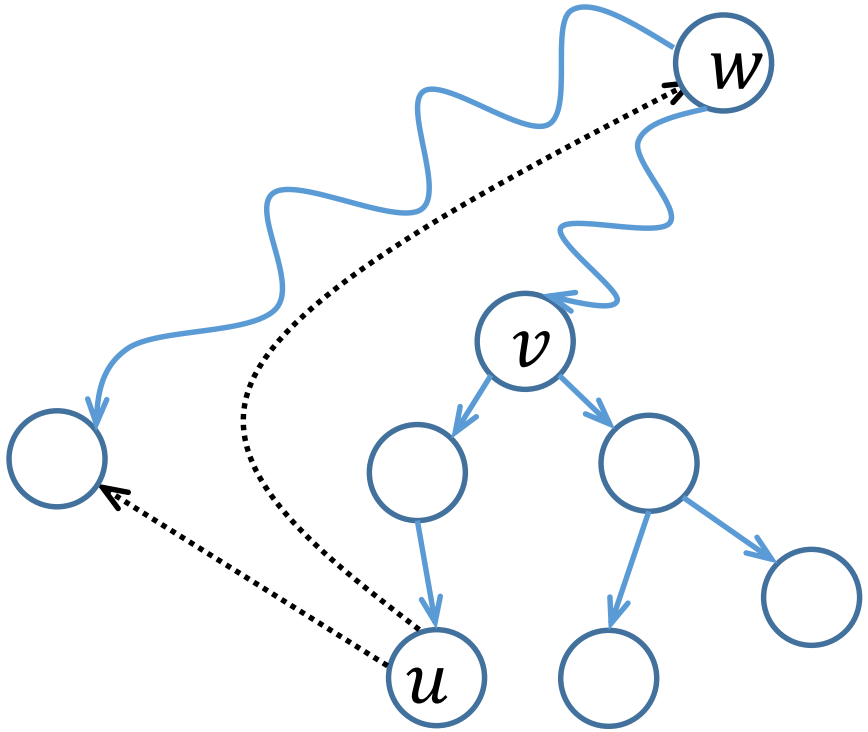
How do we identify roots ?

For each vertex v compute $low(v)$ = the smallest $w.d$ such that there is a cross edge or back edge to $w \in \boldsymbol{S}$ out of the subtree of v.
If there is no such edge then $low(v) \ = \ v.d$



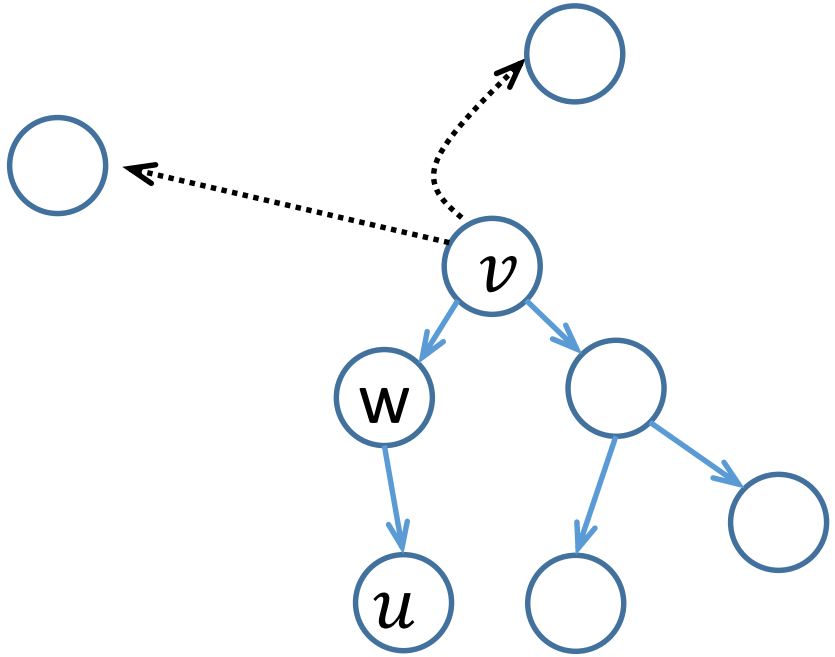
If $low(v) \ = \ v.d$
then v is the root of a component

How to compute low values ?

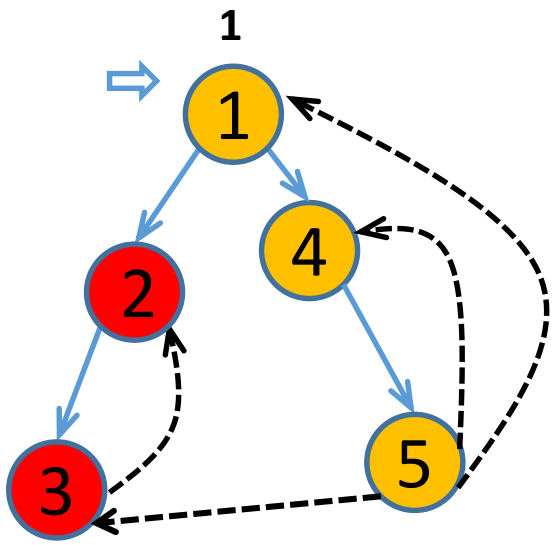
Initialize $low(v) = v.d$

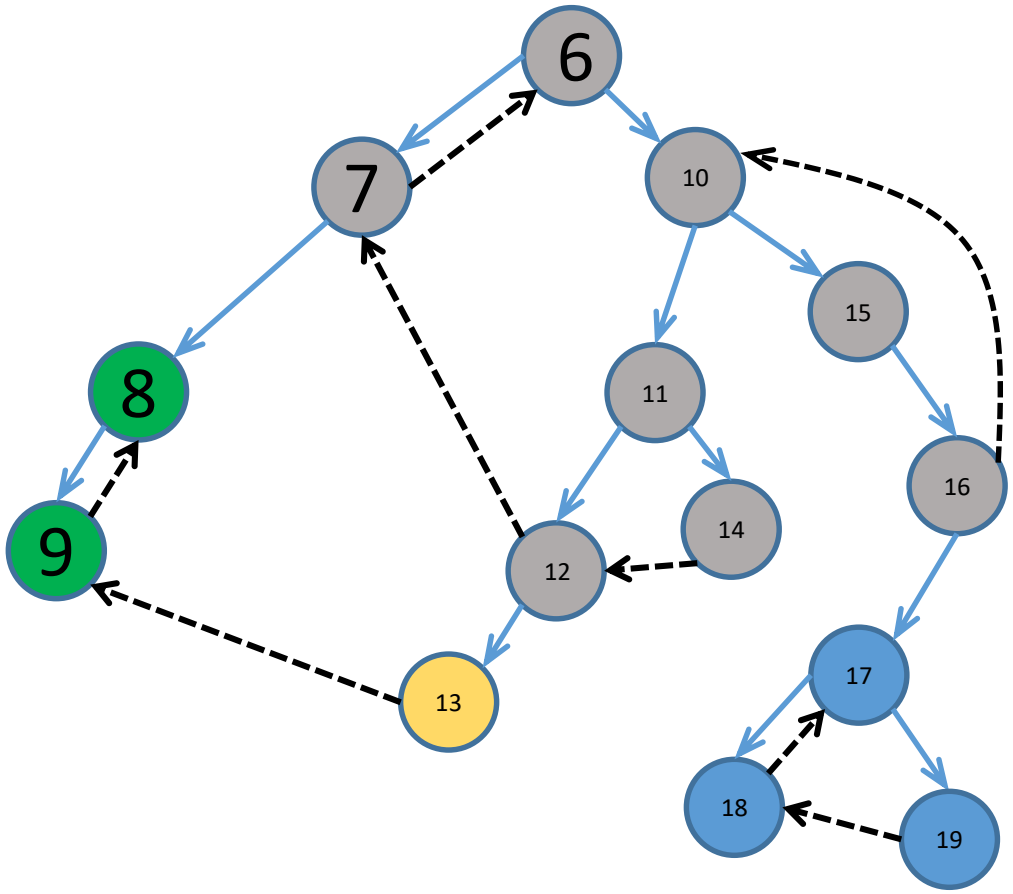
Update when you see a back edge or cross edge out of v

Update when you backtrack to v from a child w of v



Compute SCs using low values





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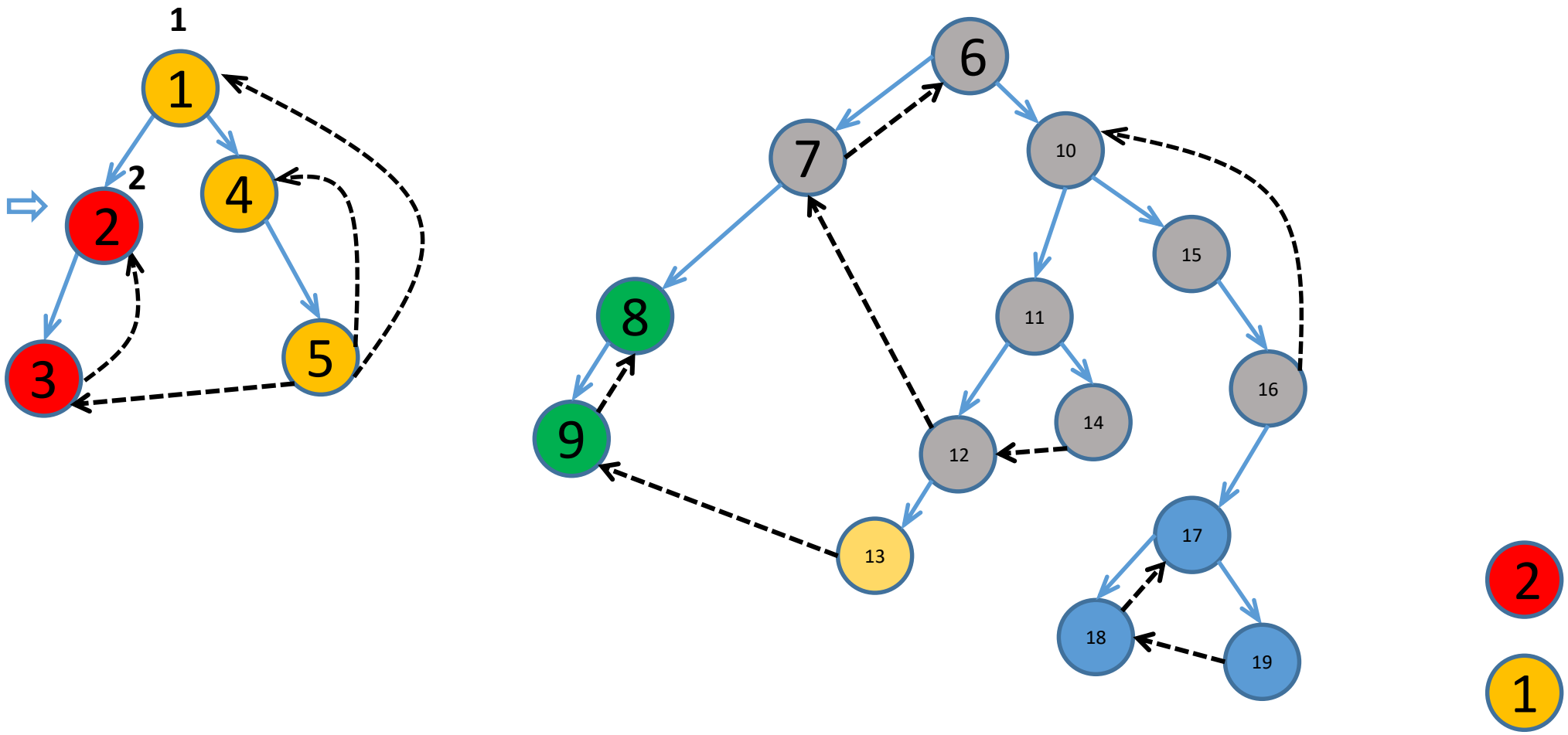

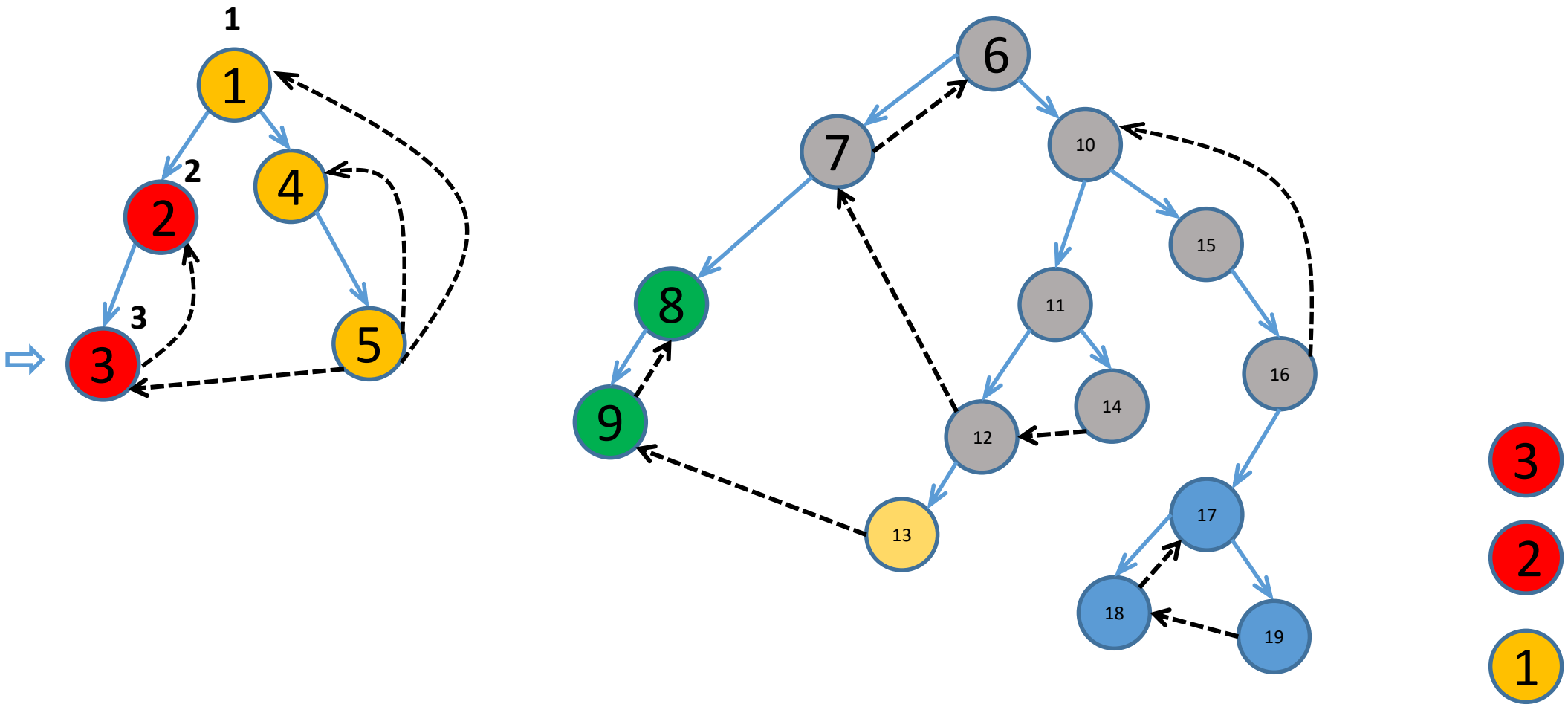
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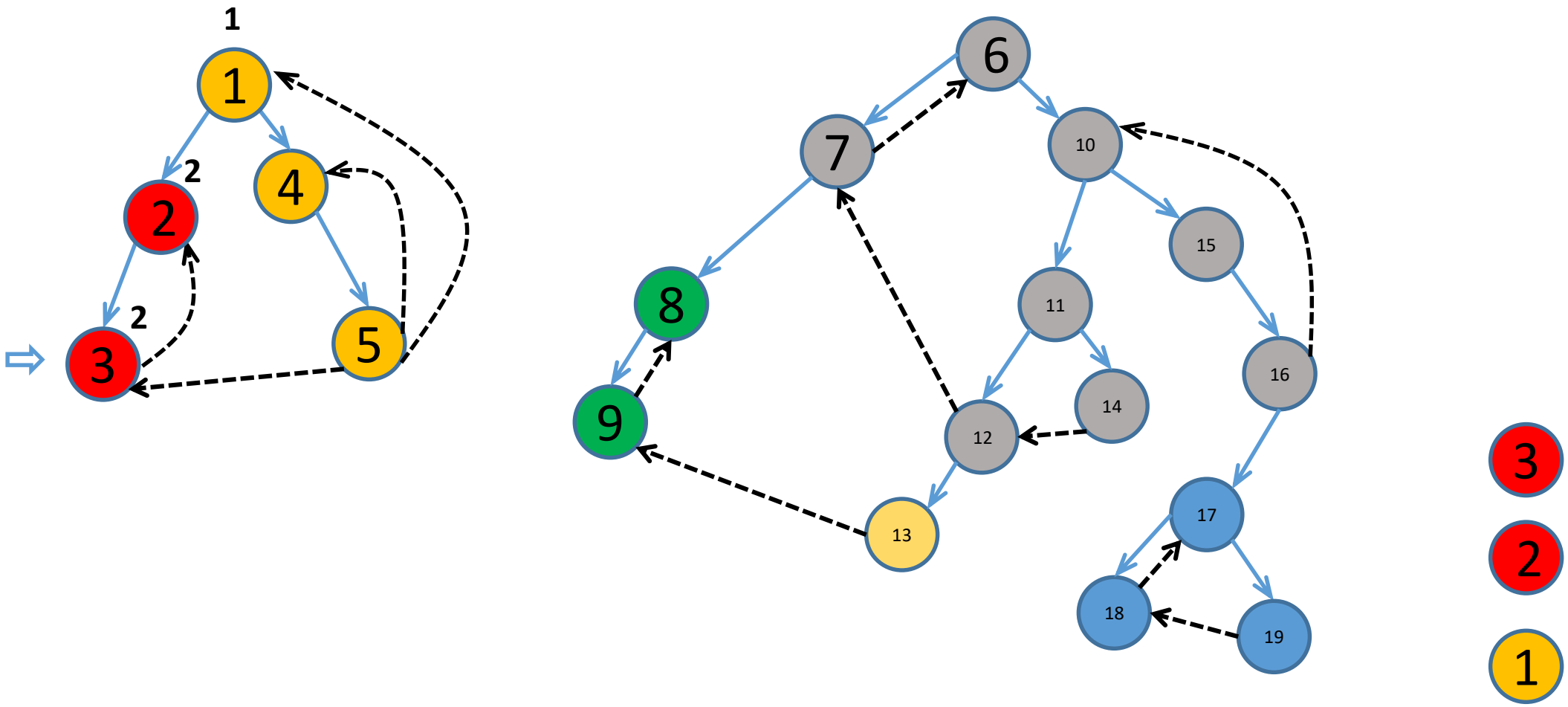
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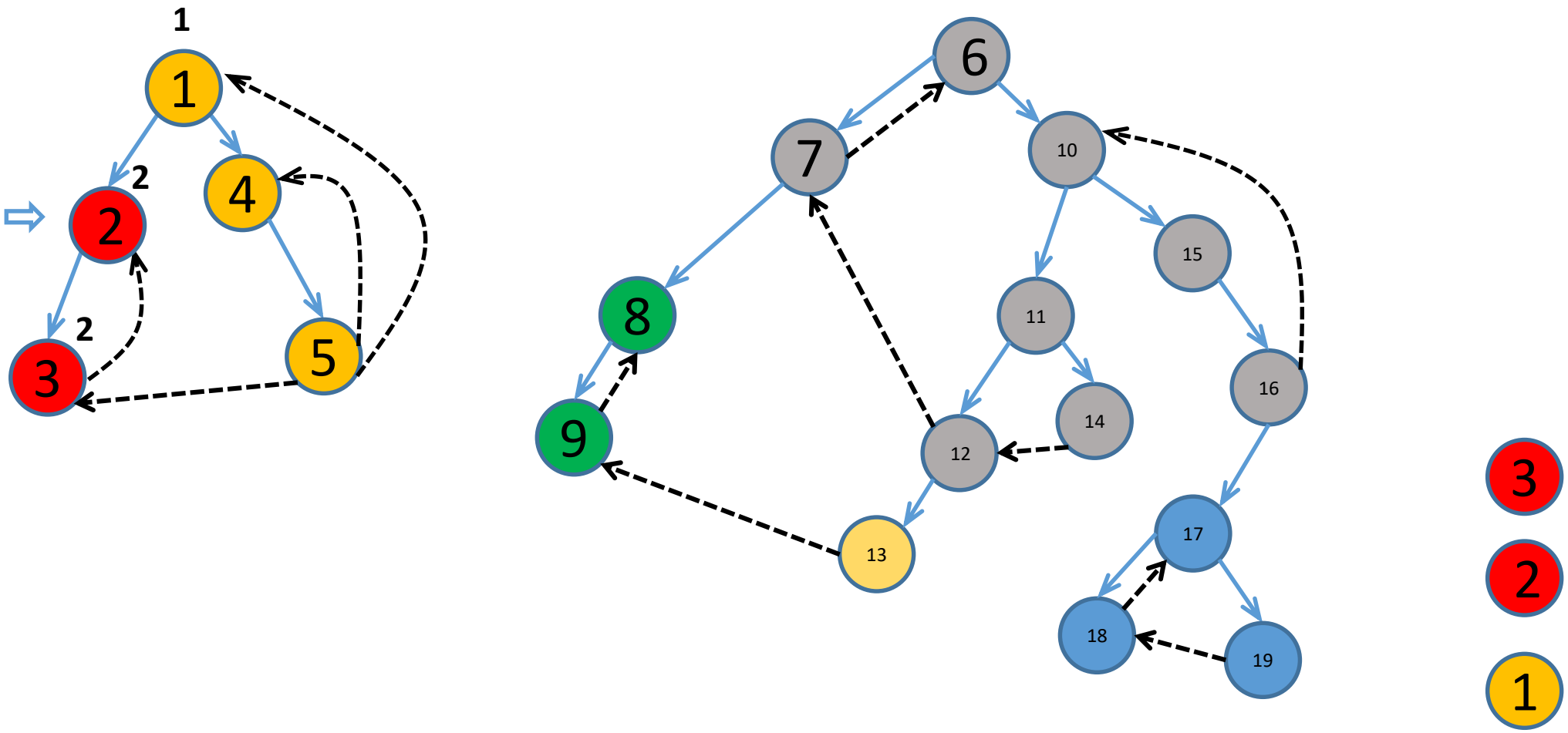
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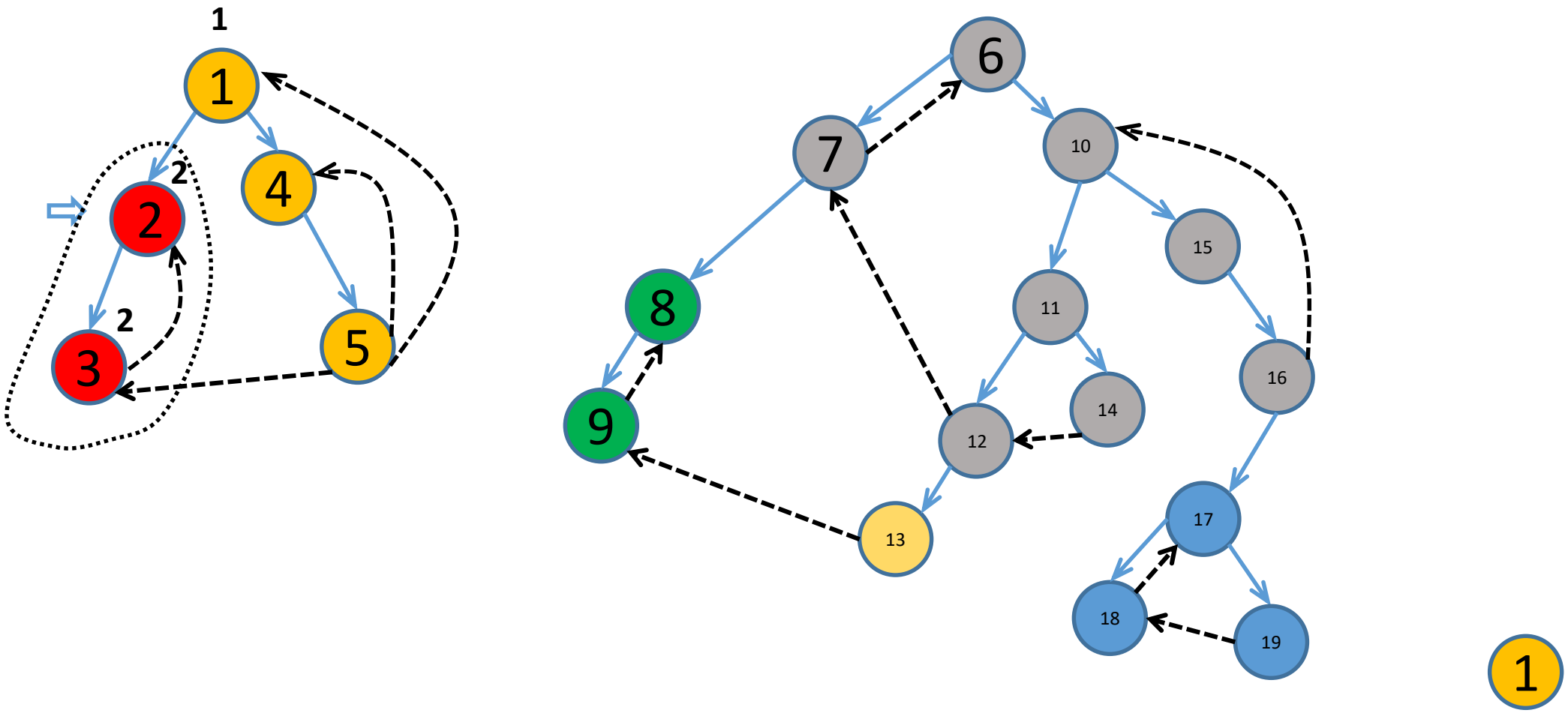
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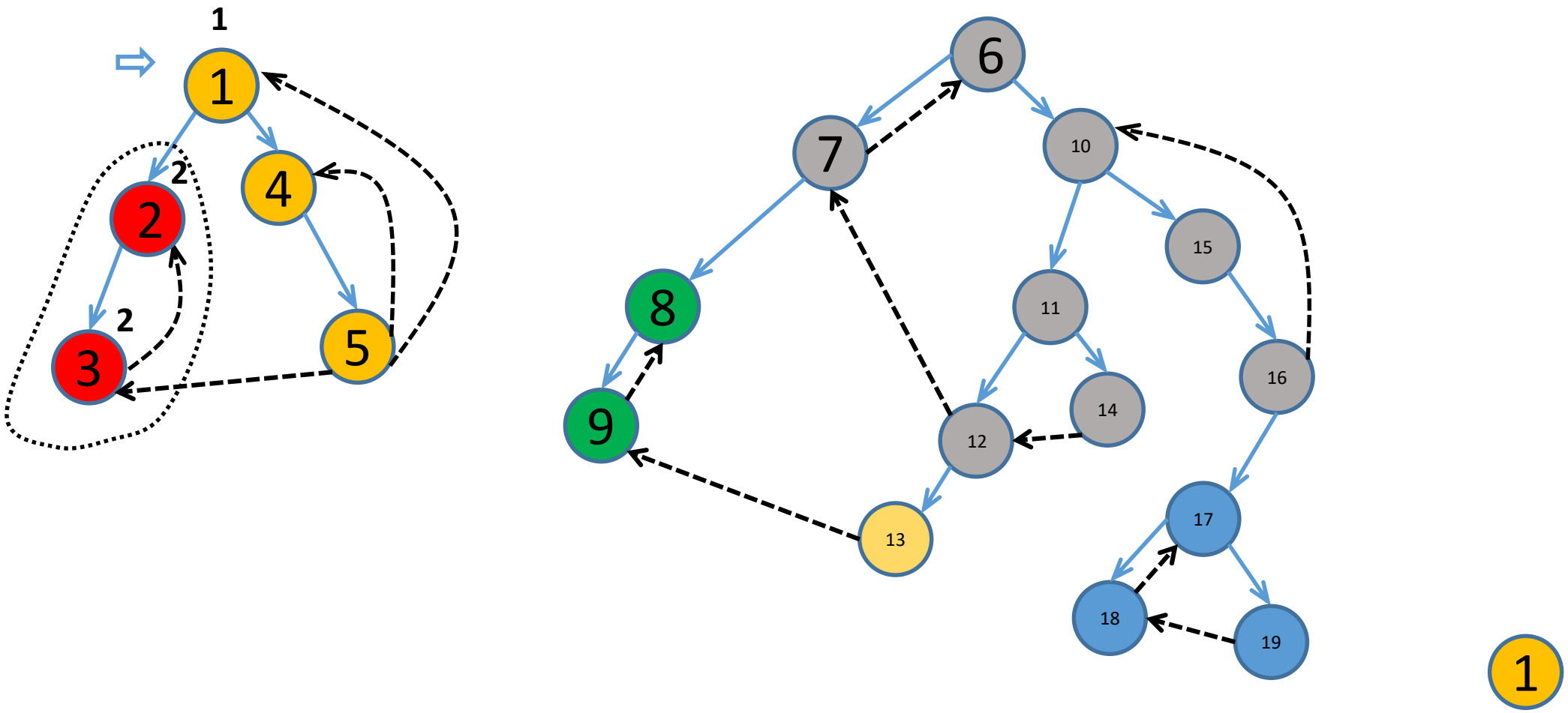
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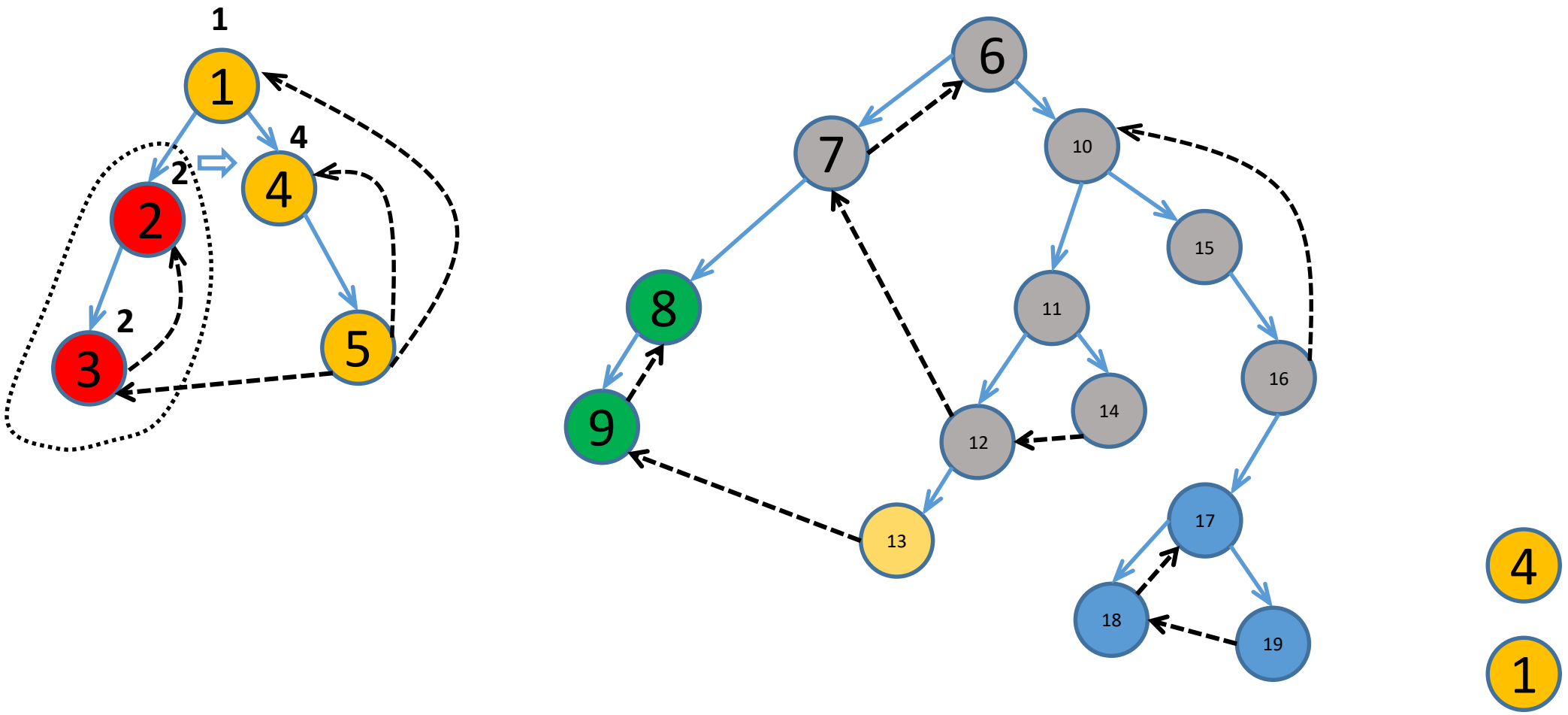
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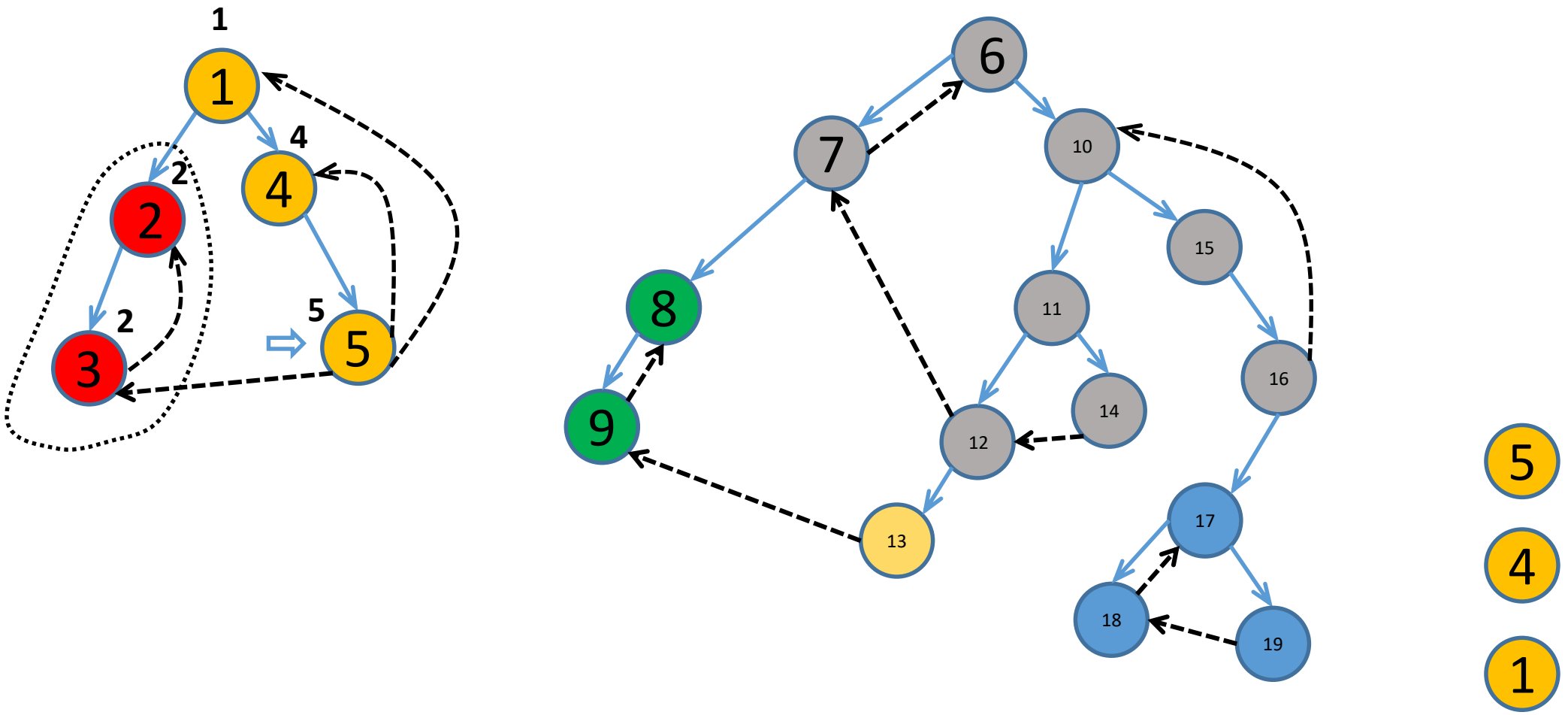
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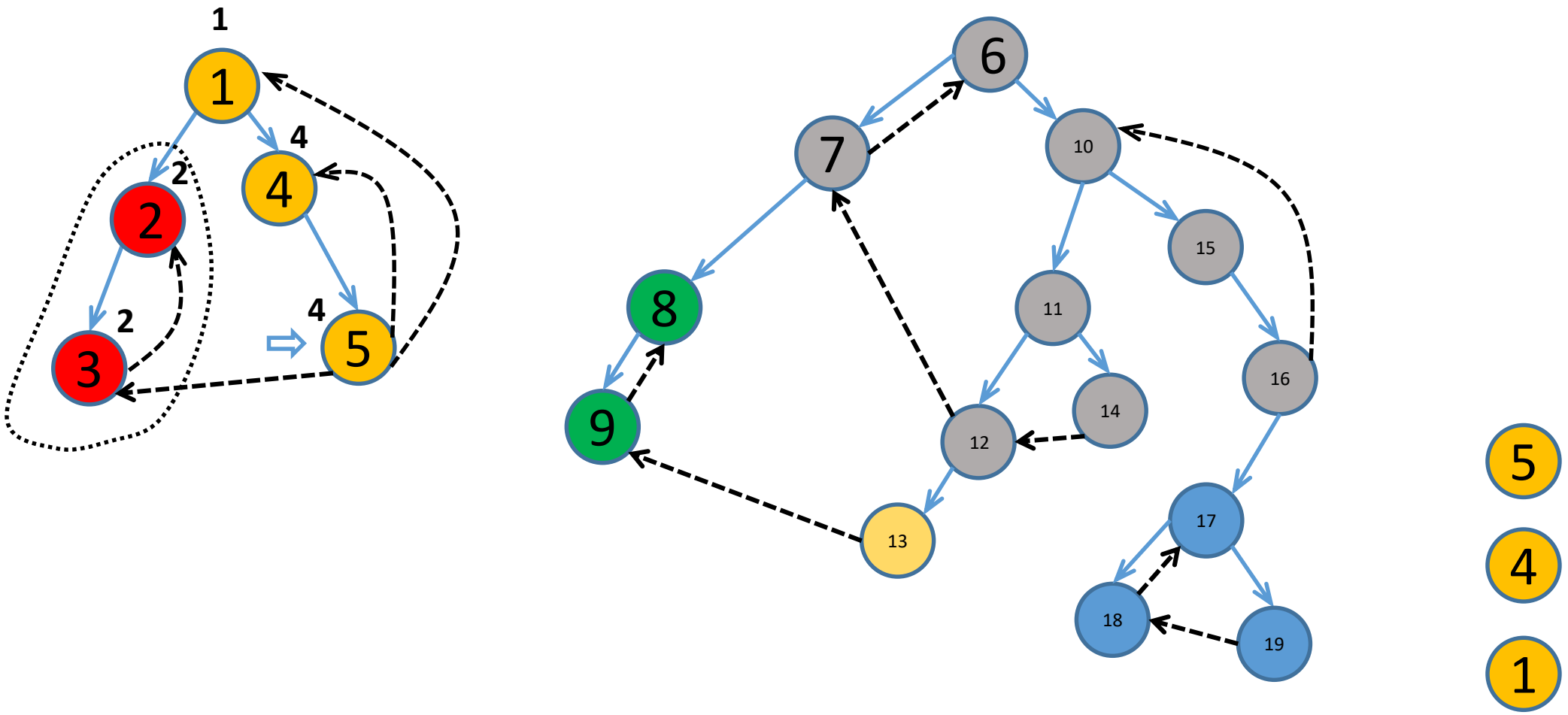
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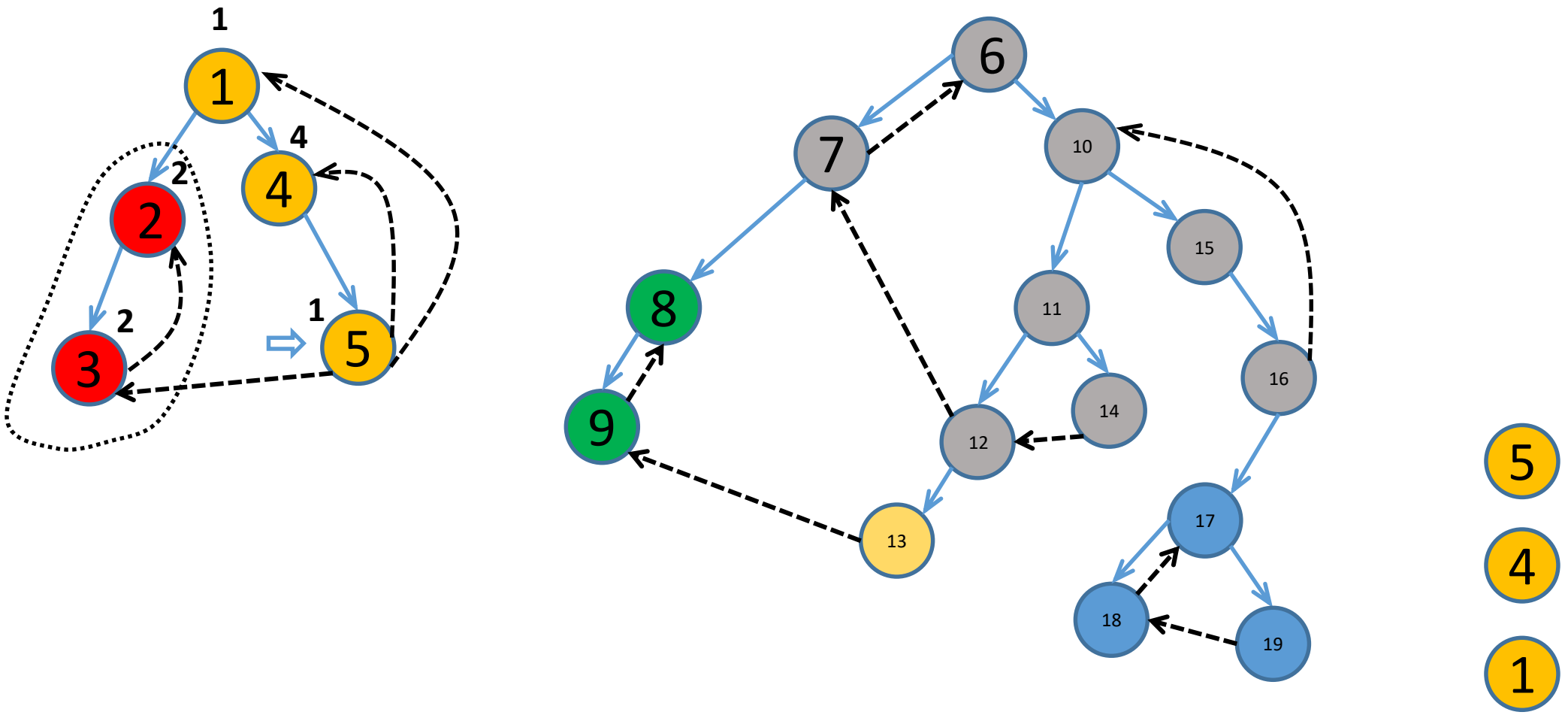
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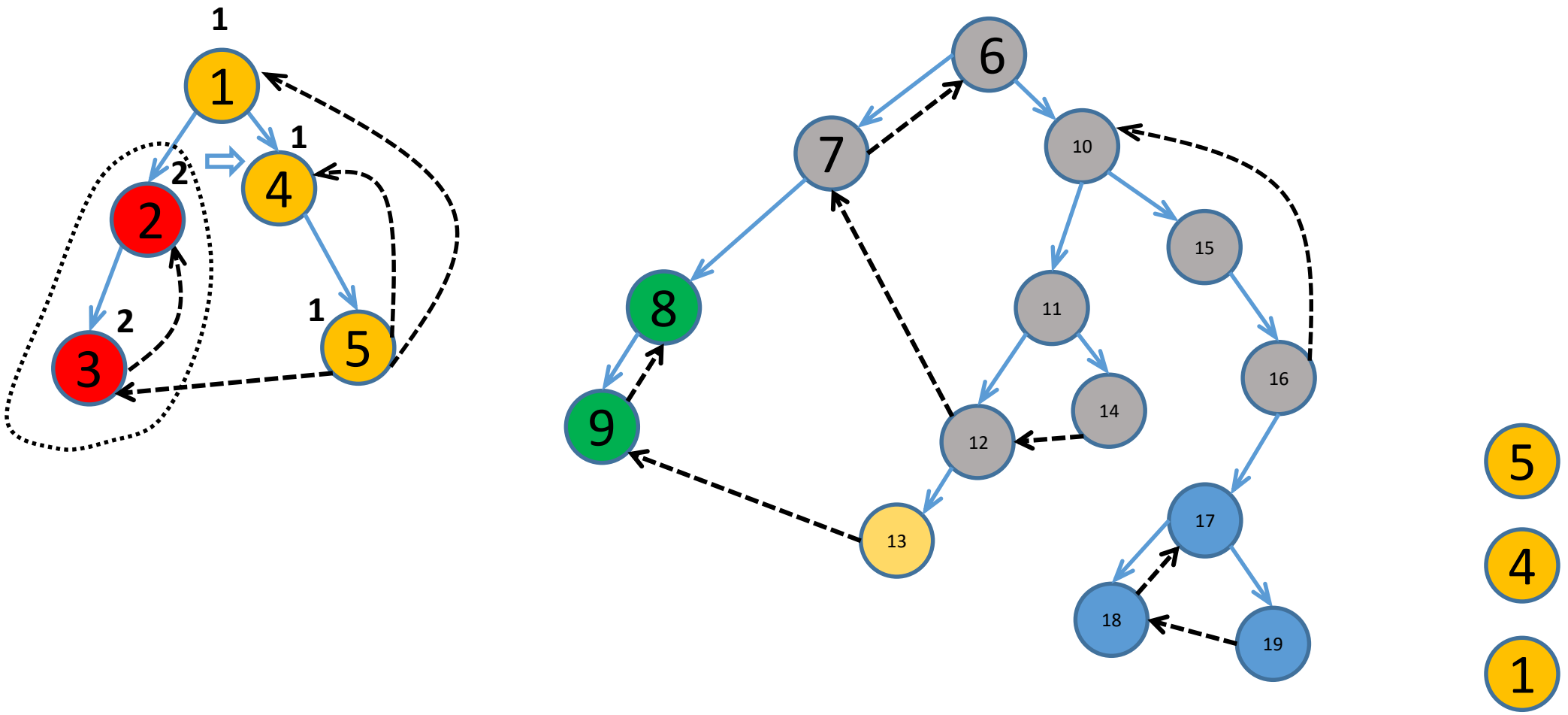
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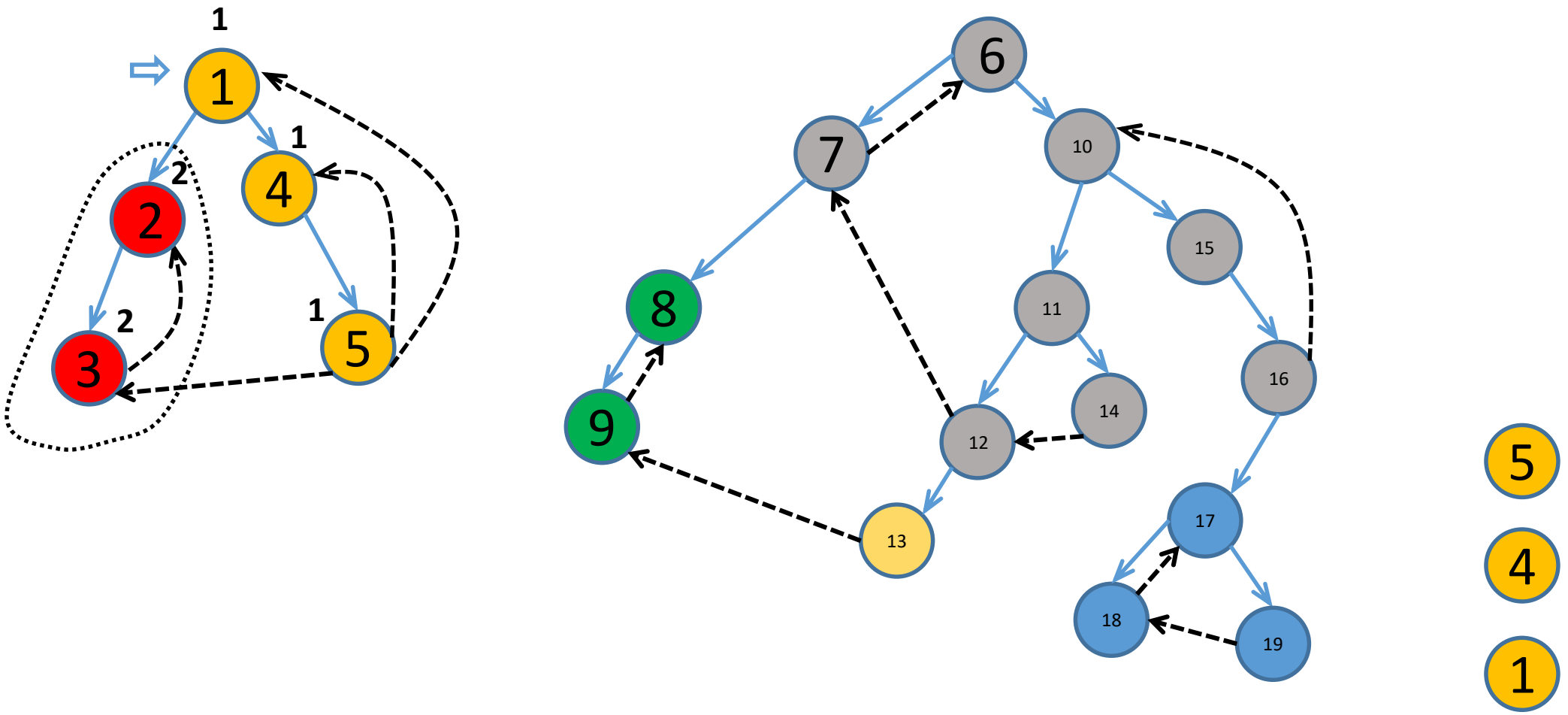
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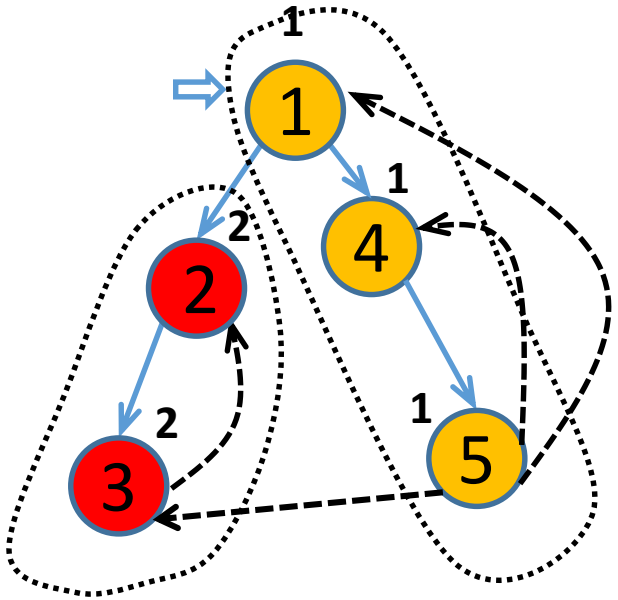
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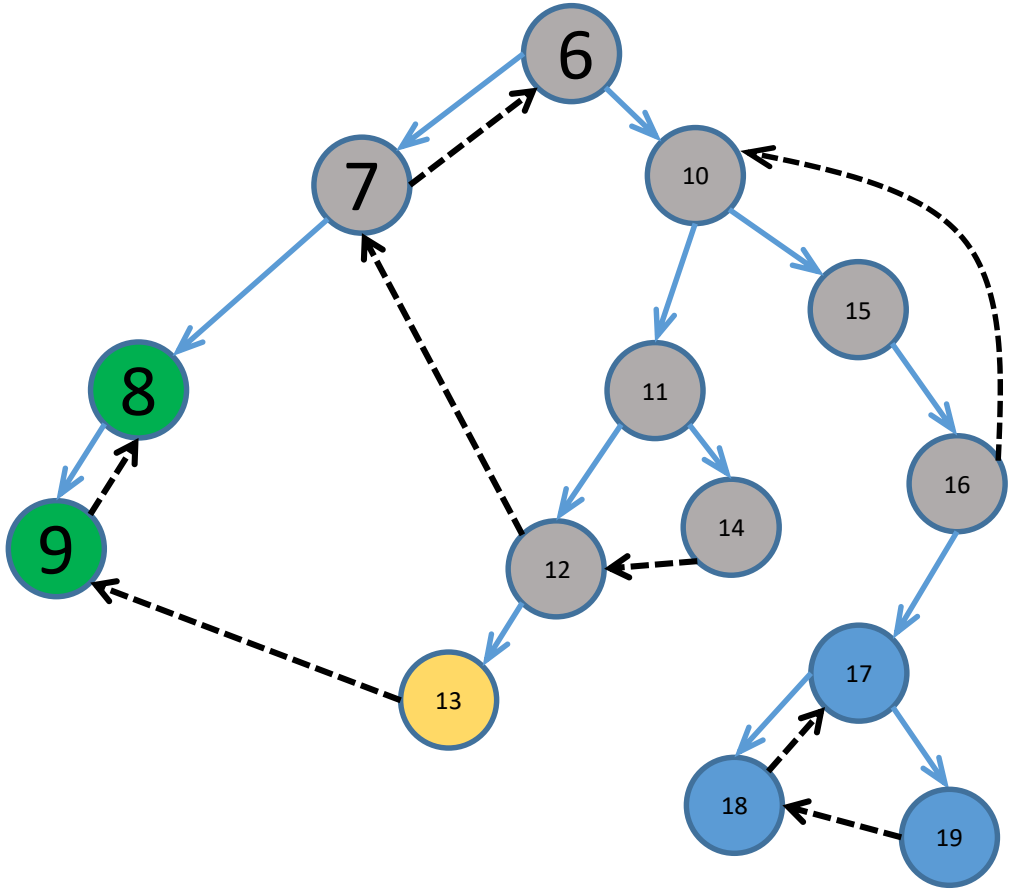
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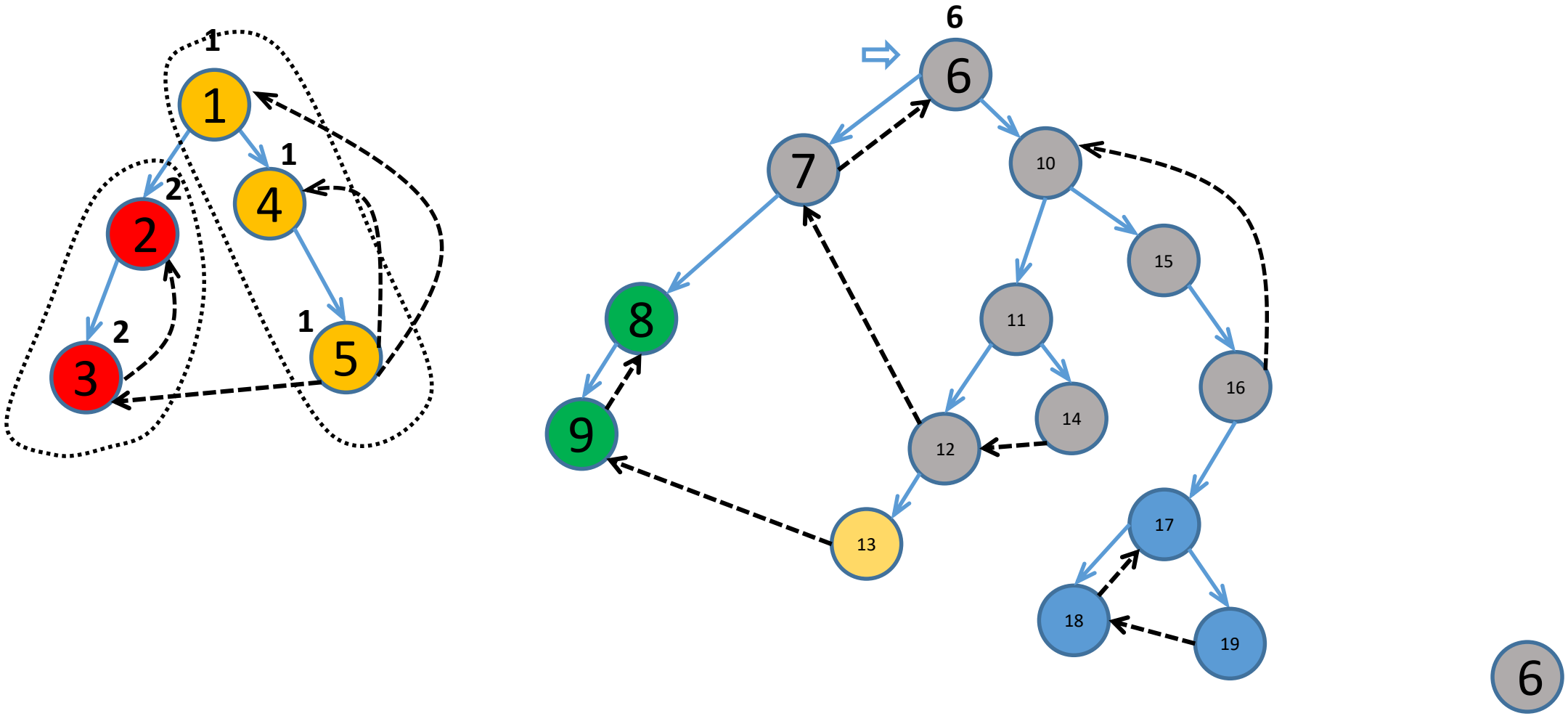
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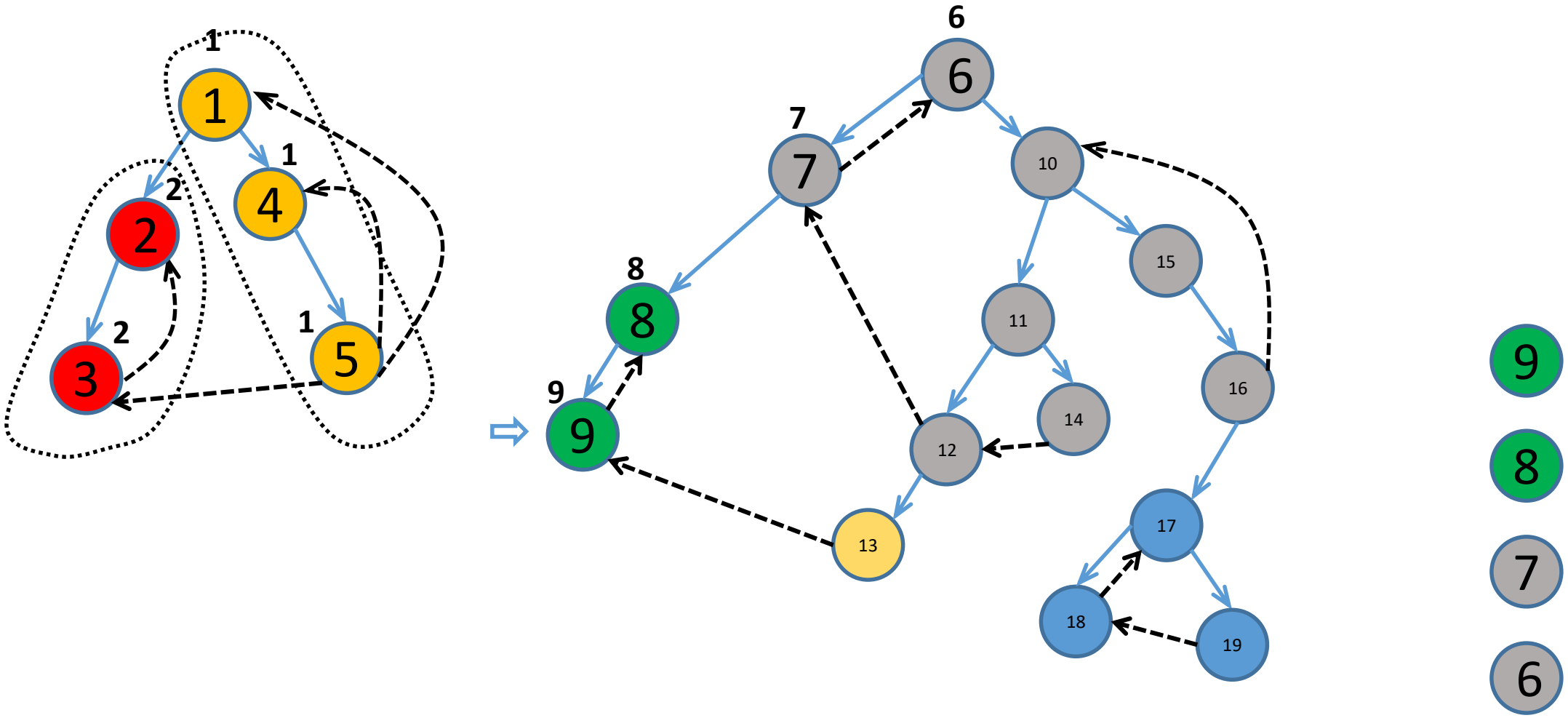
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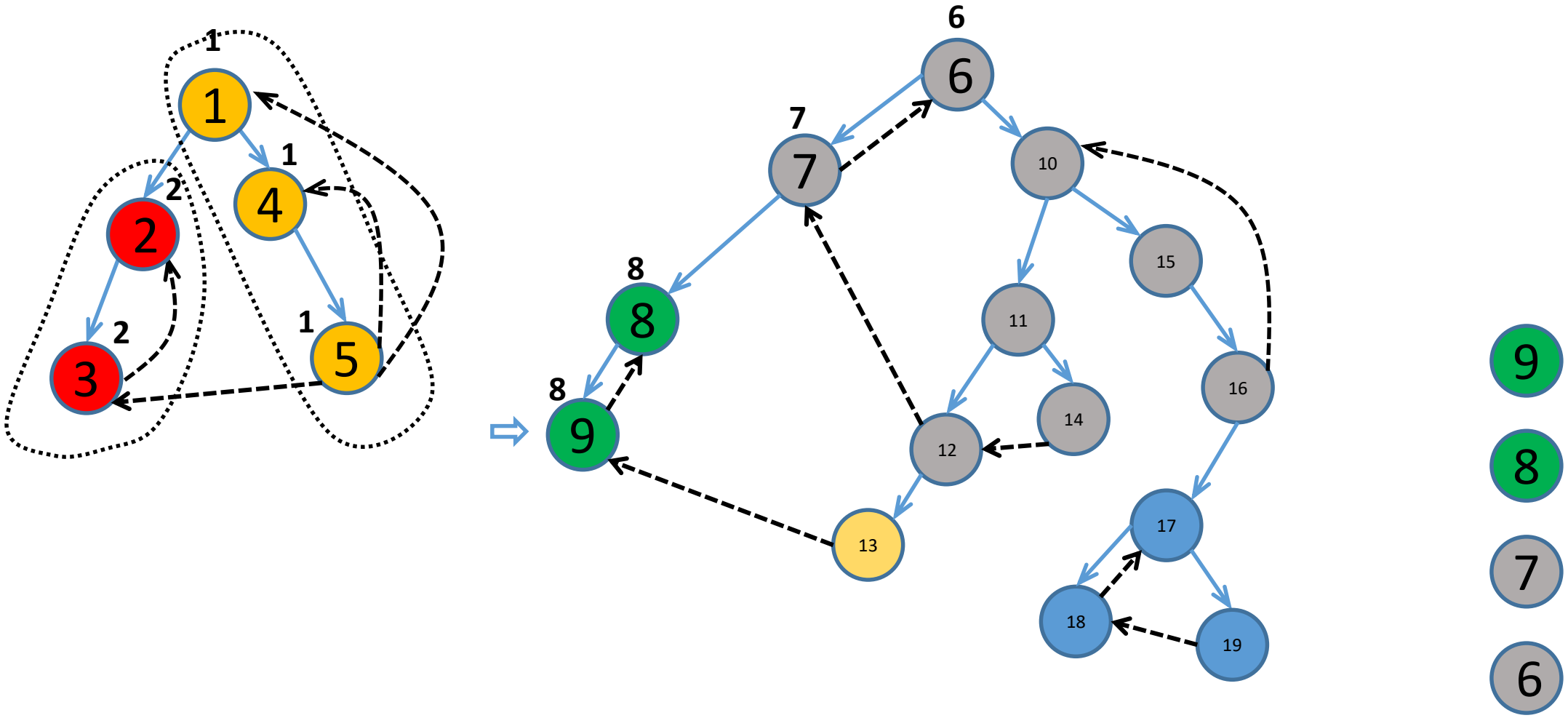
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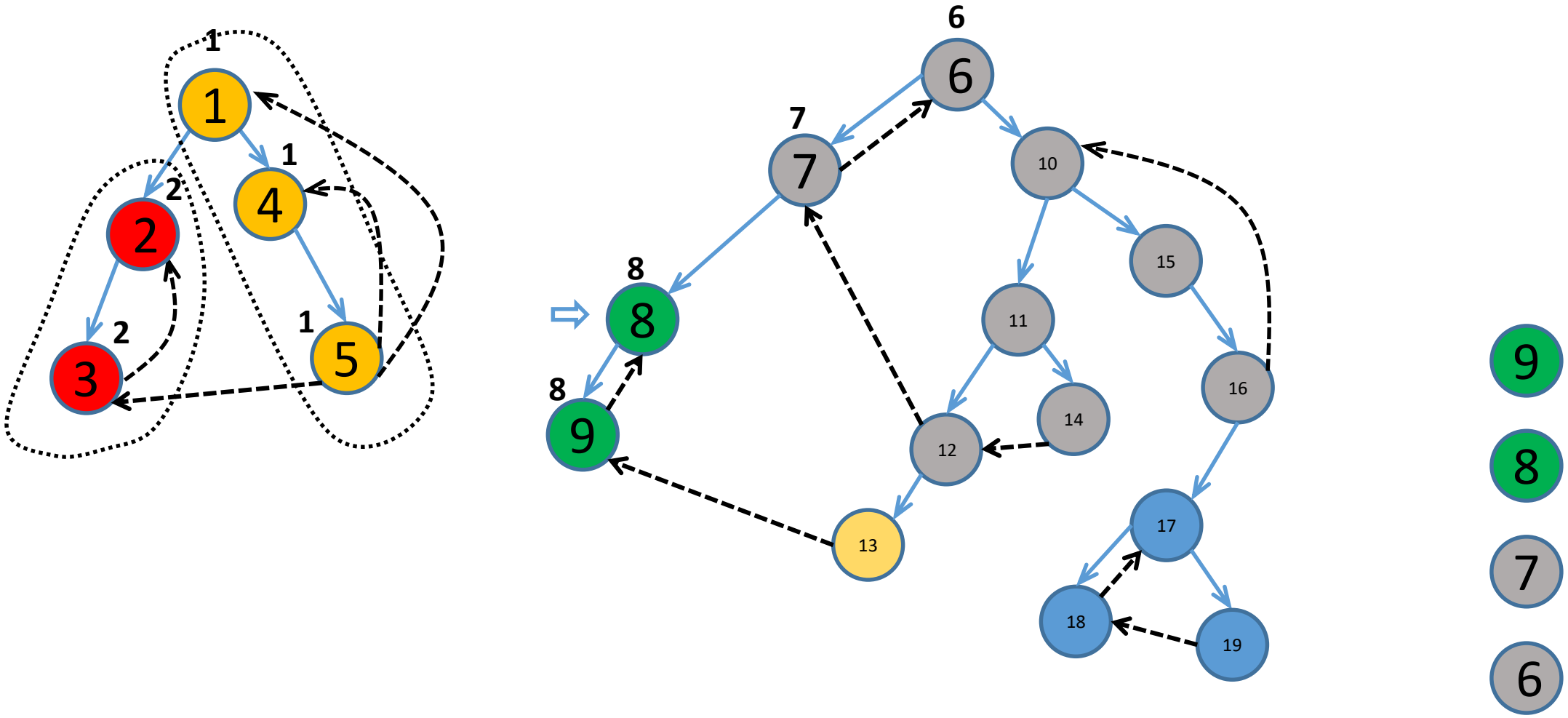
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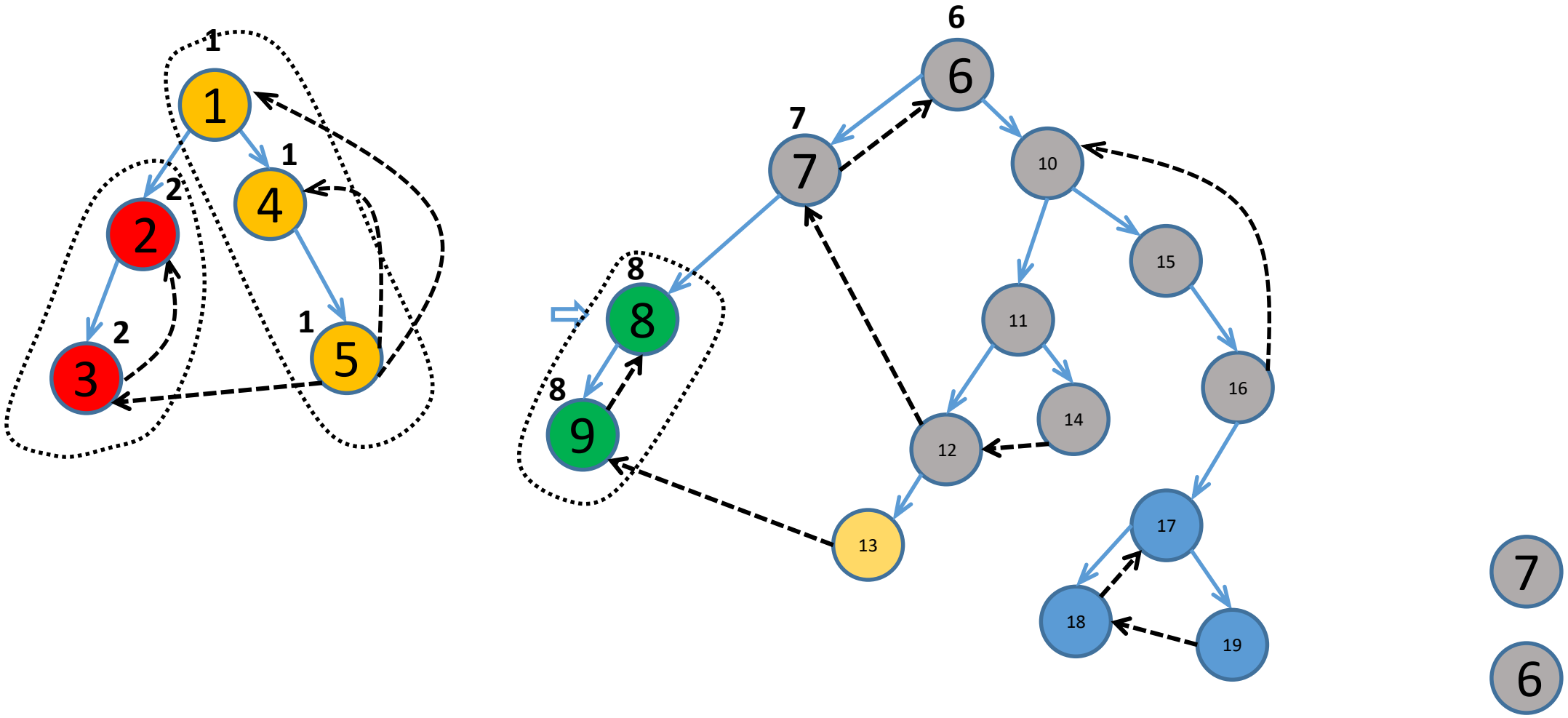
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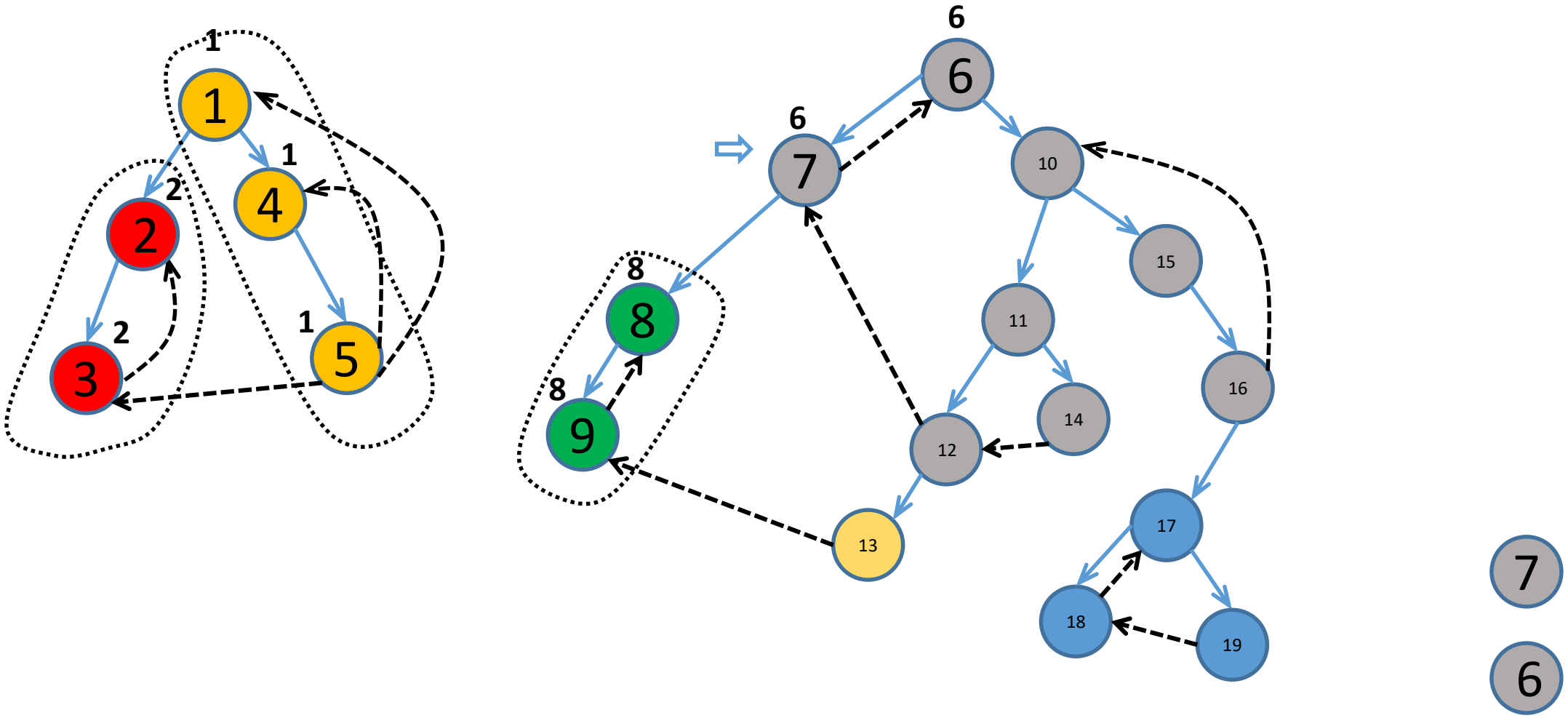
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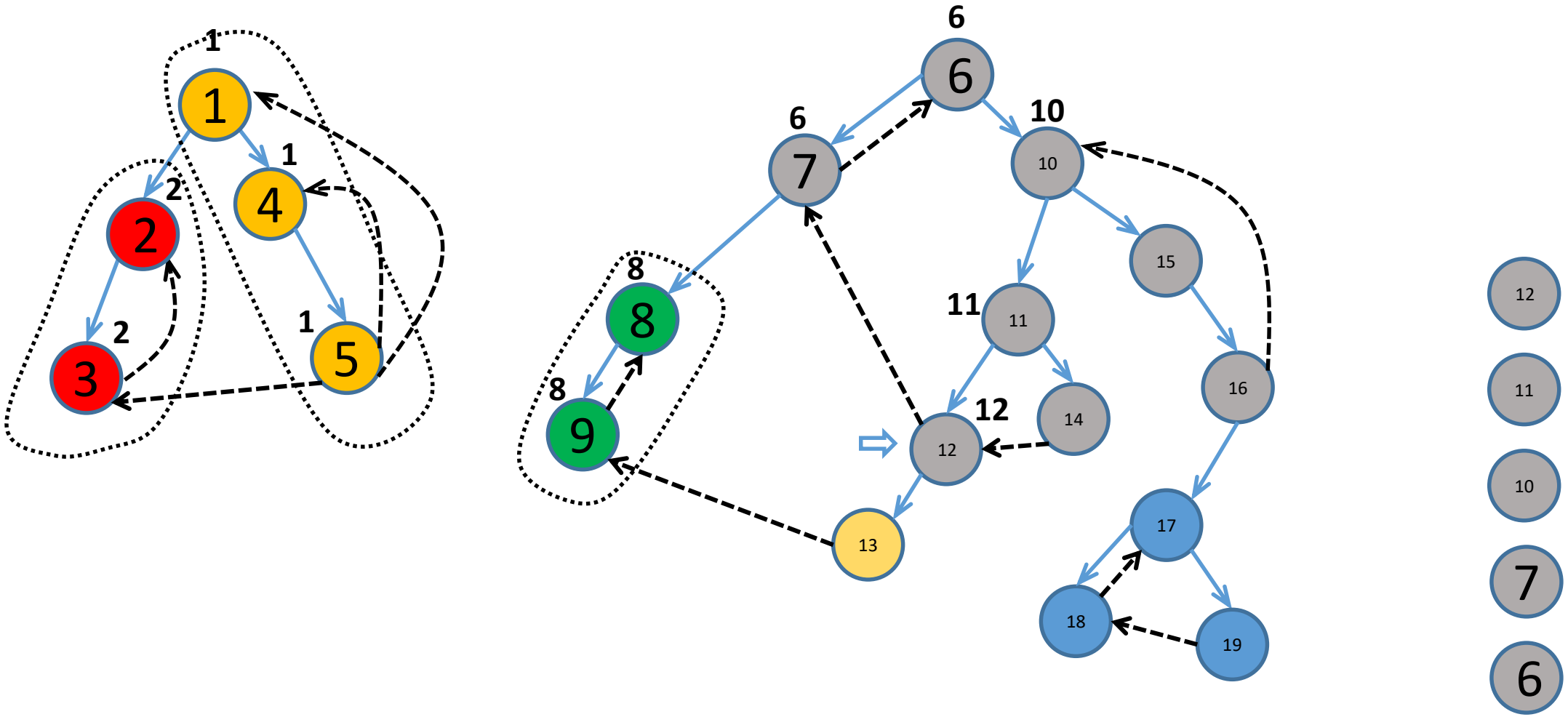
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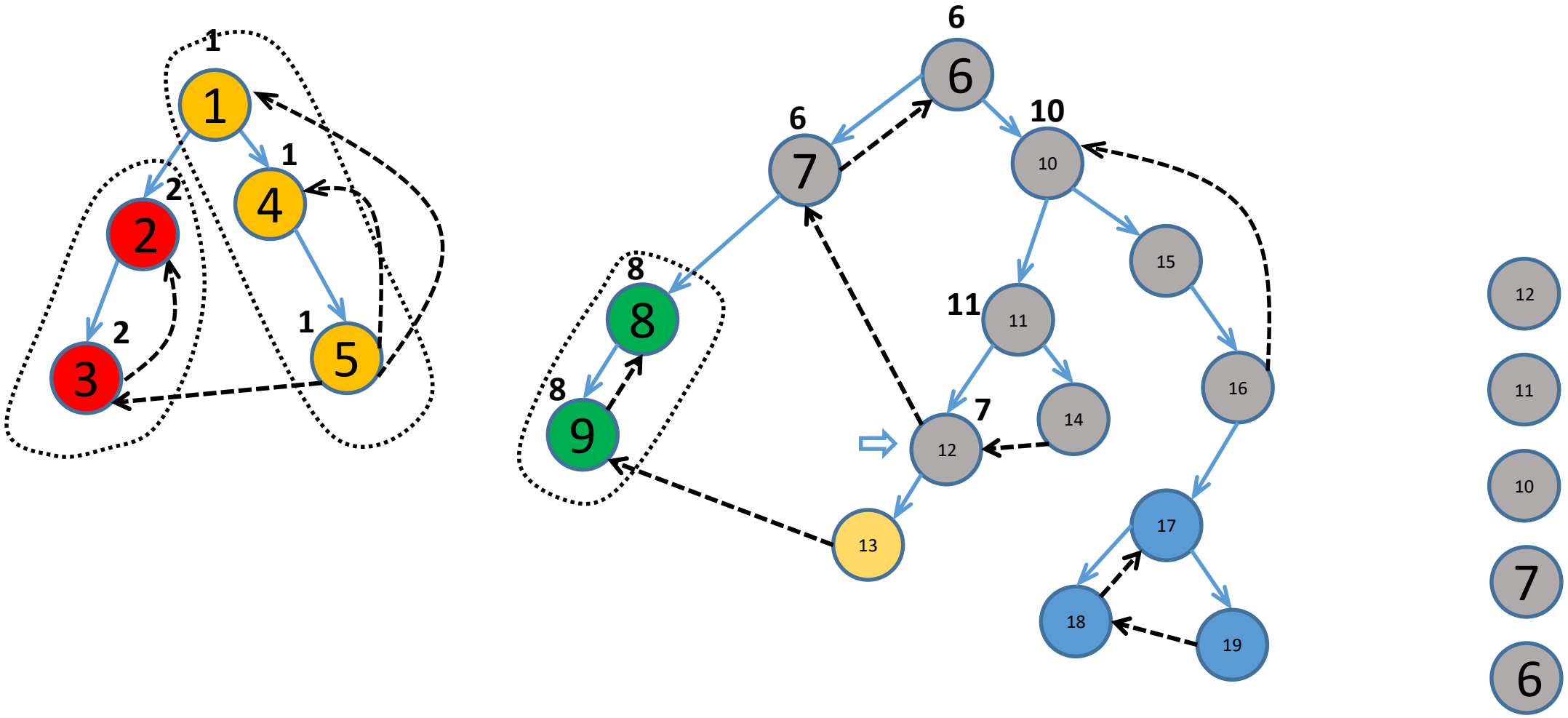
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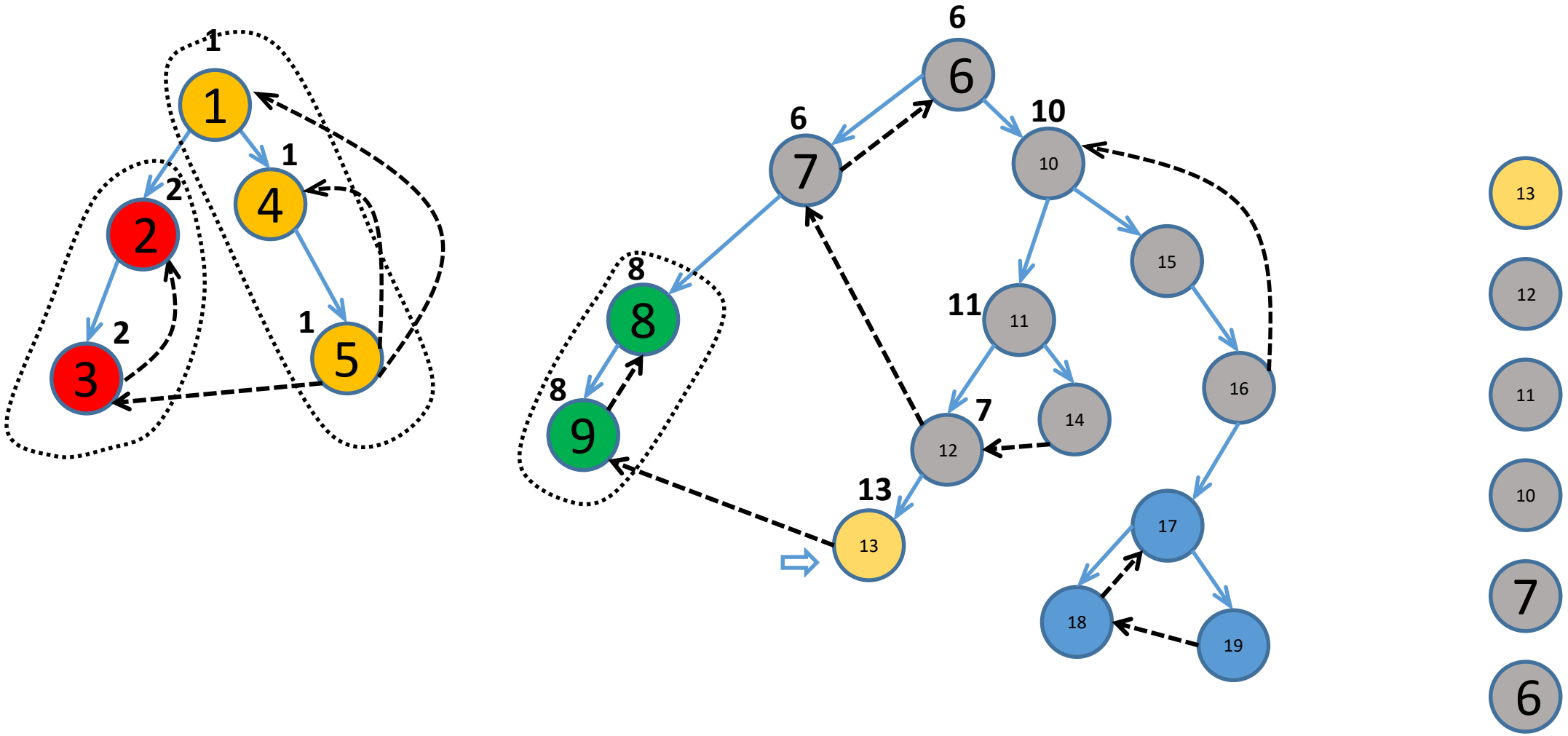
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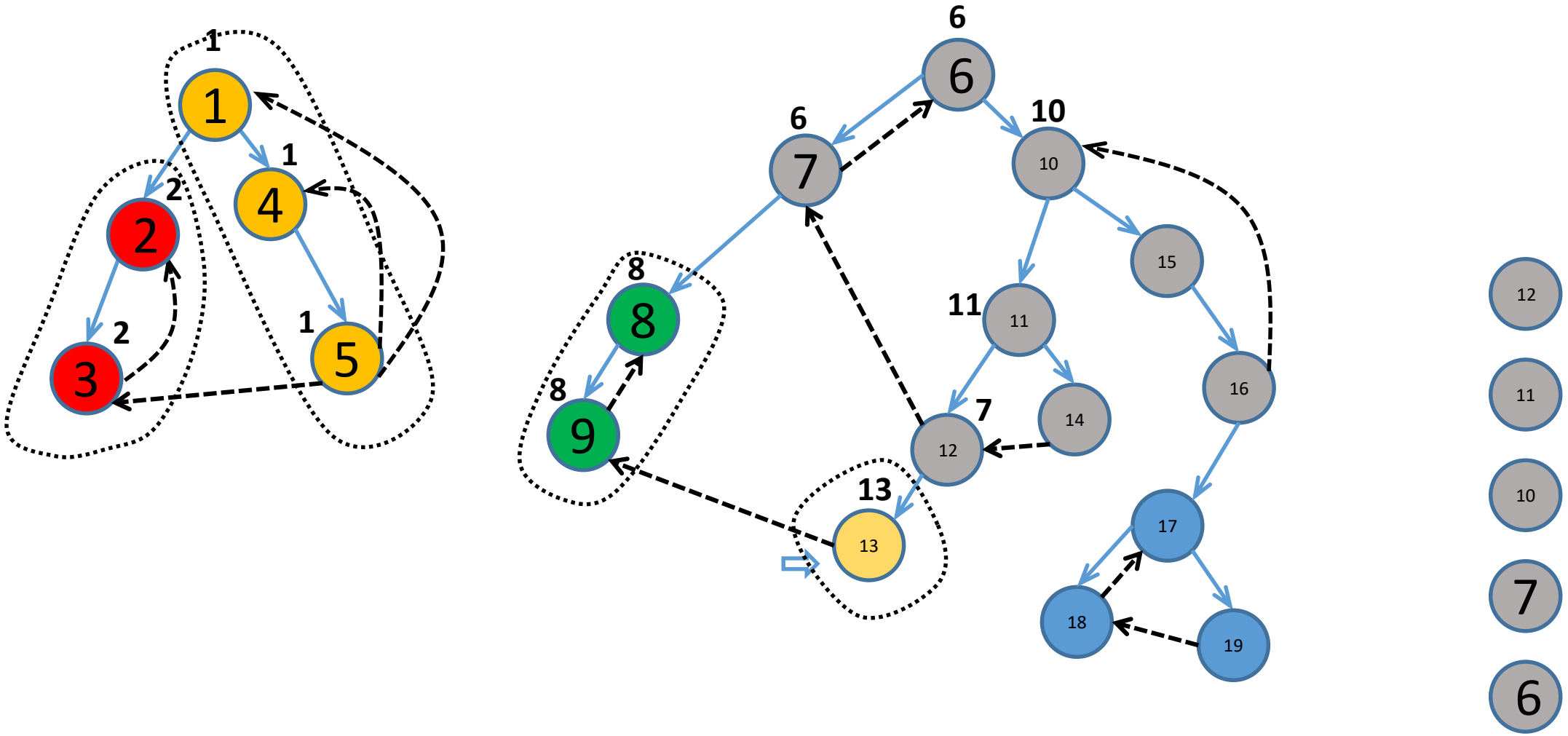
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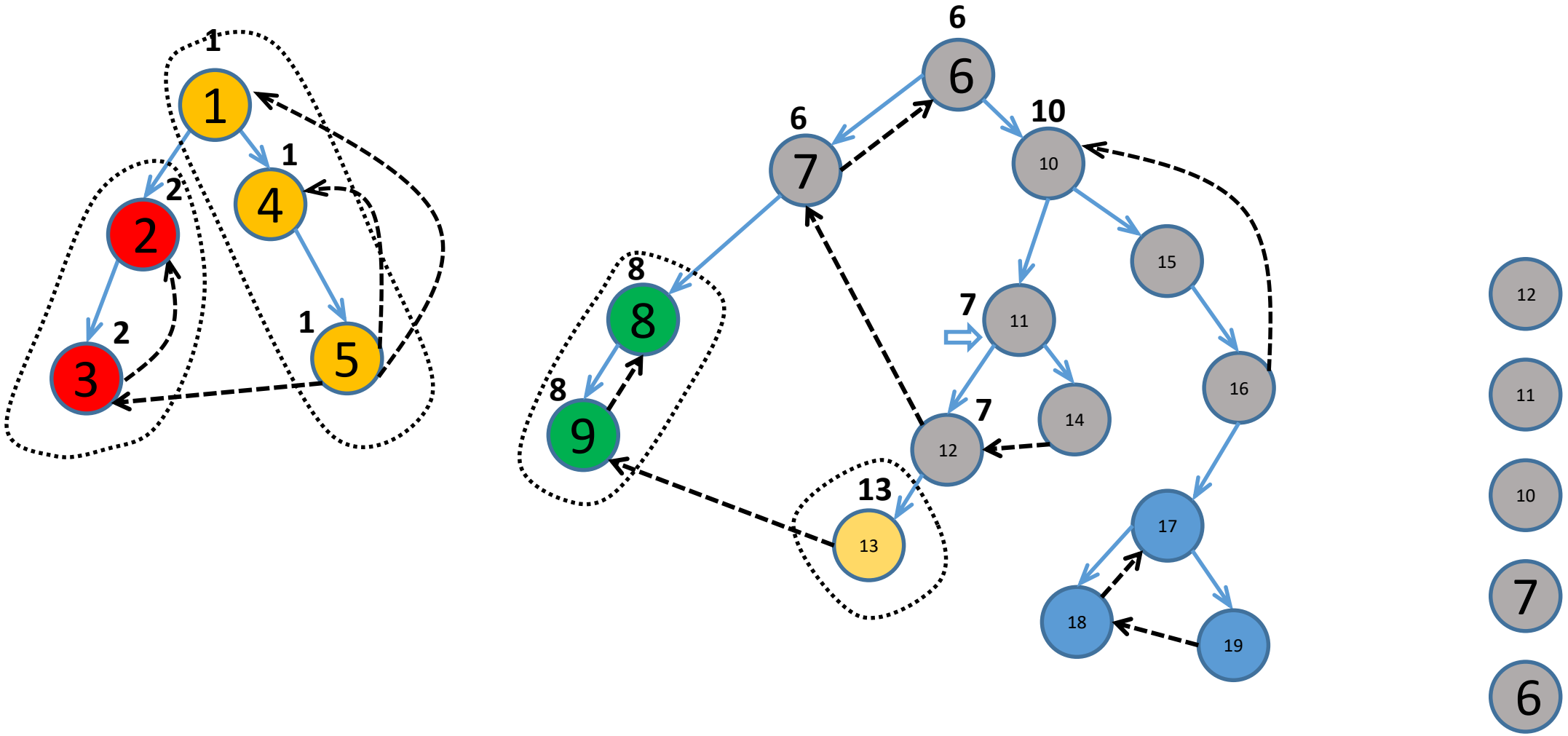
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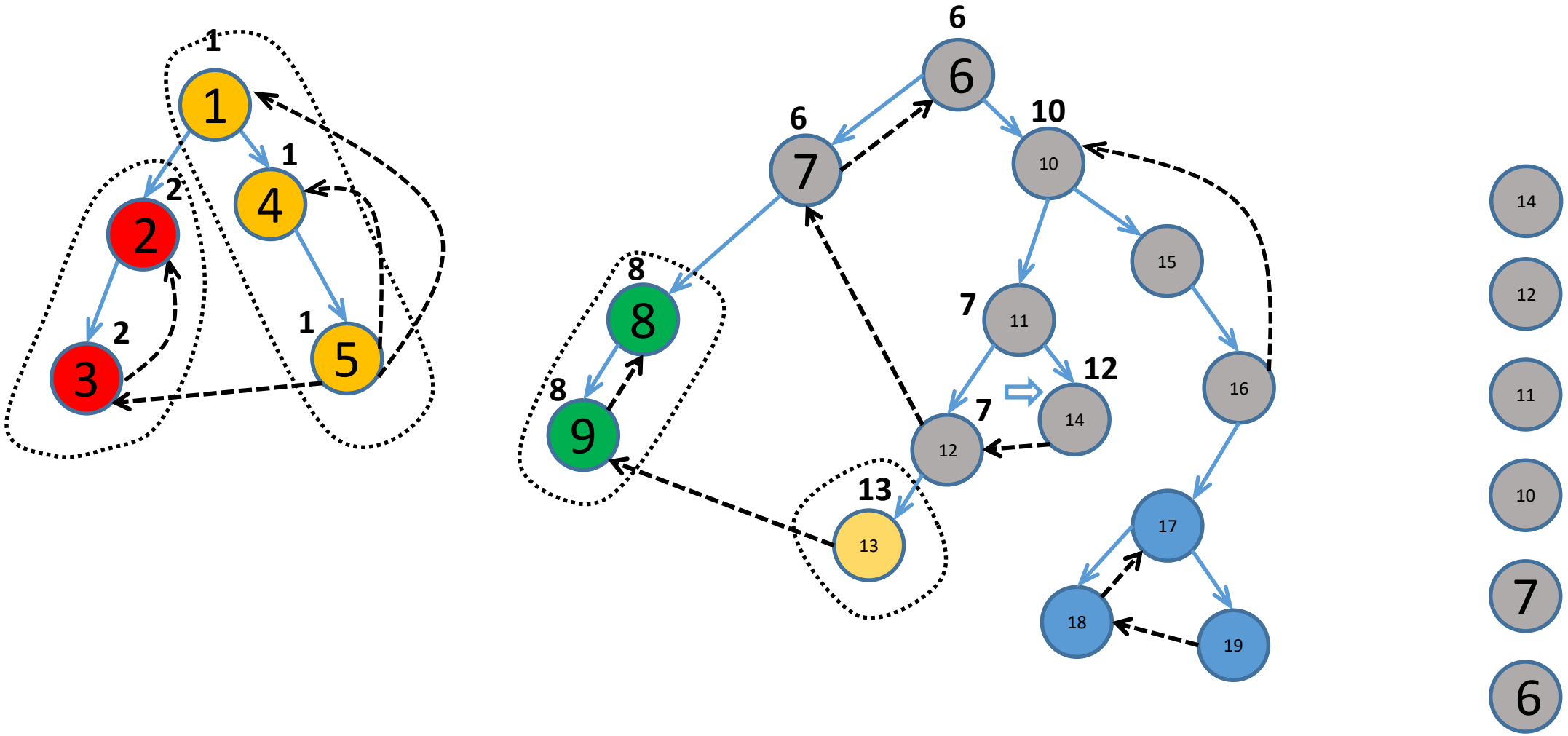
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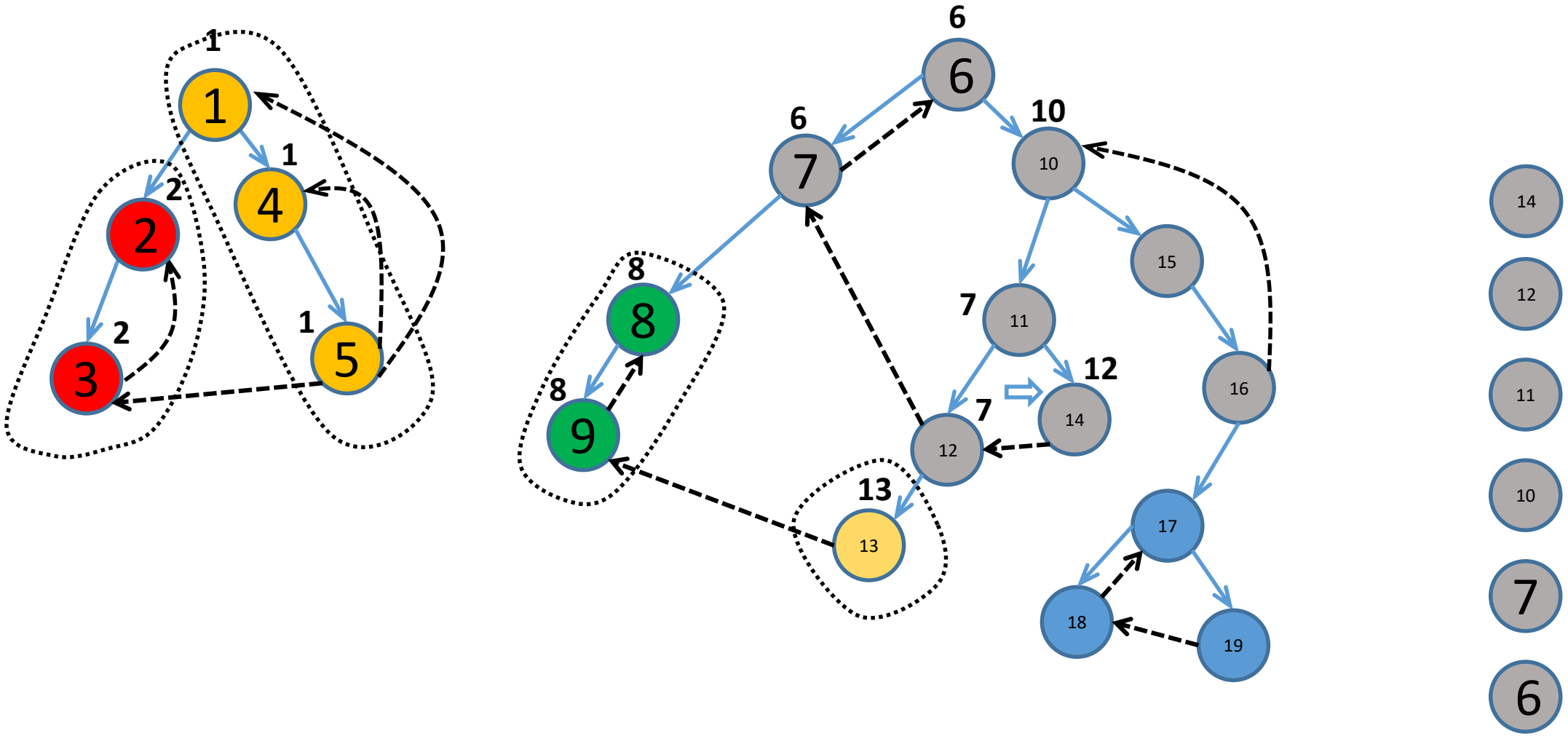
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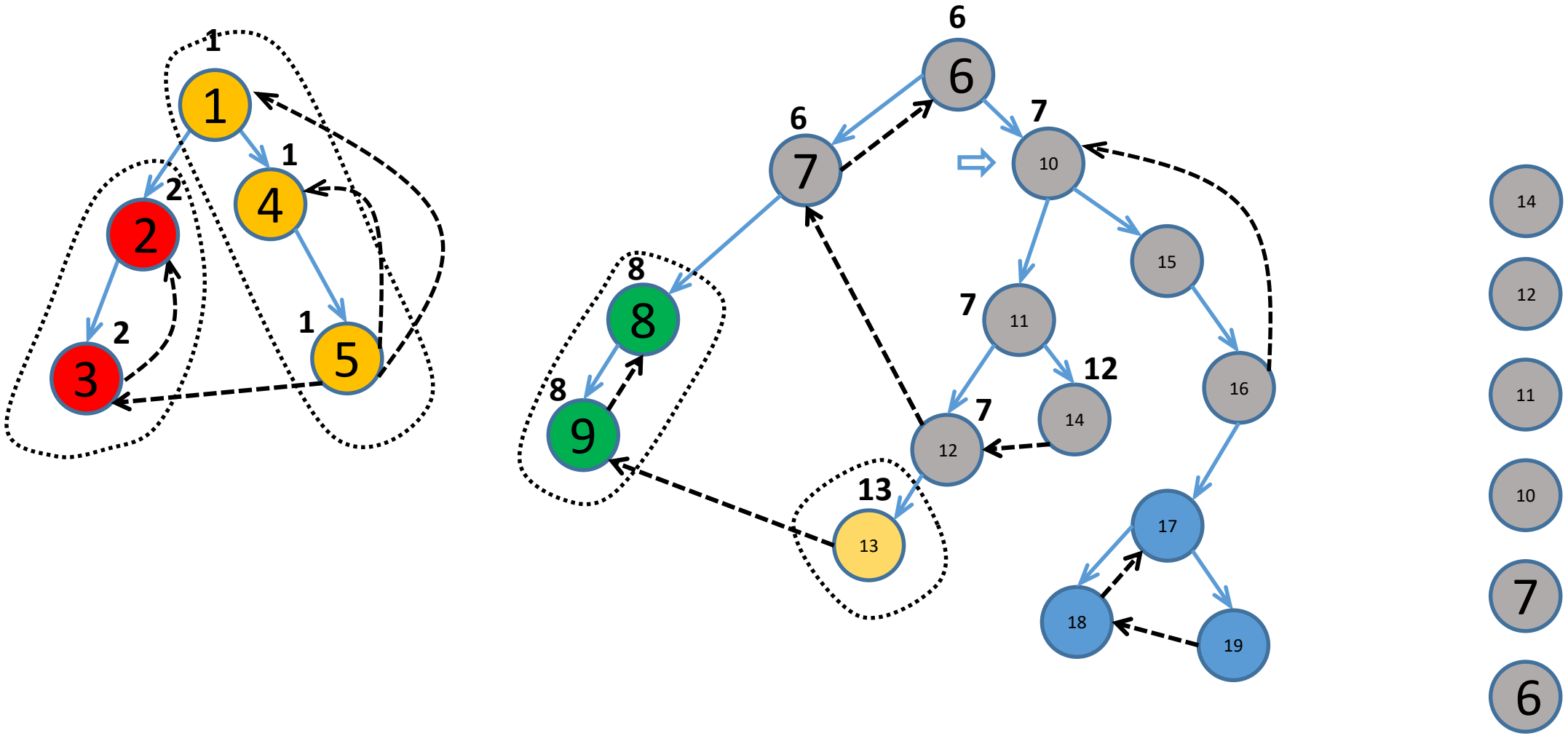
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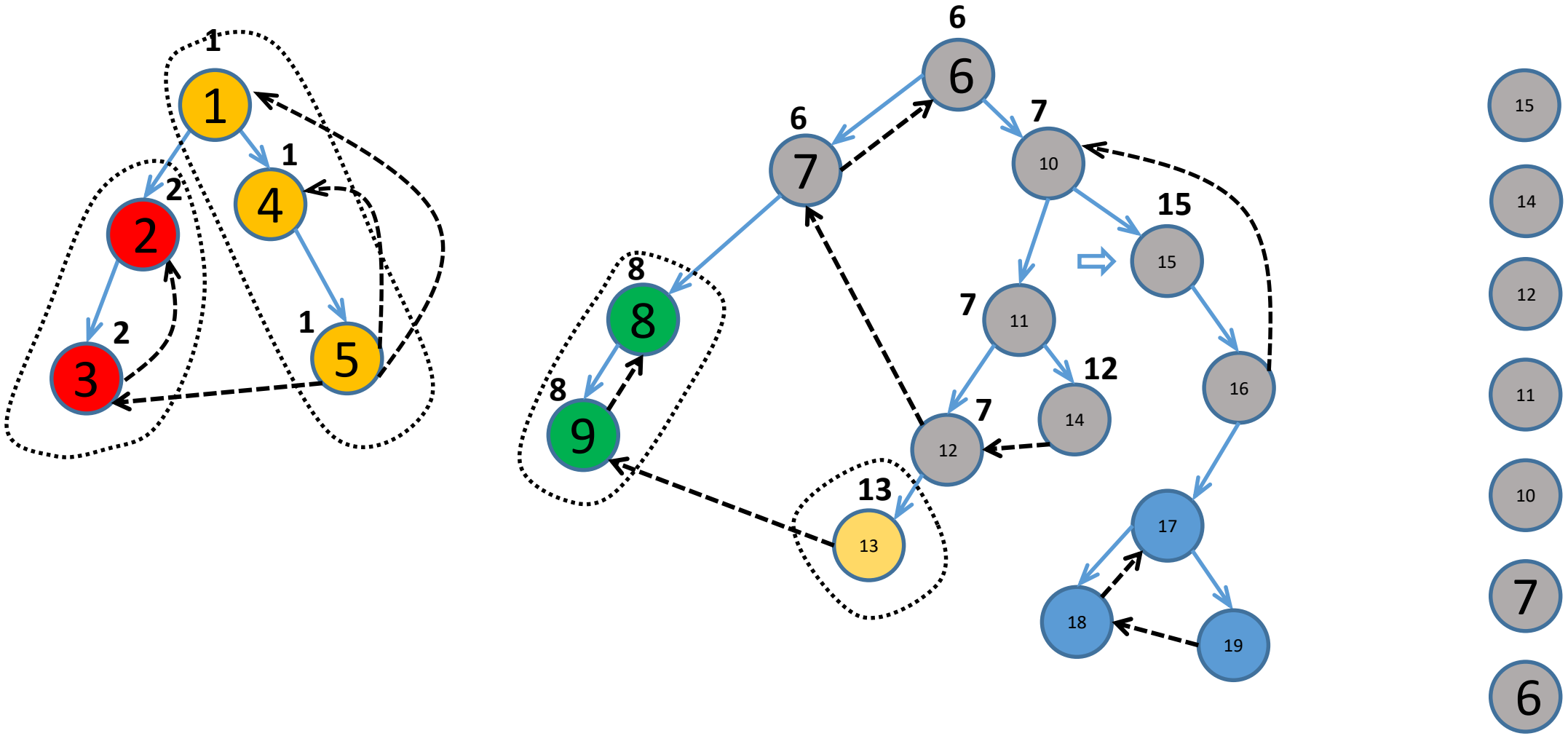
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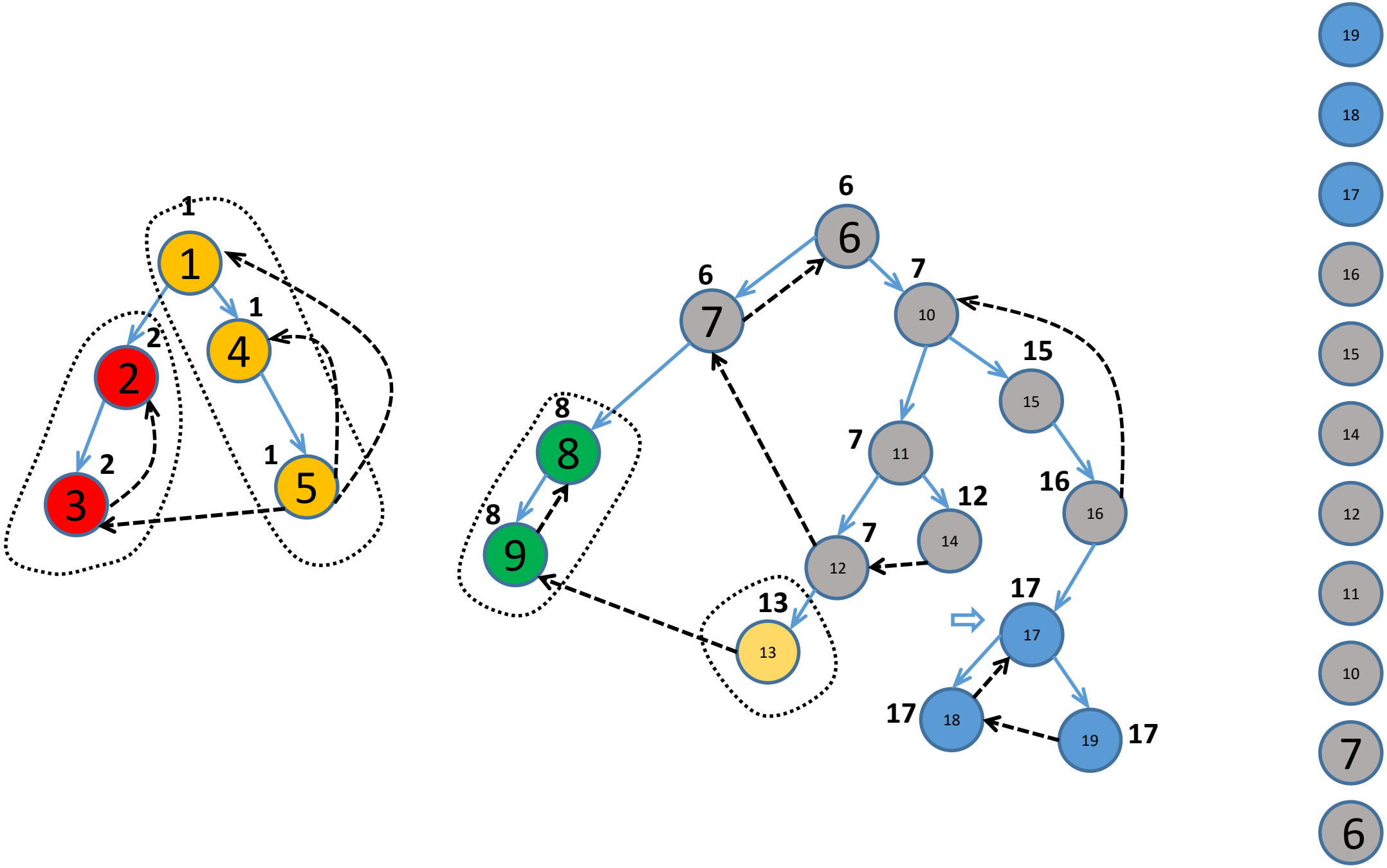
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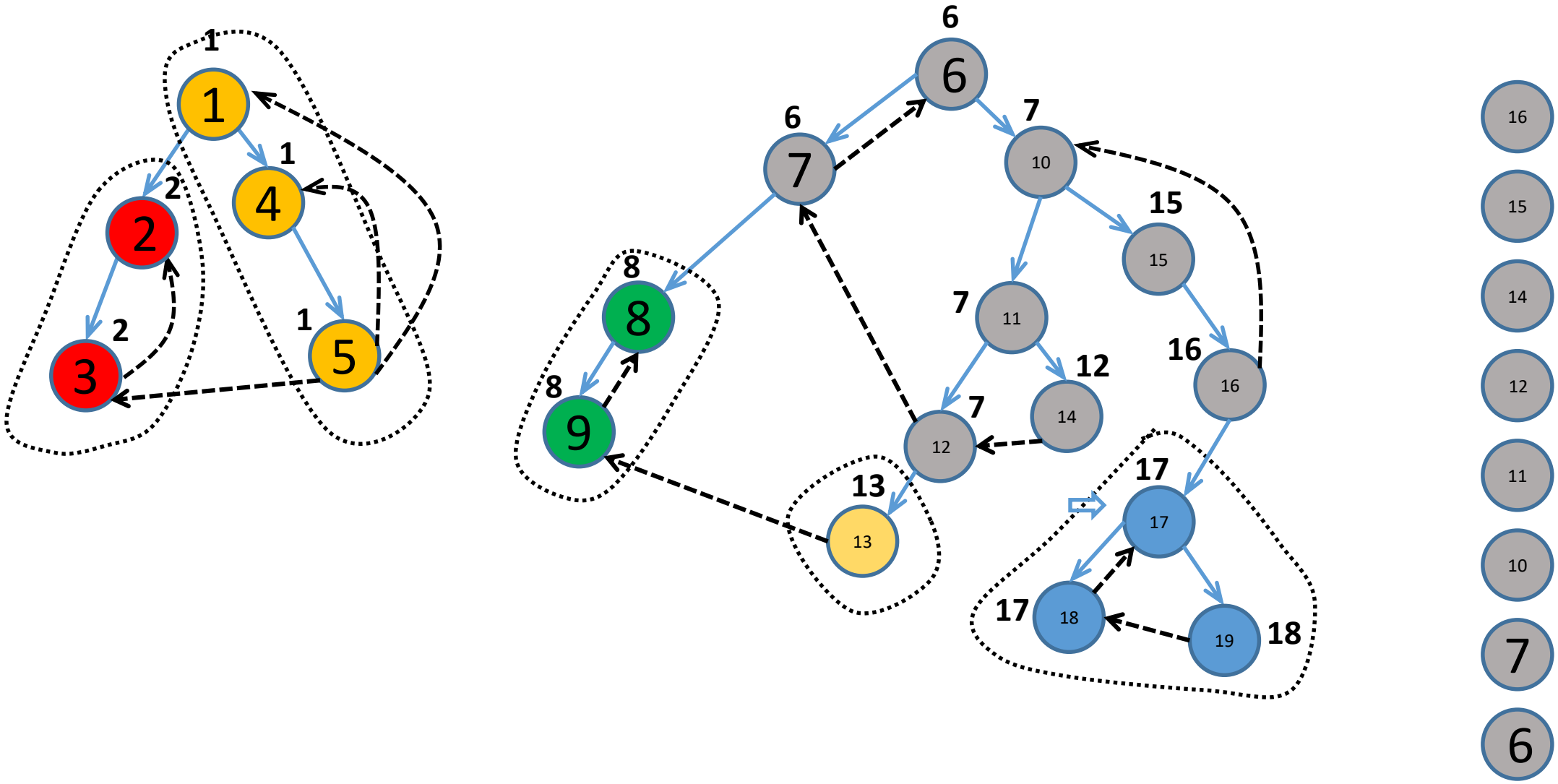
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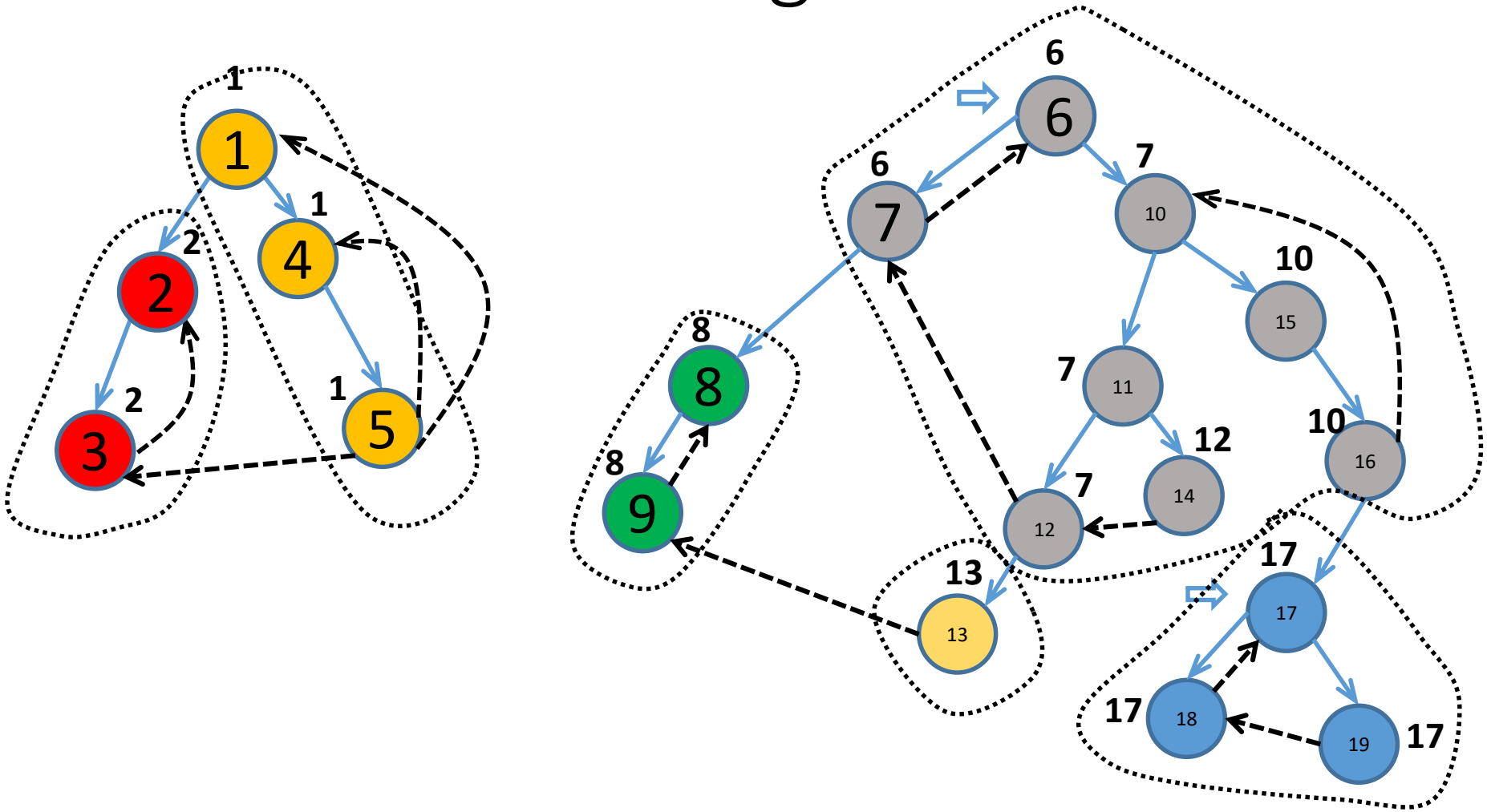
SCs using low values



Tarjan's SCC algorithm

```
SCC(G)
For each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
For each vertex u ∈ G.V
    if u.color == WHITE
        SCC-Visit(G,u)
```

```
SCC-Visit(G,u)
time = time + 1
u.d = low(u) = time
push(u,S)
u.color = GRAY
For each vertex v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        SCC-Visit(G,v)
        low(u) = min{low(u), low(v)}
    else if v ∈ S
        low(u) = min{v.d, low(u)}
u.color = BLACK
time = time + 1
u.f = time
if low(u) = u.d then peform pop(S) until u
  is poped, the set poped is a SC
```