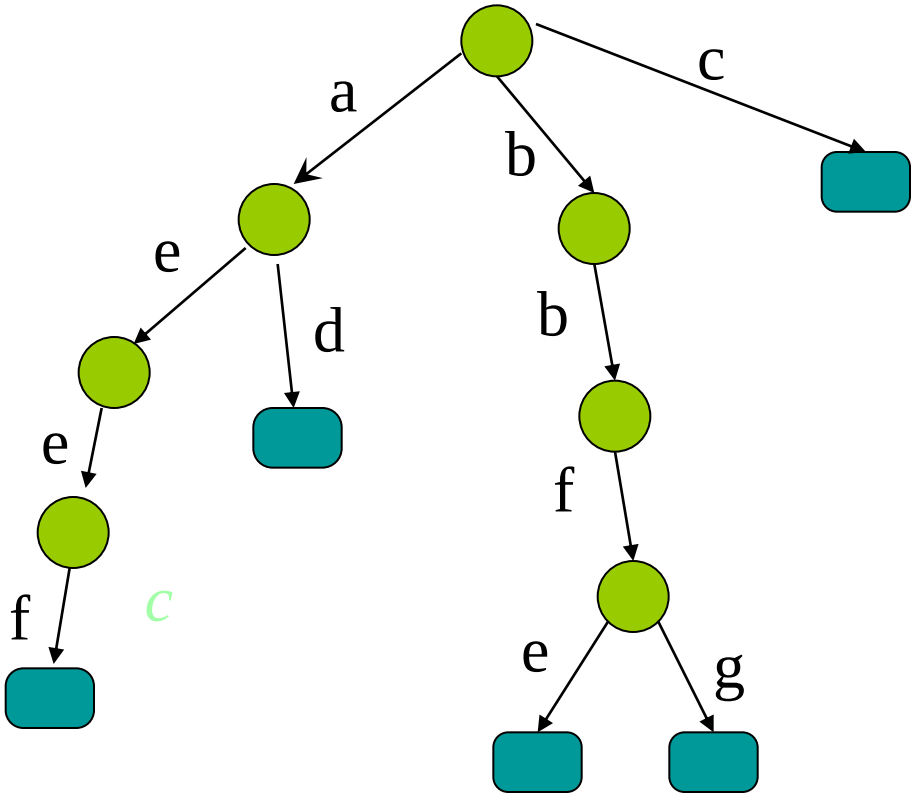
Suffix trees

Trie

- A tree representing a set of strings.

{
aeef
ad
bbfe
bbfg
c
}

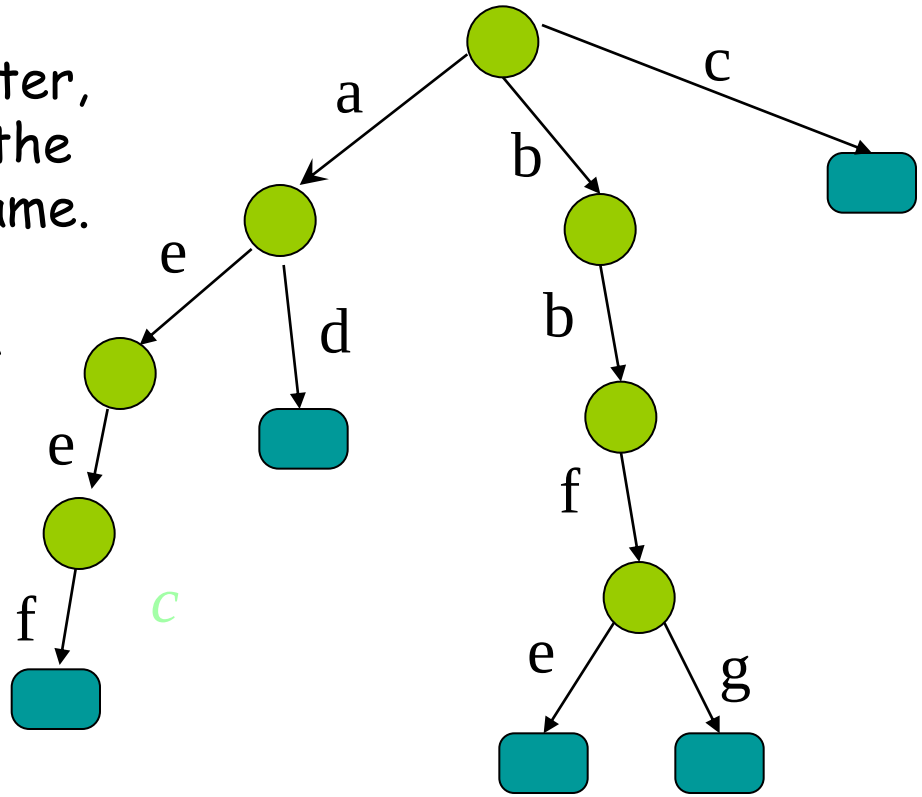


Trie (Cont)

- Assume no string is a prefix of another

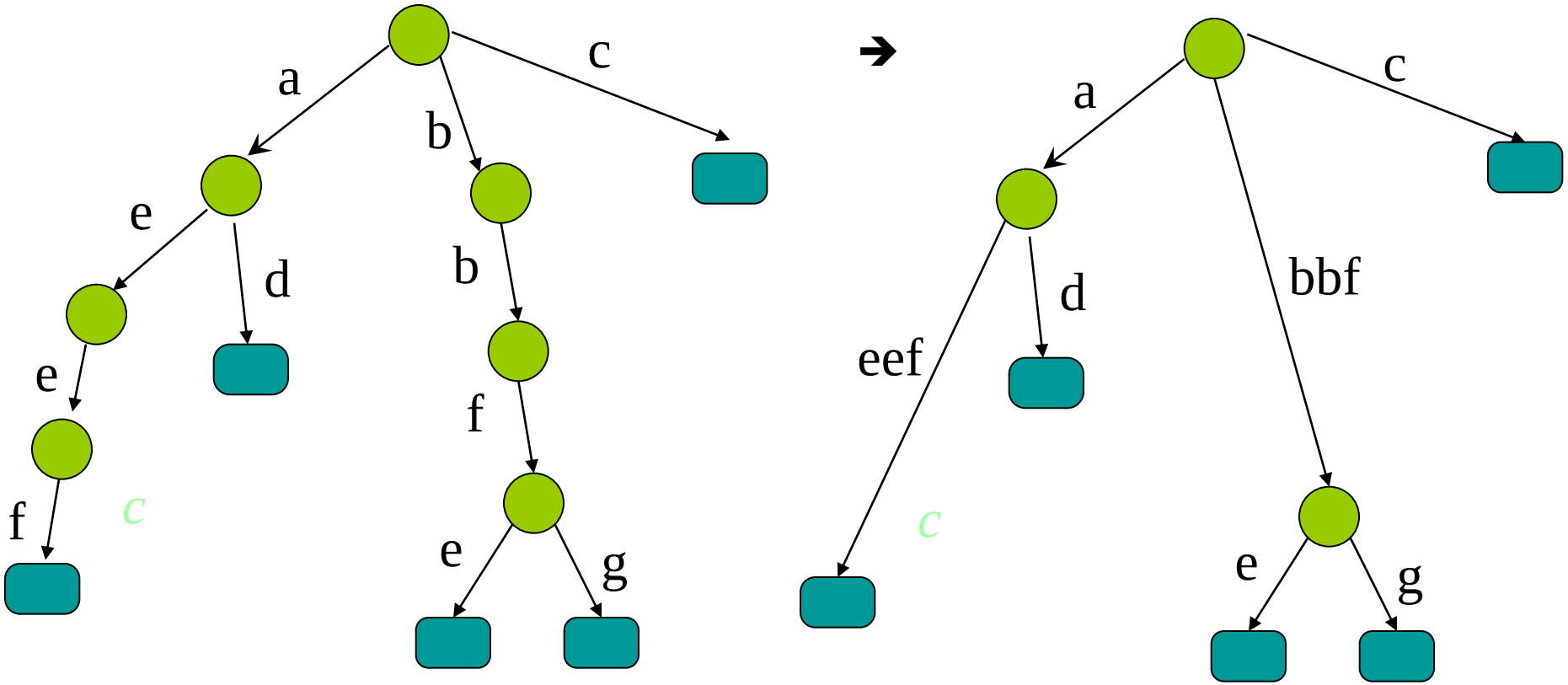
Each edge is labeled by a letter, no two edges outgoing from the same node are labeled the same.

Each string corresponds to a leaf.



Compressed Trie

- Compress unary nodes, label edges by strings



Suffix tree

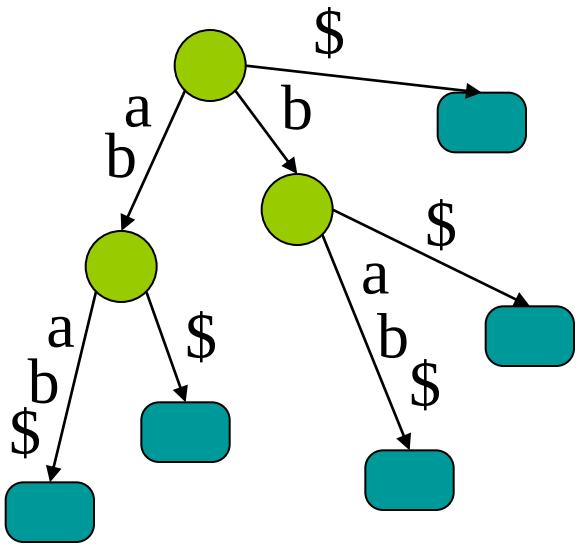
Given a string s a suffix tree of s is a compressed trie of all suffixes of s

To make these suffixes prefix-free we add a special character, say $\$$, at the end of s

Suffix tree (Example)

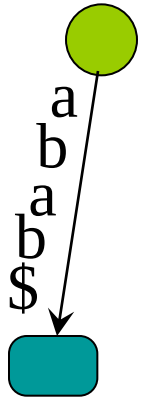
Let s=abab, a suffix tree of s is a compressed trie of all suffixes of s=abab$

{
  $
  b$
  ab$
  bab$
  abab$
}

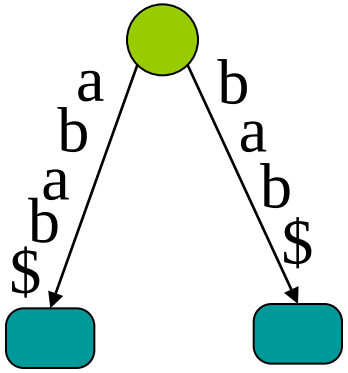


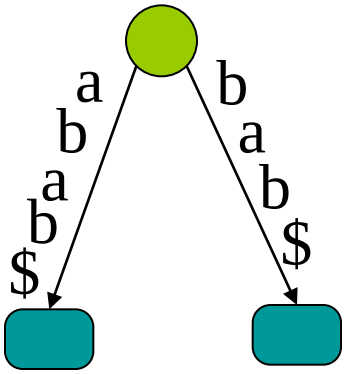
Trivial algorithm to build a Suffix tree

Put the largest suffix in

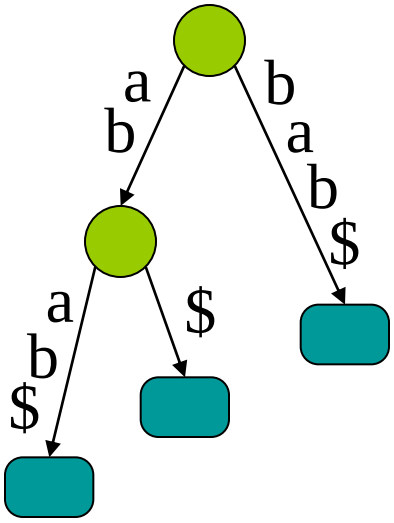


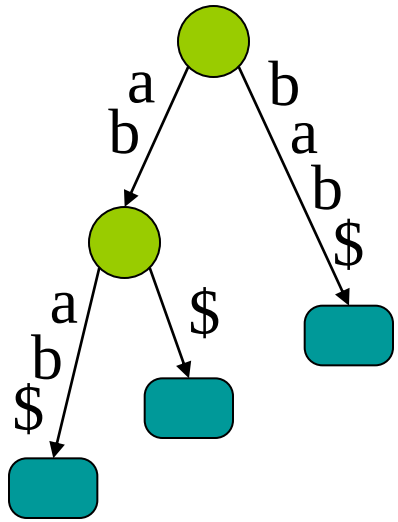
Put the suffix bab$ in



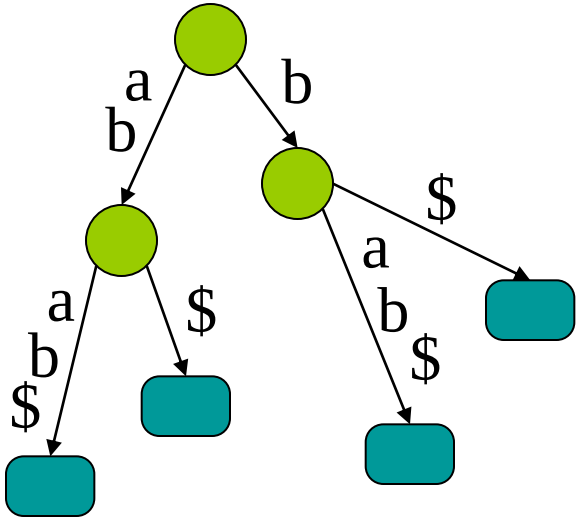


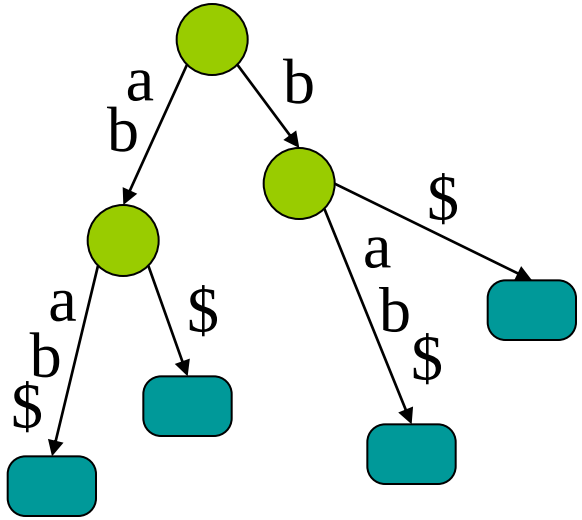
Put the suffix ab$ in



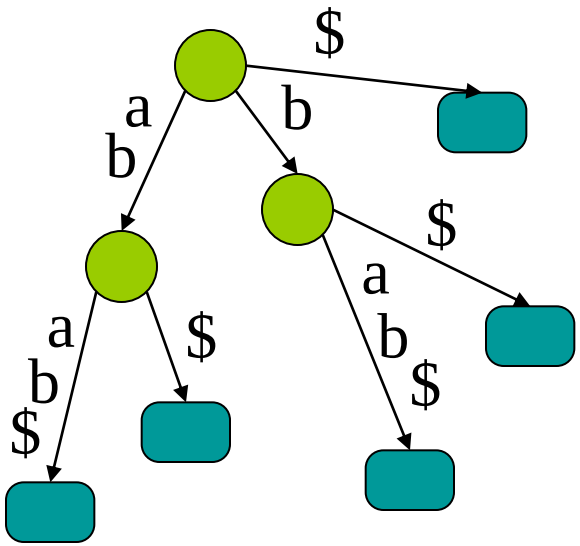


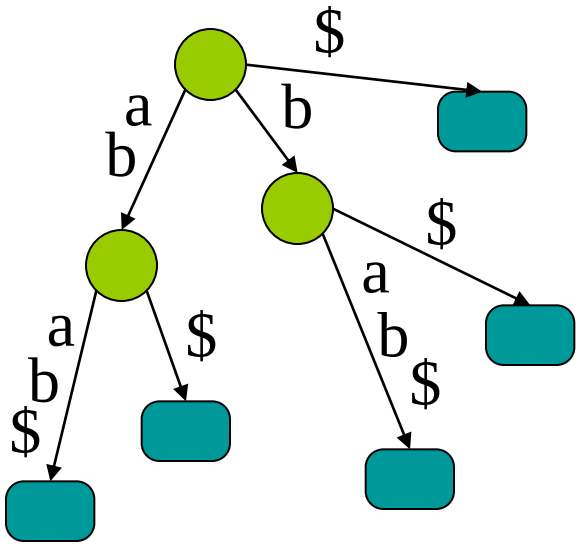
Put the suffix b$ in



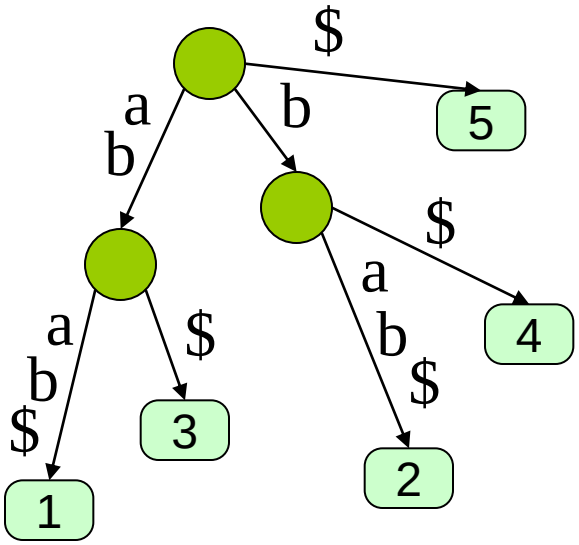


Put the suffix $ in





We will also label each leaf with the starting point of the corres. suffix.



Analysis

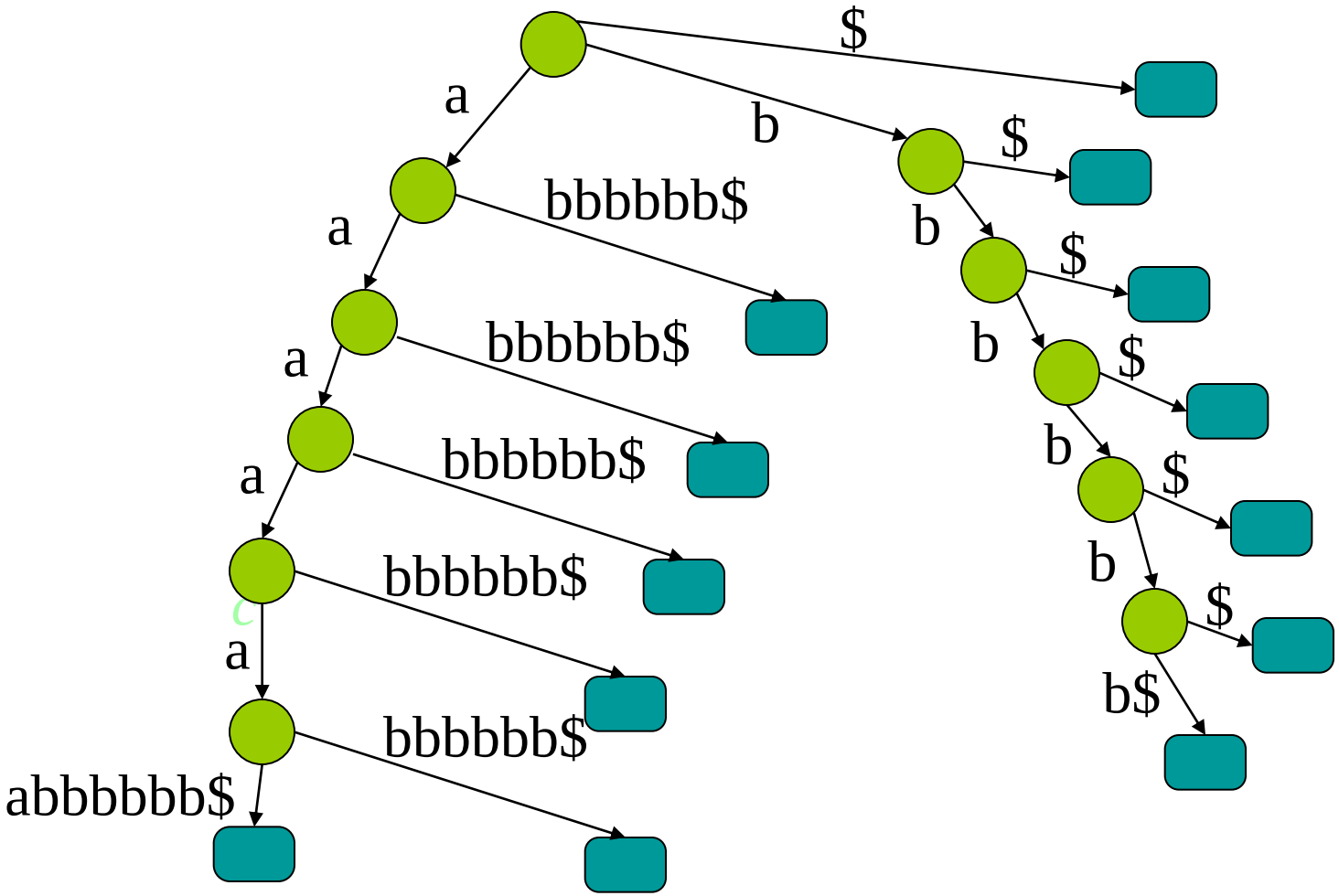
Takes $O(n^2)$ time to build.

You can do it in $O(n)$ time

But, how come? does it take $O(n)$ space ?

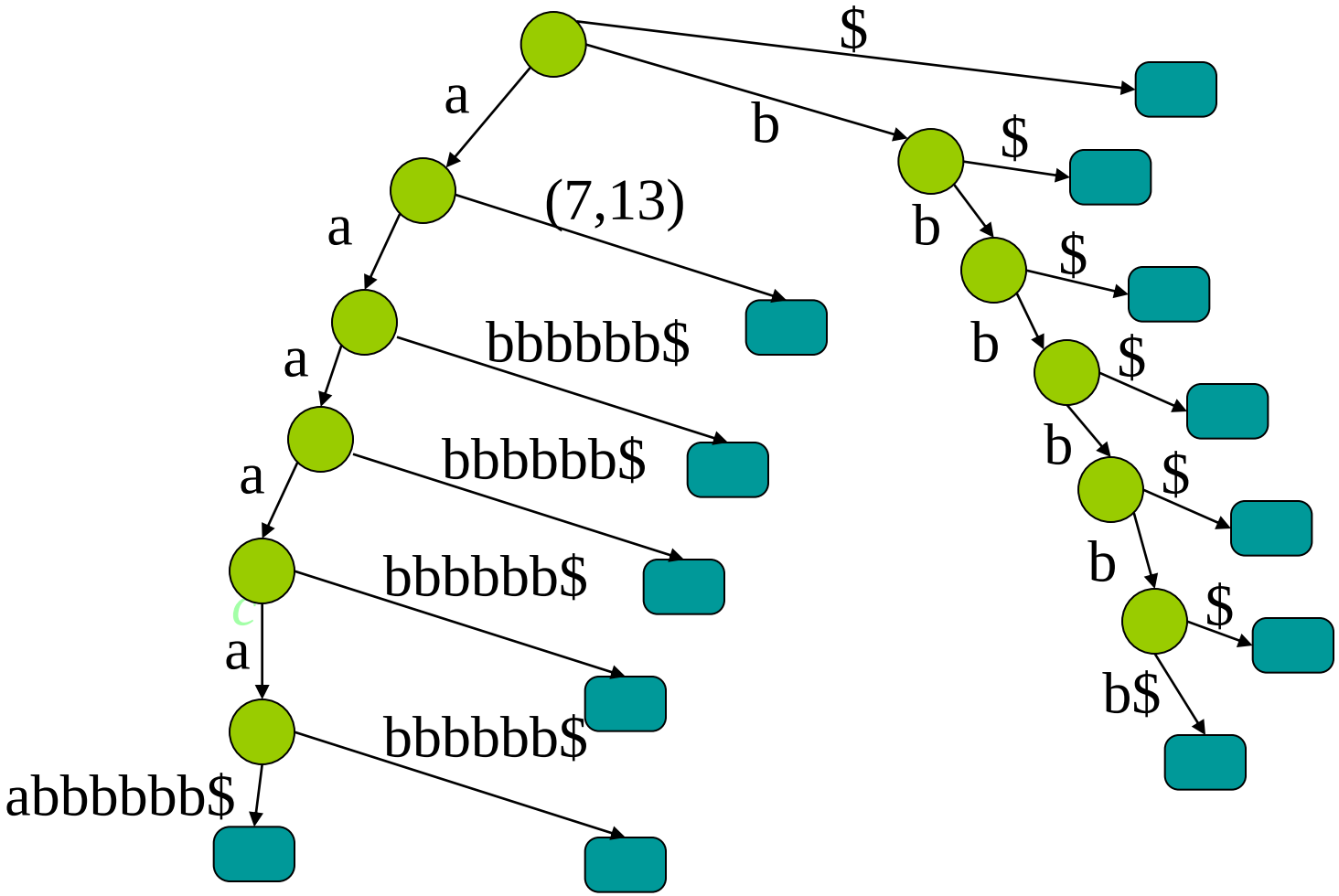
Linear space ?

- Consider the string aaaaaabbbbbb



To use only O(n) space encode the edge-labels

- Consider the string aaaaaabbbbbb



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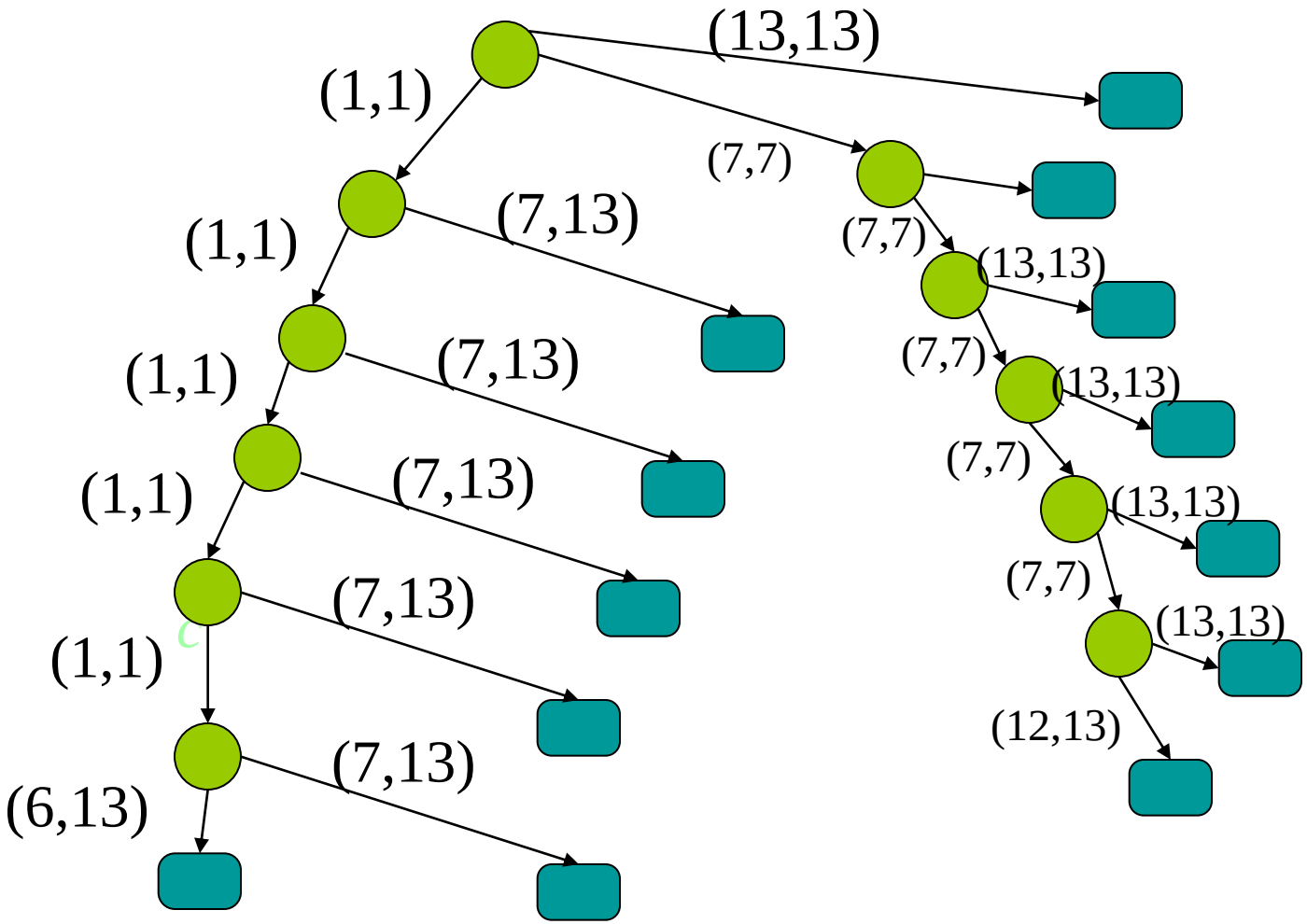


What can we do with it ?

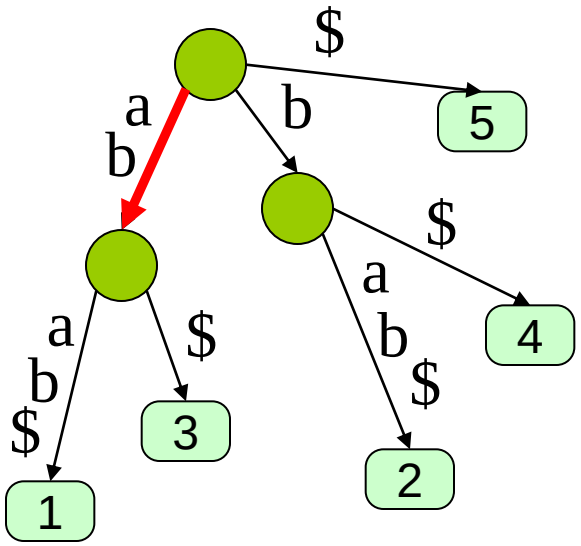
Exact string matching:

Given a Text T, |T| = n, preprocess it such that when a pattern P, |P|=m, arrives you can quickly decide when it occurs in T.

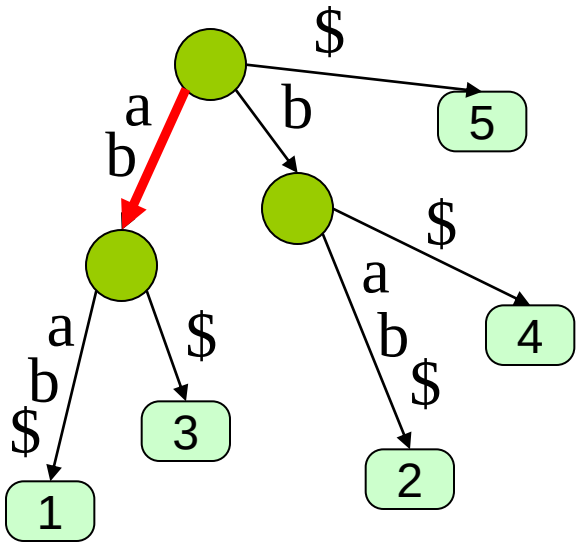
We may also want to find all occurrences of P in T

Exact string matching

In preprocessing we just build a suffix tree in O(n) time



Given a pattern P = ab we traverse the tree according to the pattern.



If we did not get stuck traversing the pattern then the pattern occurs in the text.

Each leaf in the subtree below the node we reach corresponds to an occurrence.

By traversing this subtree we get all k occurrences in O(n+k) time

Generalized suffix tree

Given a set of strings S a *generalized suffix tree* of S is a compressed trie of all suffixes of $s \in S$

To associate each suffix with a unique string in S add a different special char to each s

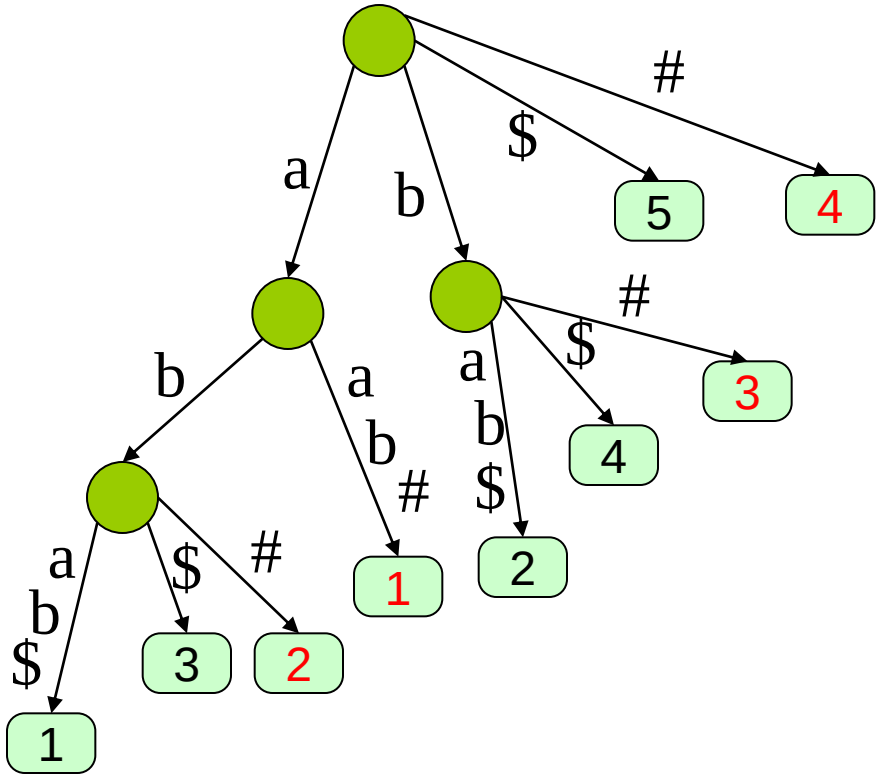
Generalized suffix tree (Example)

Let s_1=abab and s_2=aab here is a generalized suffix tree for s_1 and s_2

{

$	#
b$	b#
ab$	ab#
bab$	aab#
abab$	

}



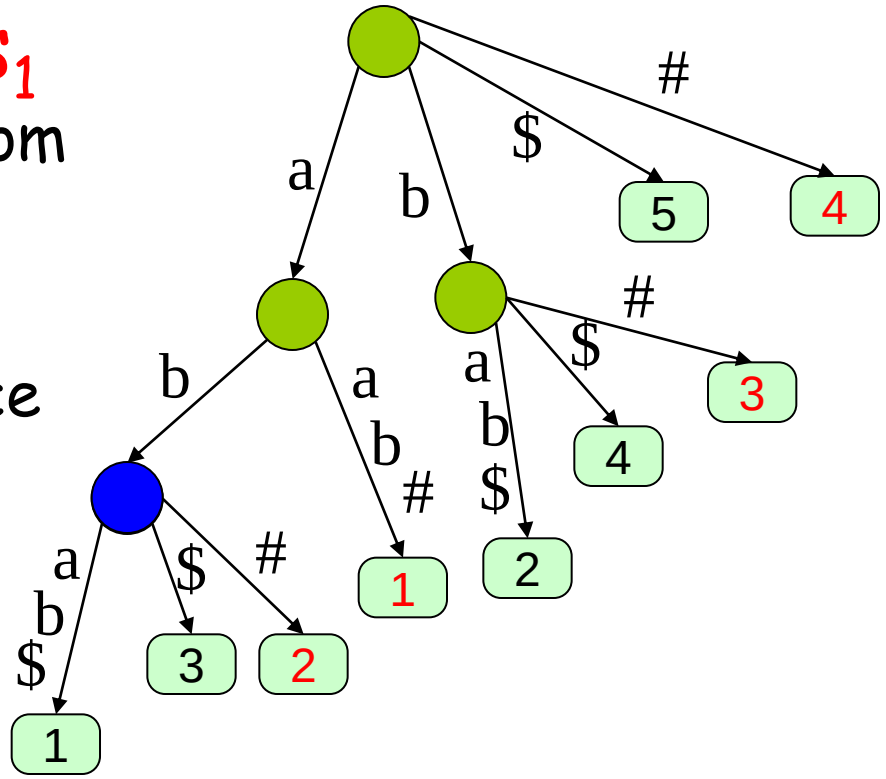
So what can we do with it ?

Matching a pattern against a database of strings

Longest common substring (of two strings)

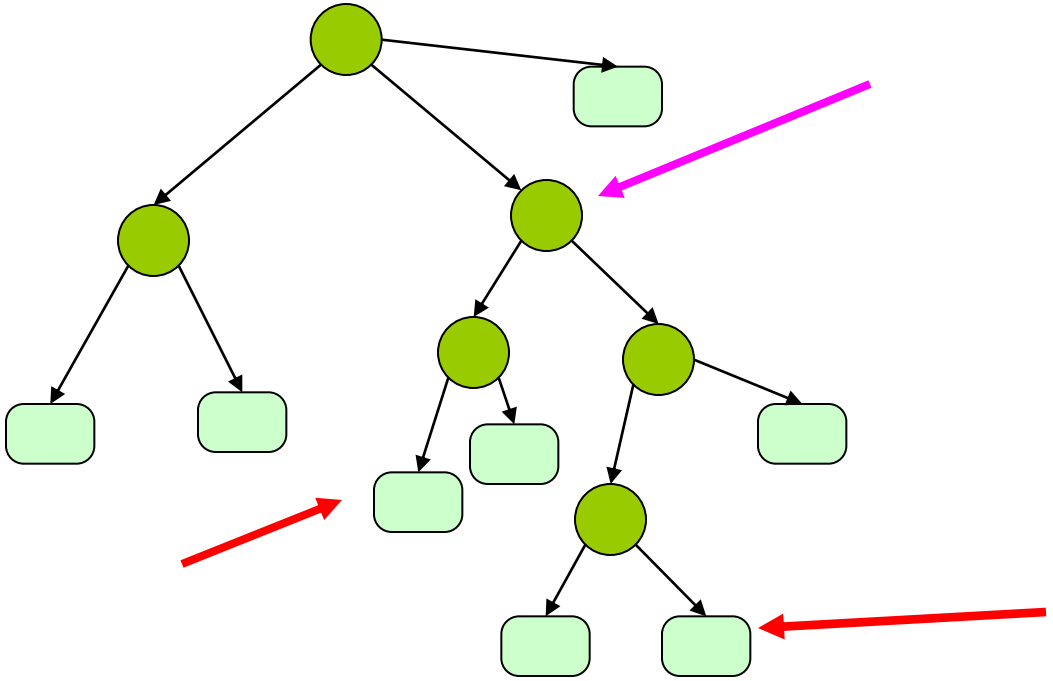
Every node with a leaf descendant from string s_1 and a leaf descendant from string s_2 represents a maximal common substring and vice versa.

Find such node with largest "string depth"



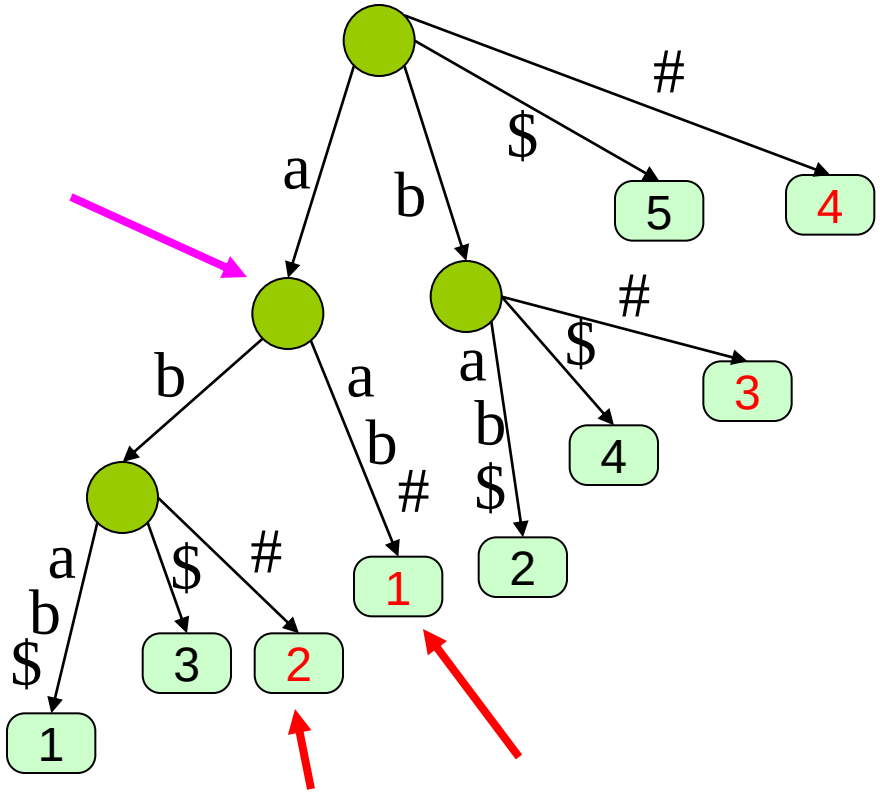
Lowest common ancetors

A lot more can be gained from the suffix tree if we preprocess it so that we can answer LCA queries on it



Why?

The LCA of two leaves represents the longest common prefix (LCP) of these 2 suffixes



Finding maximal palindromes

- A palindrome: caabaac, cbaabc
- Want to find all maximal palindromes in a string s

Let s = cbaaba

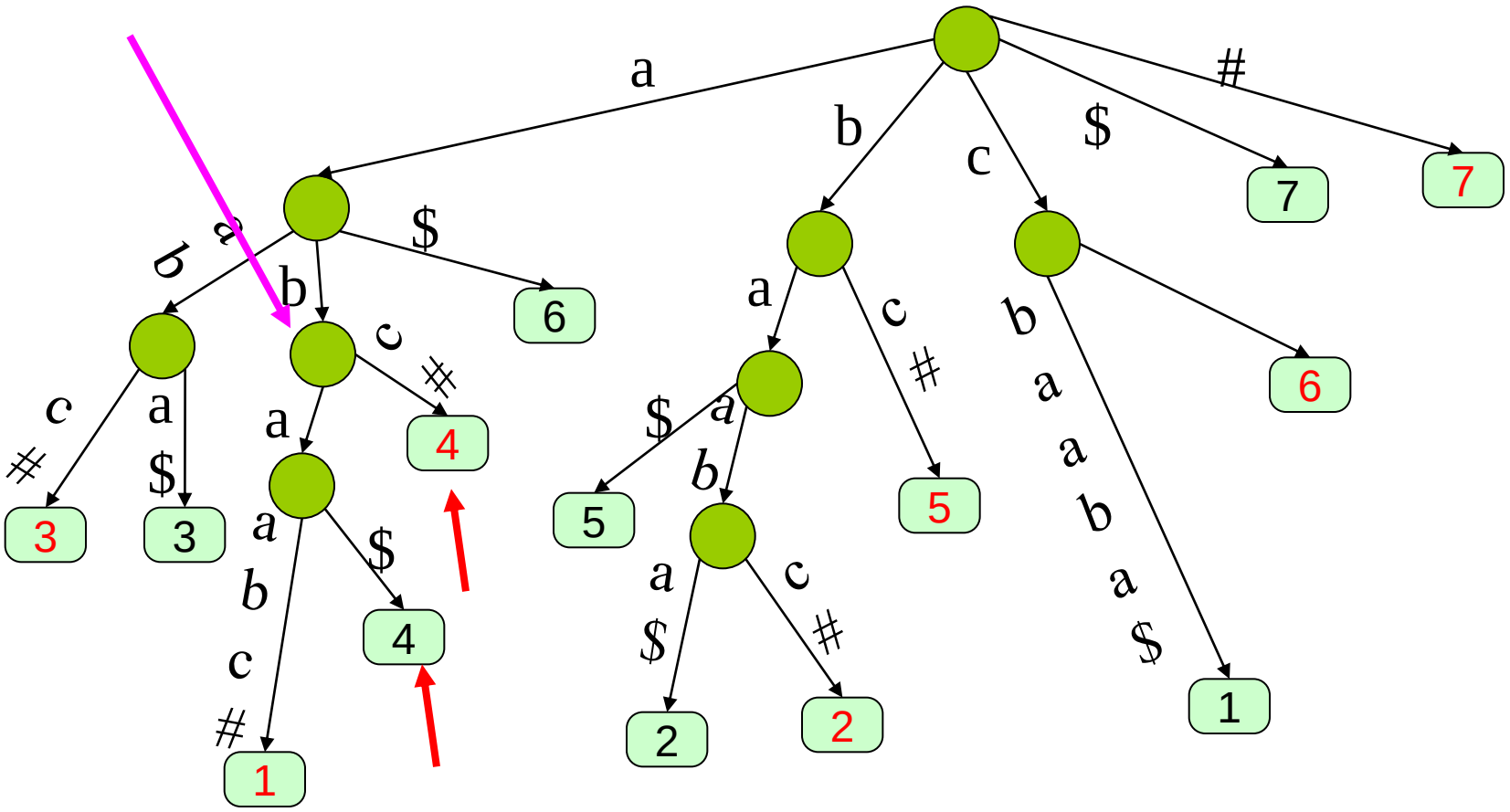
The maximal palindrome with center between i-1 and i is the LCP of the suffix at position i of s and the suffix at position m-i+1 of s^r

Maximal palindromes algorithm

Prepare a generalized suffix tree for
s = cbaaba$ and s^r = abaabc#

For every i find the LCA of suffix i of s
and suffix m-i+1 of s^r

Let s = cbaaba\$ then s^r = abaabc#



Analysis

O(n) time to identify all palindromes

Can we construct a suffix tree in linear time ?

Ukkonen's *linear* time construction

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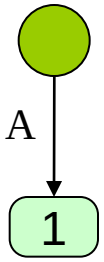

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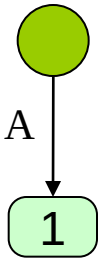

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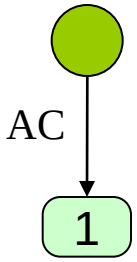

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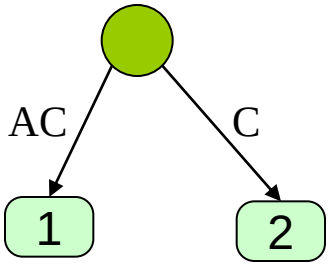

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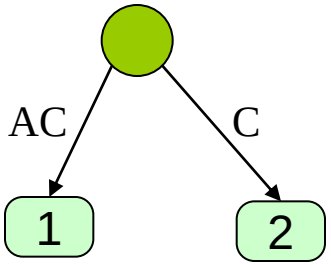

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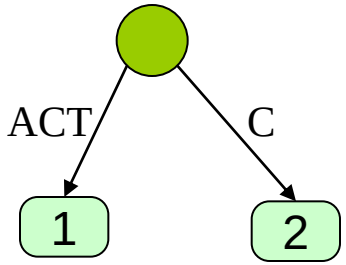

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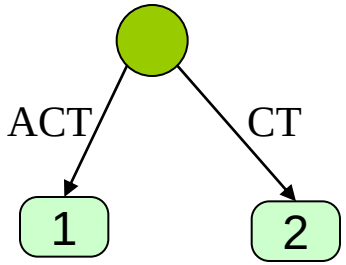

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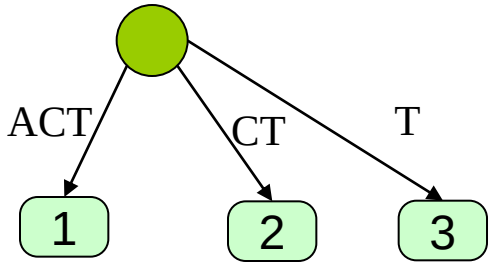

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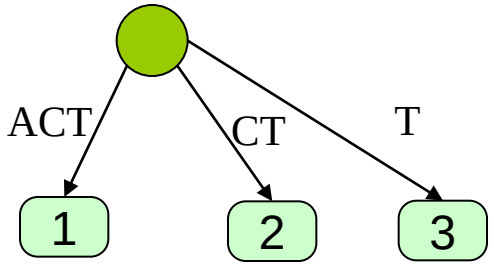

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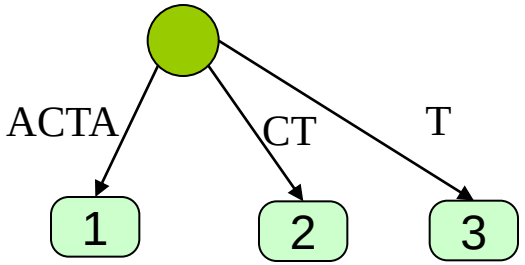

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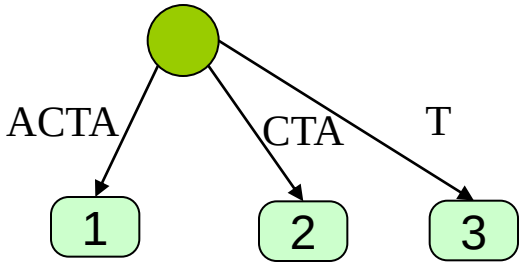

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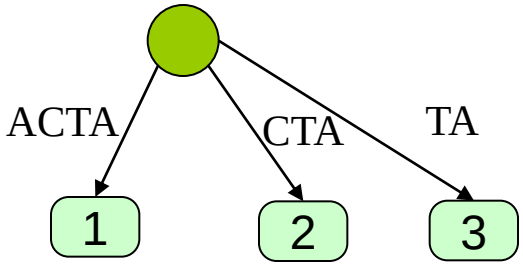

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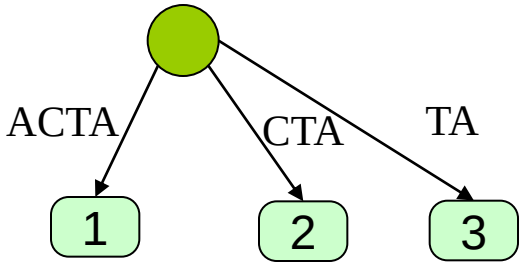

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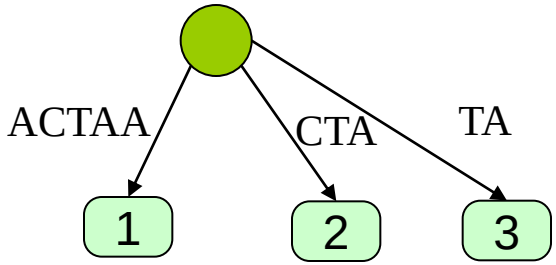

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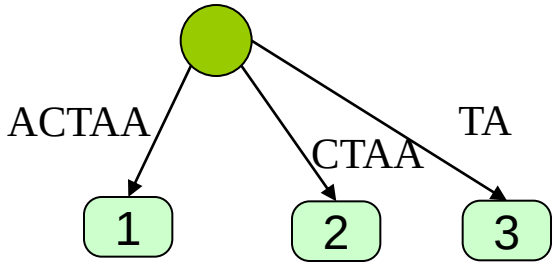

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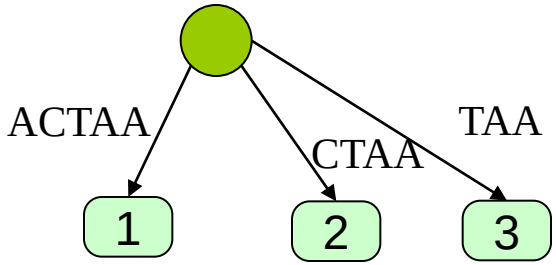

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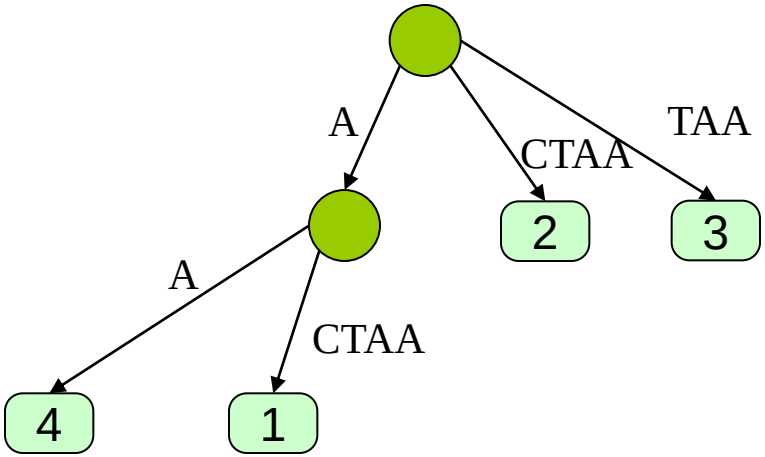

Phases & extensions

- Phase i is when we add character i

- In phase i we have i extensions of suffixes

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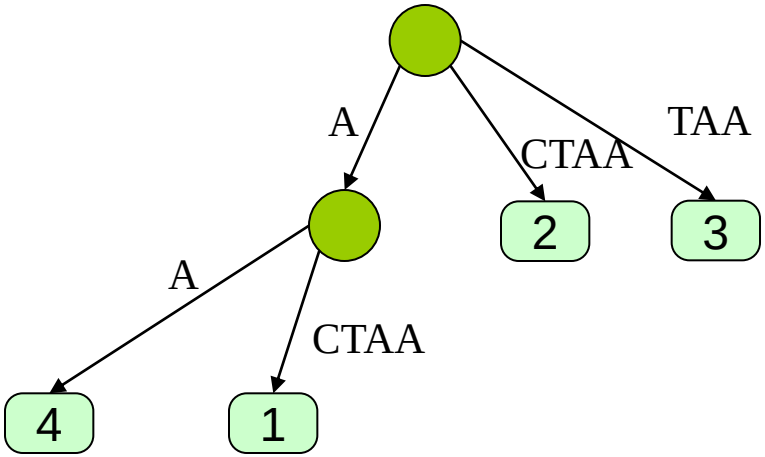

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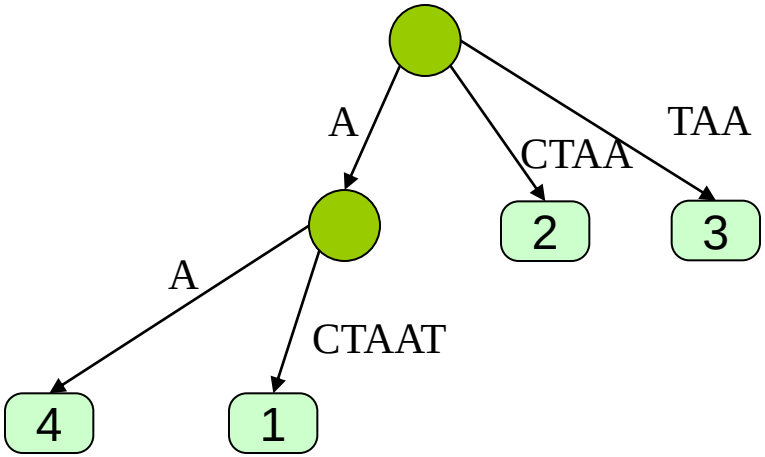

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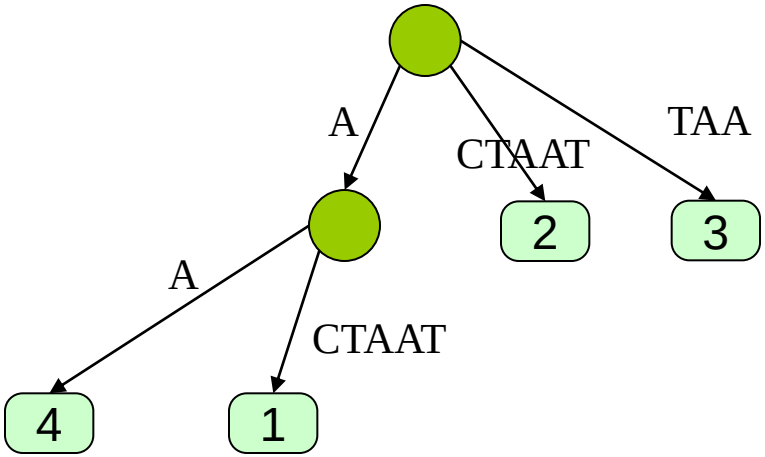

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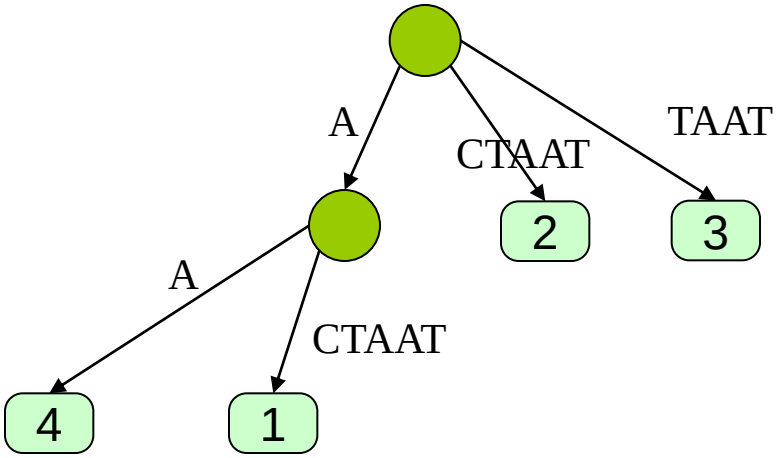

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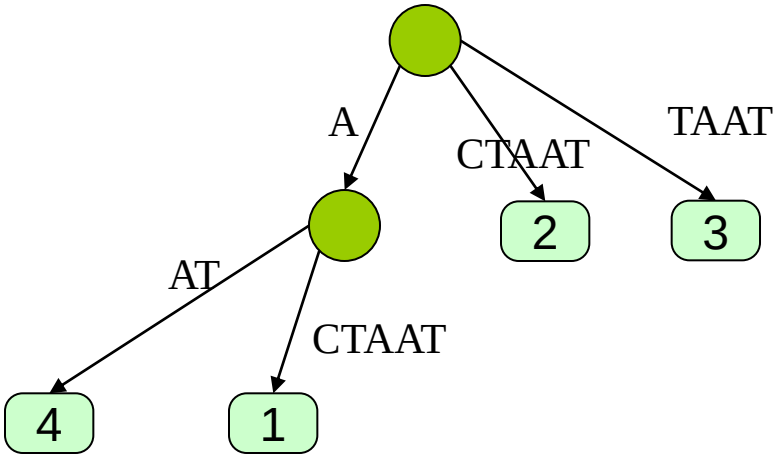

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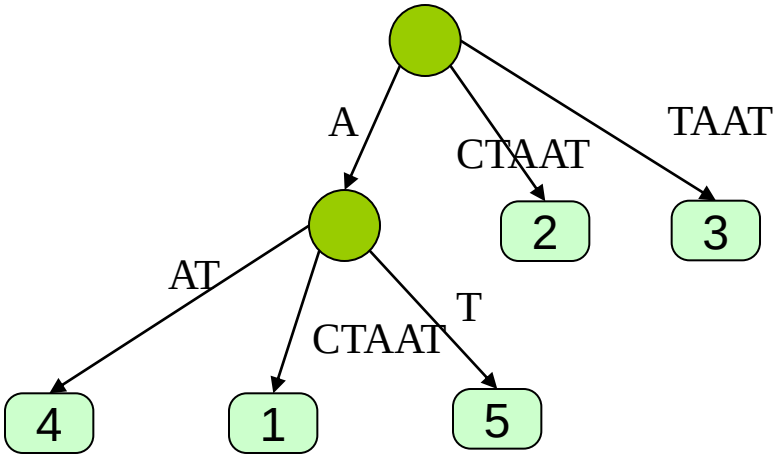

Extension rules

- Rule 1: The suffix ends at a leaf, you add a character on the edge entering the leaf
- Rule 2: The suffix ended internally and the extended suffix does not exist, you add a leaf and possibly an internal node
- Rule 3: The suffix exists and the extended suffix exists, you do nothing

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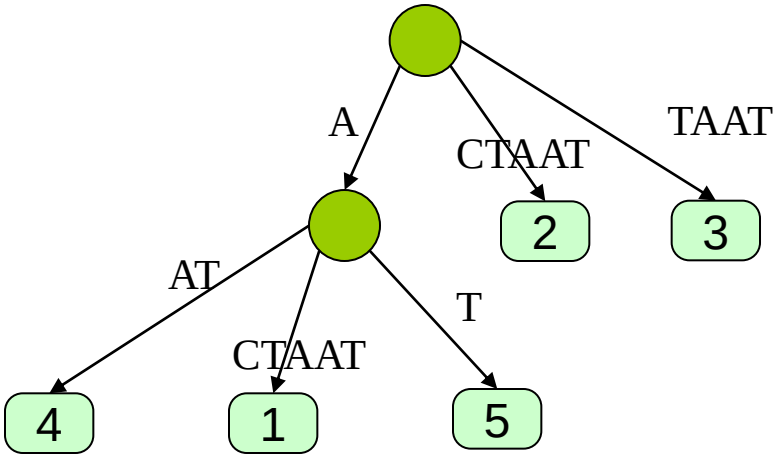

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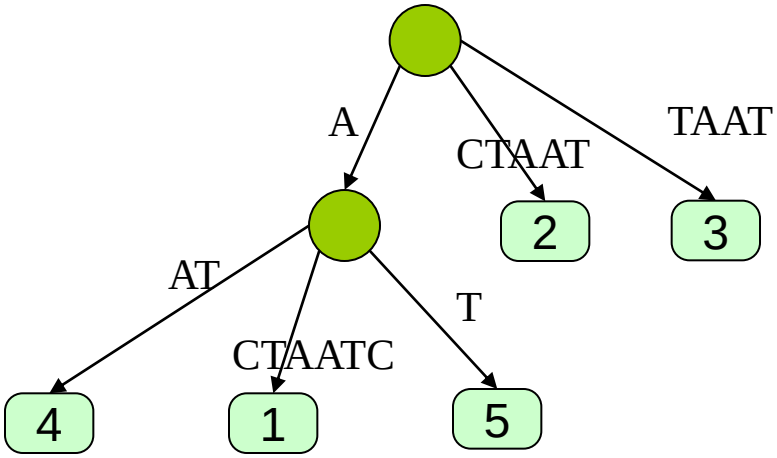

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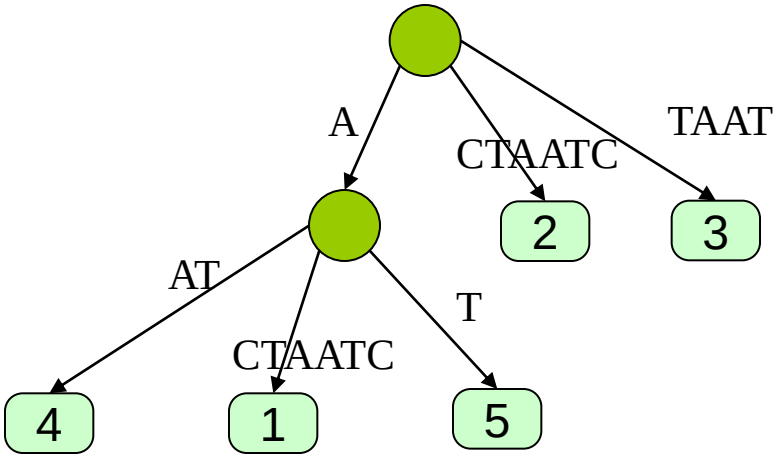

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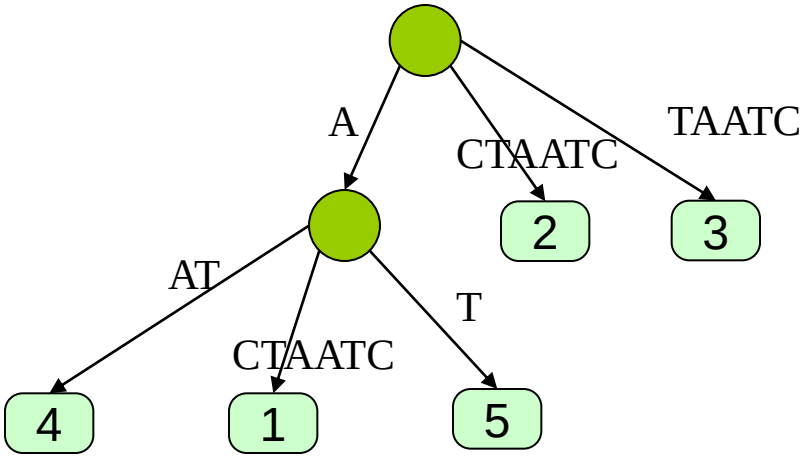

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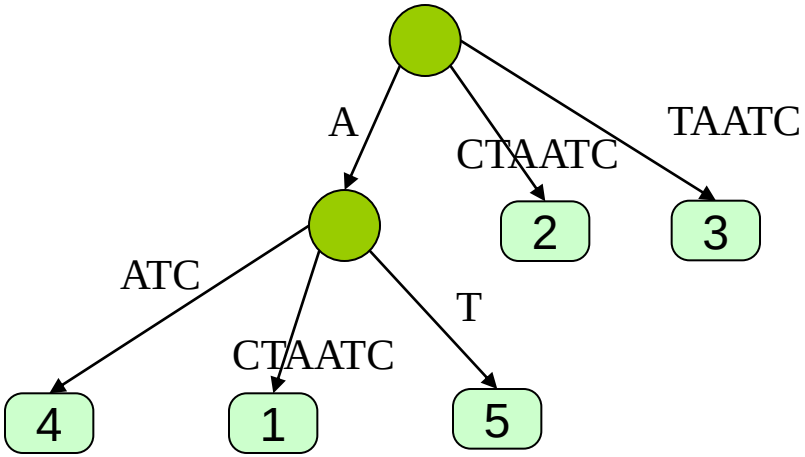

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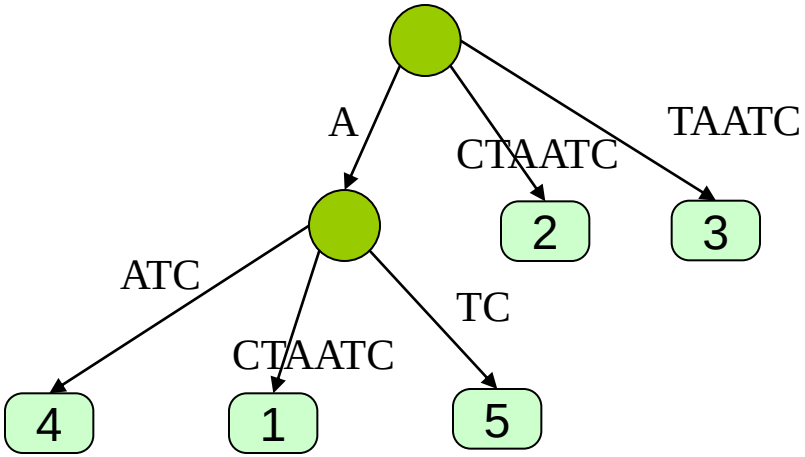

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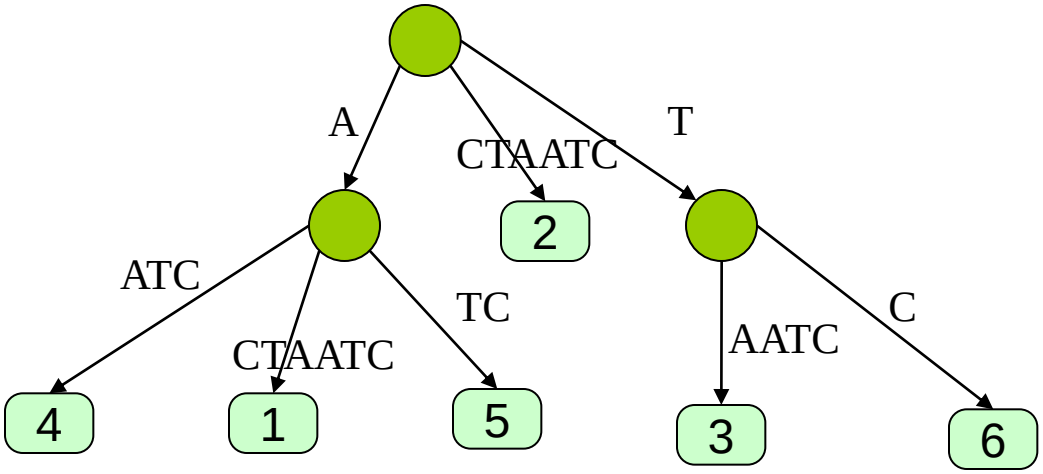

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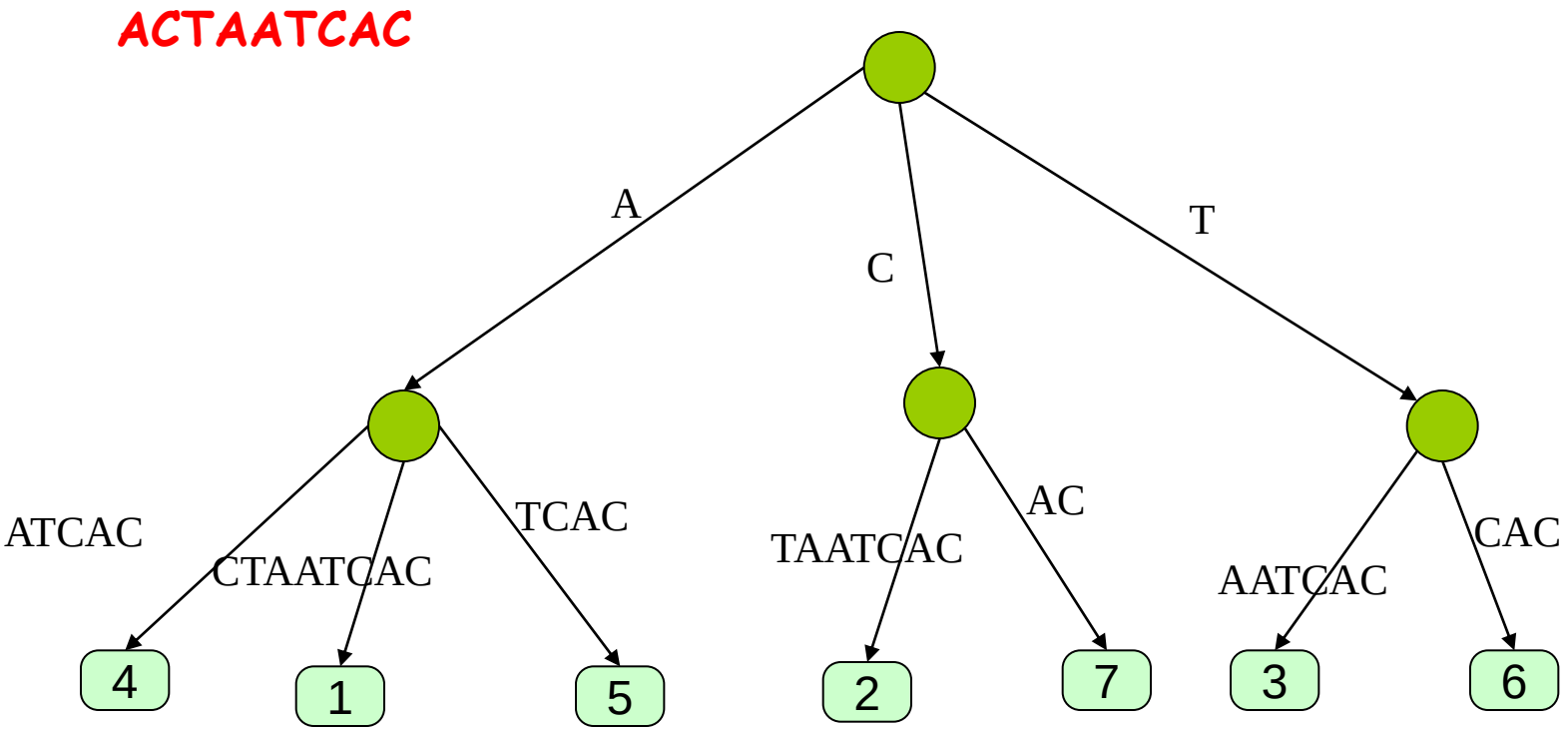

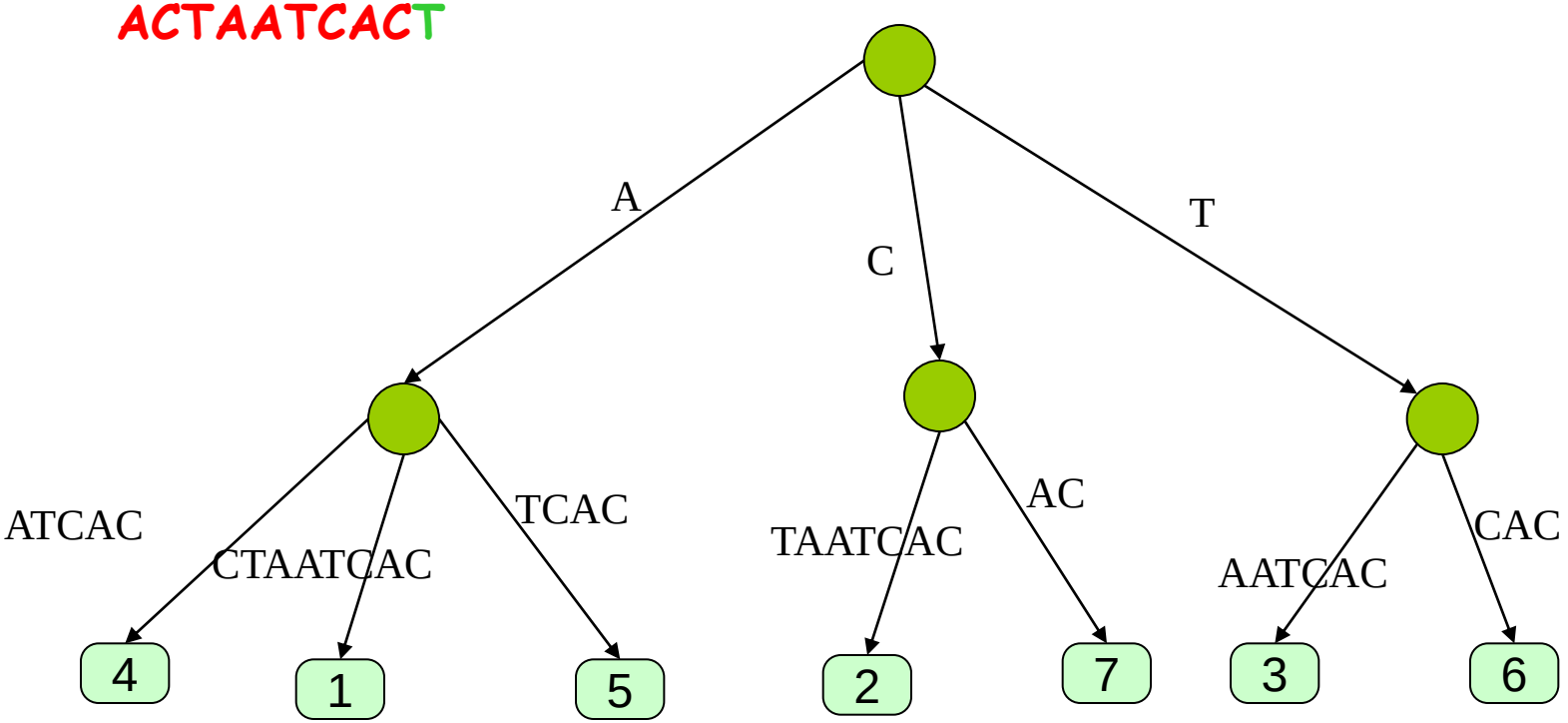
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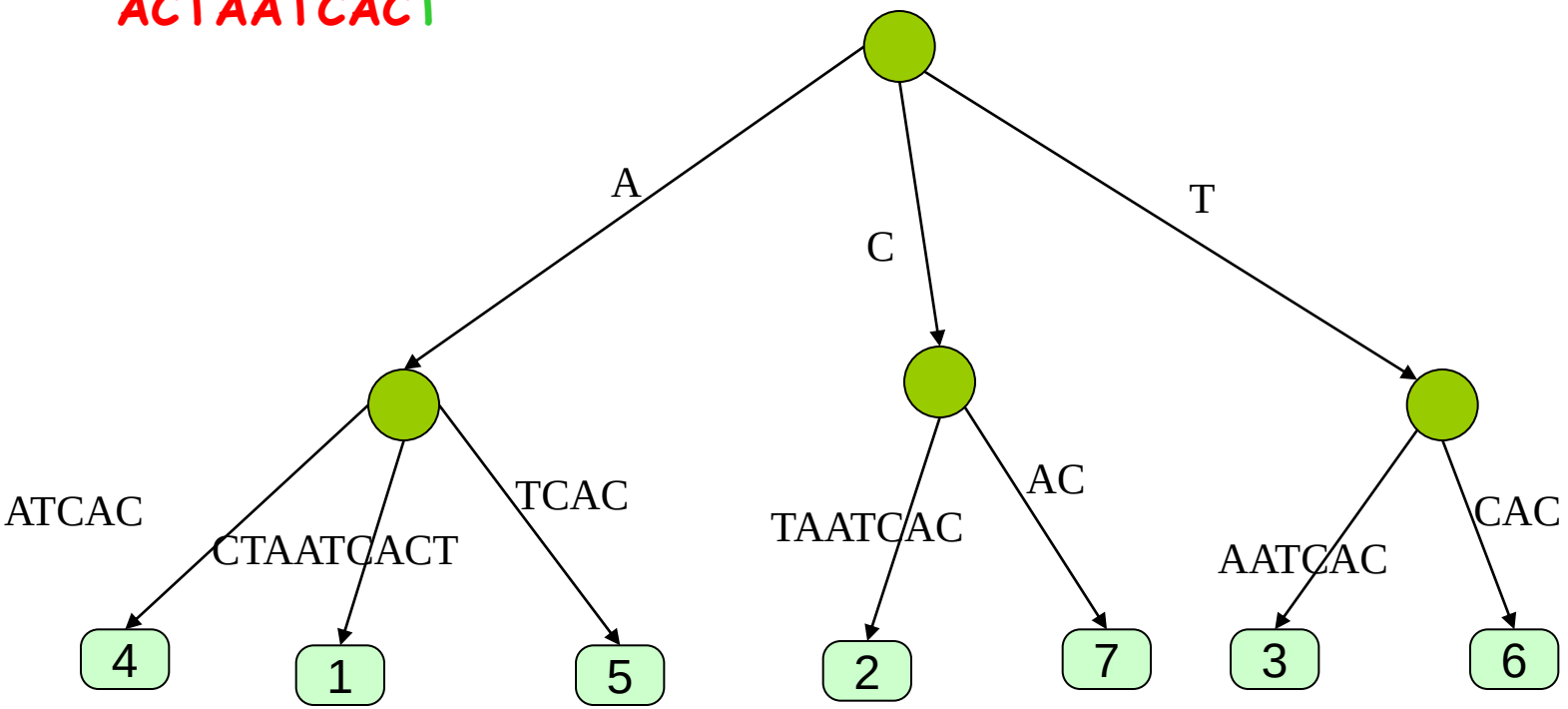

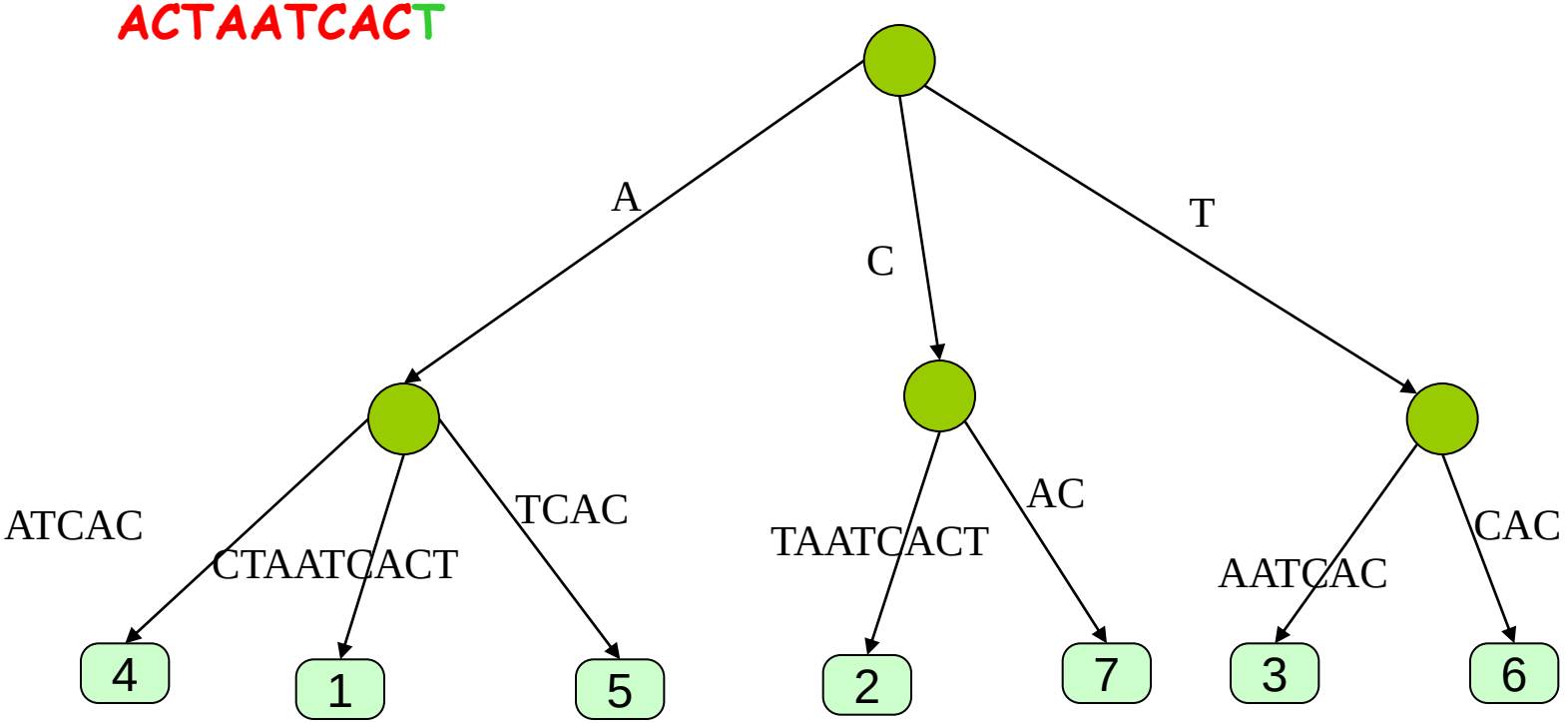
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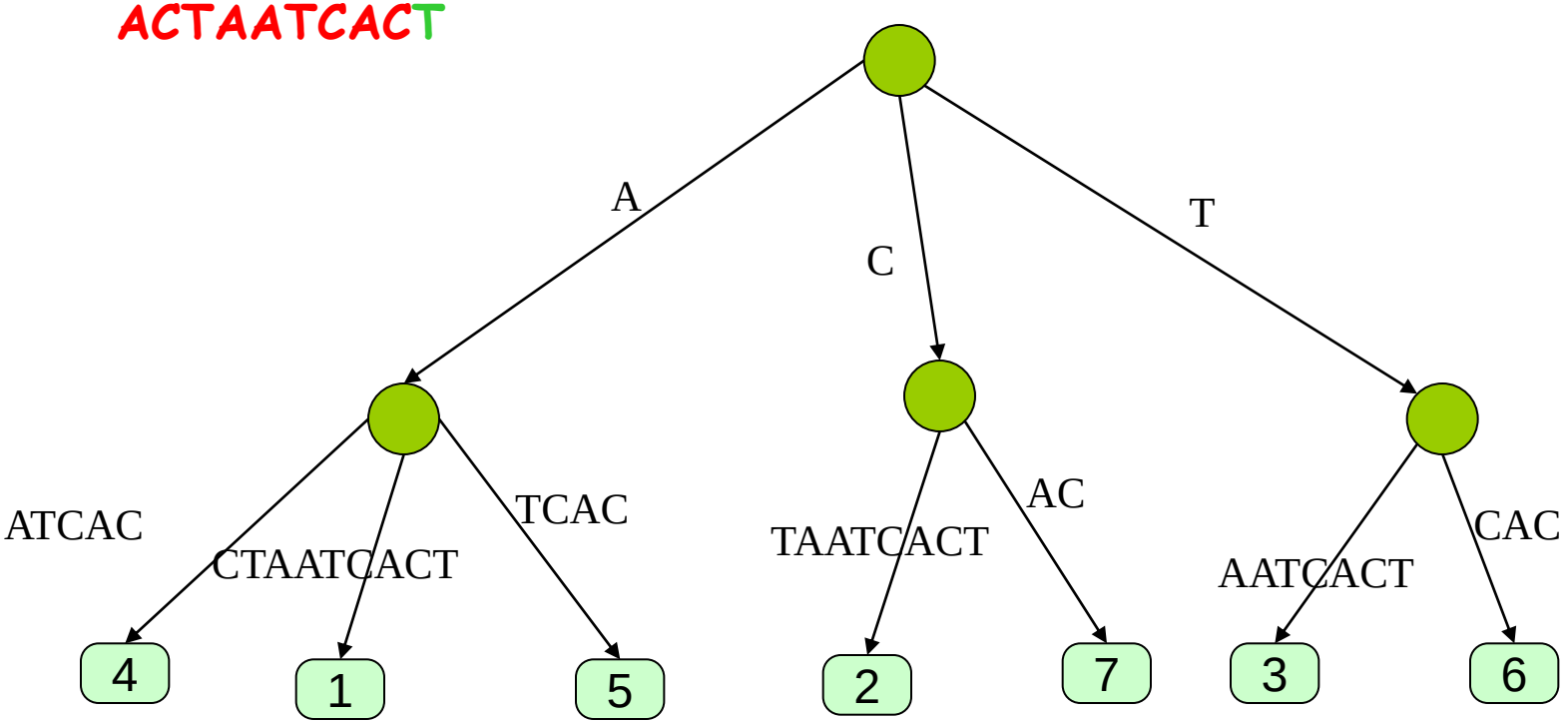

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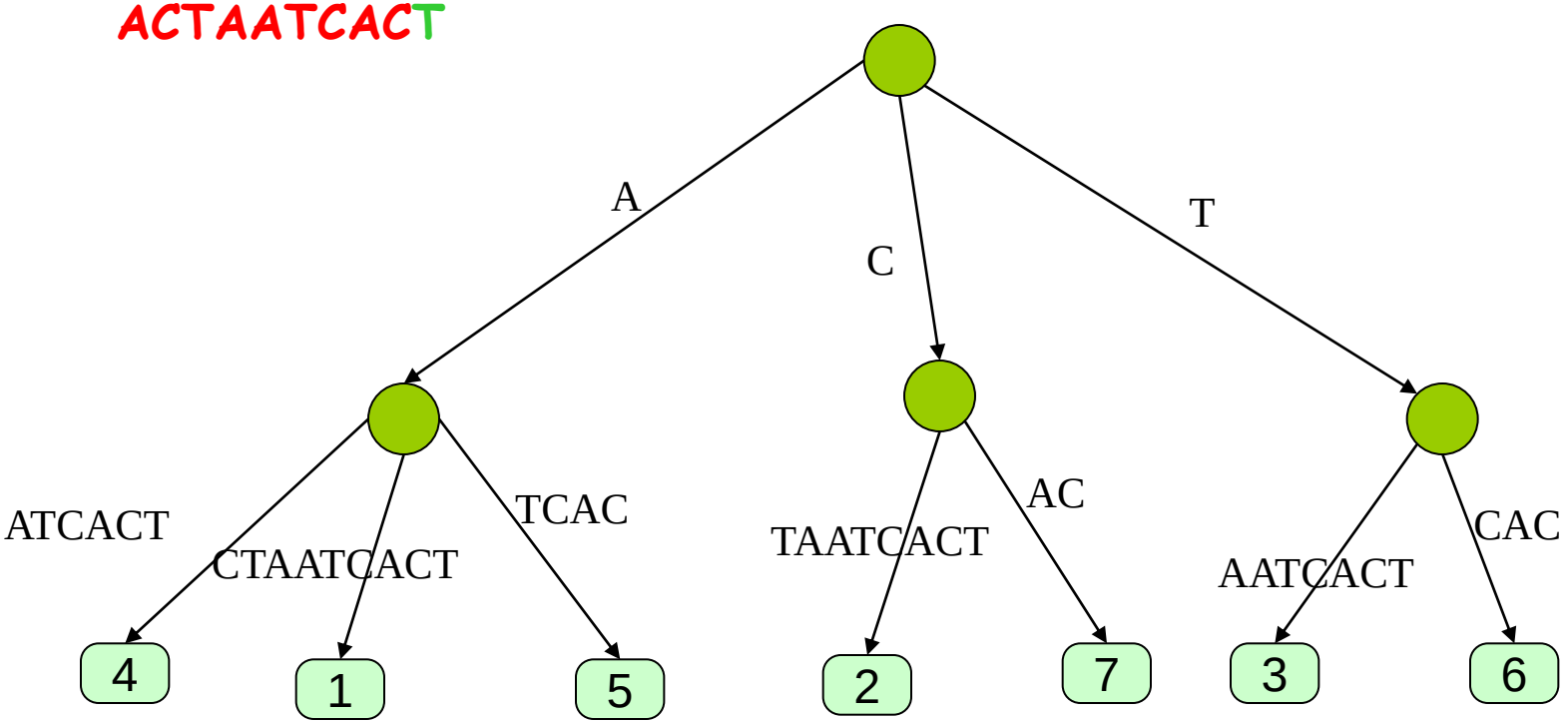

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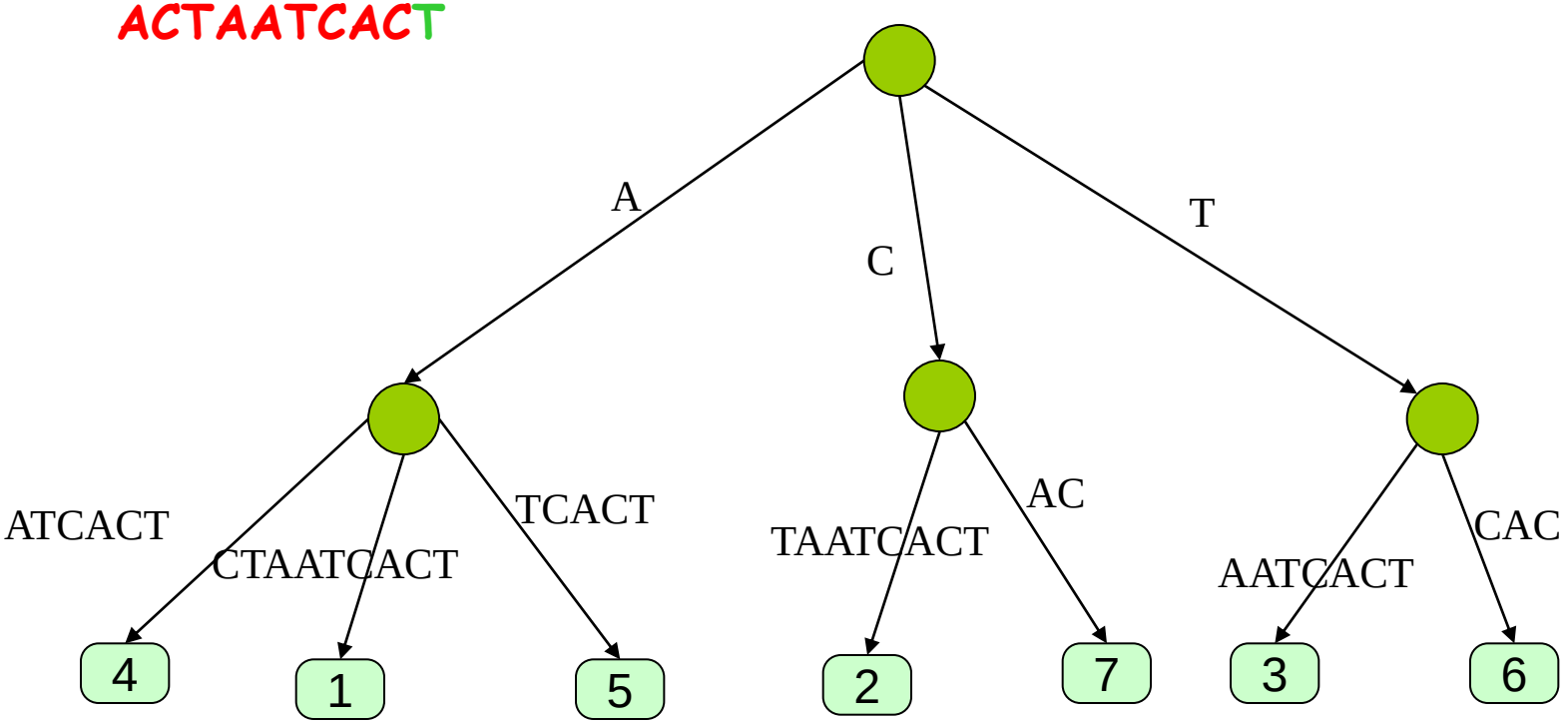
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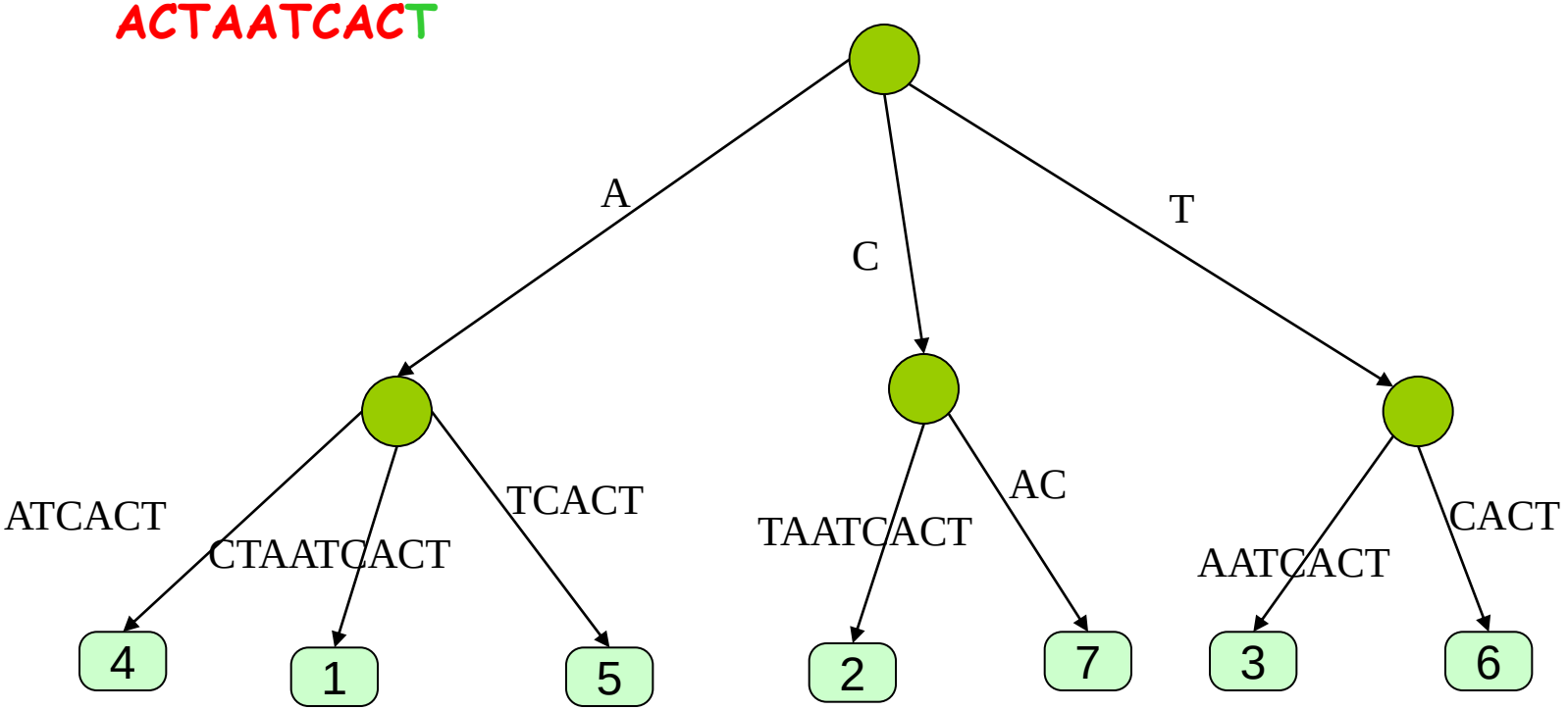
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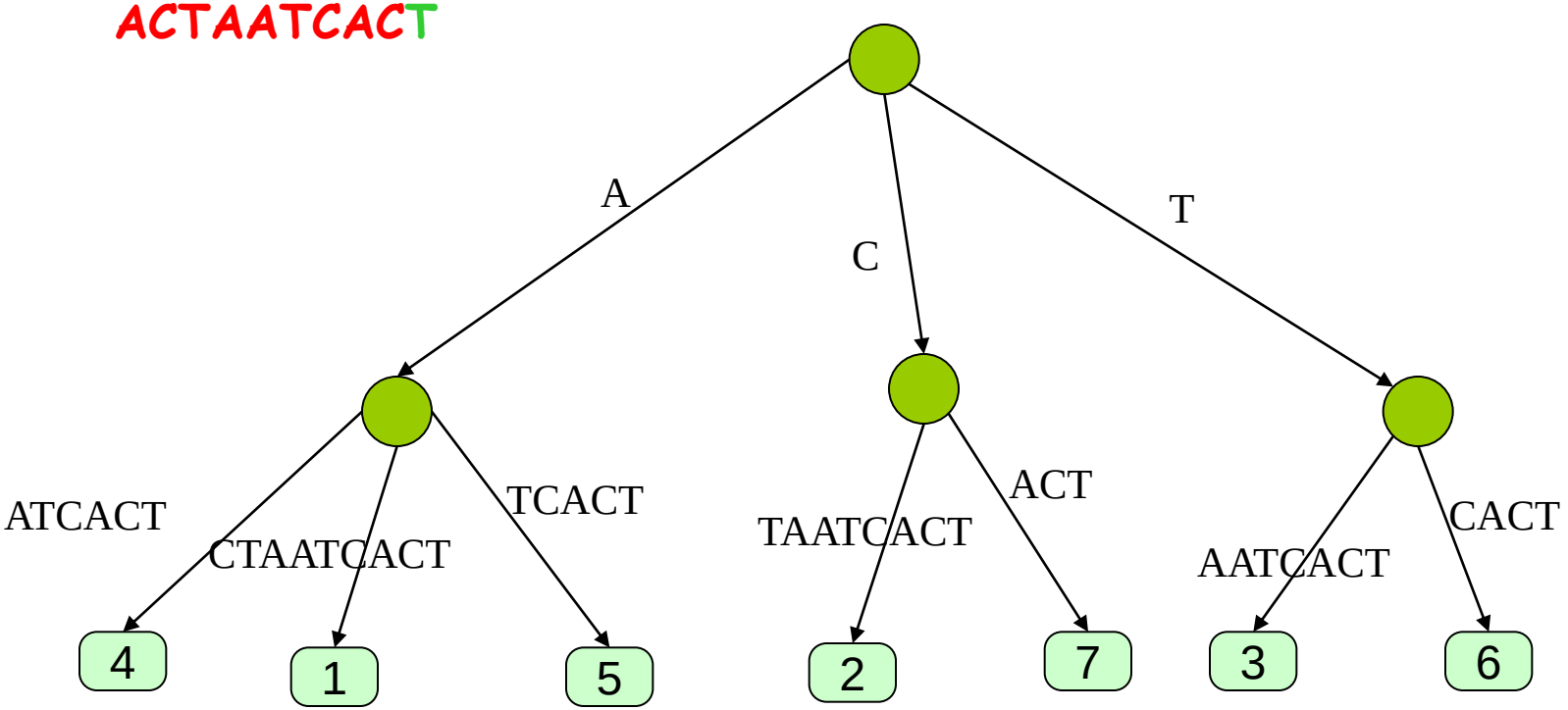
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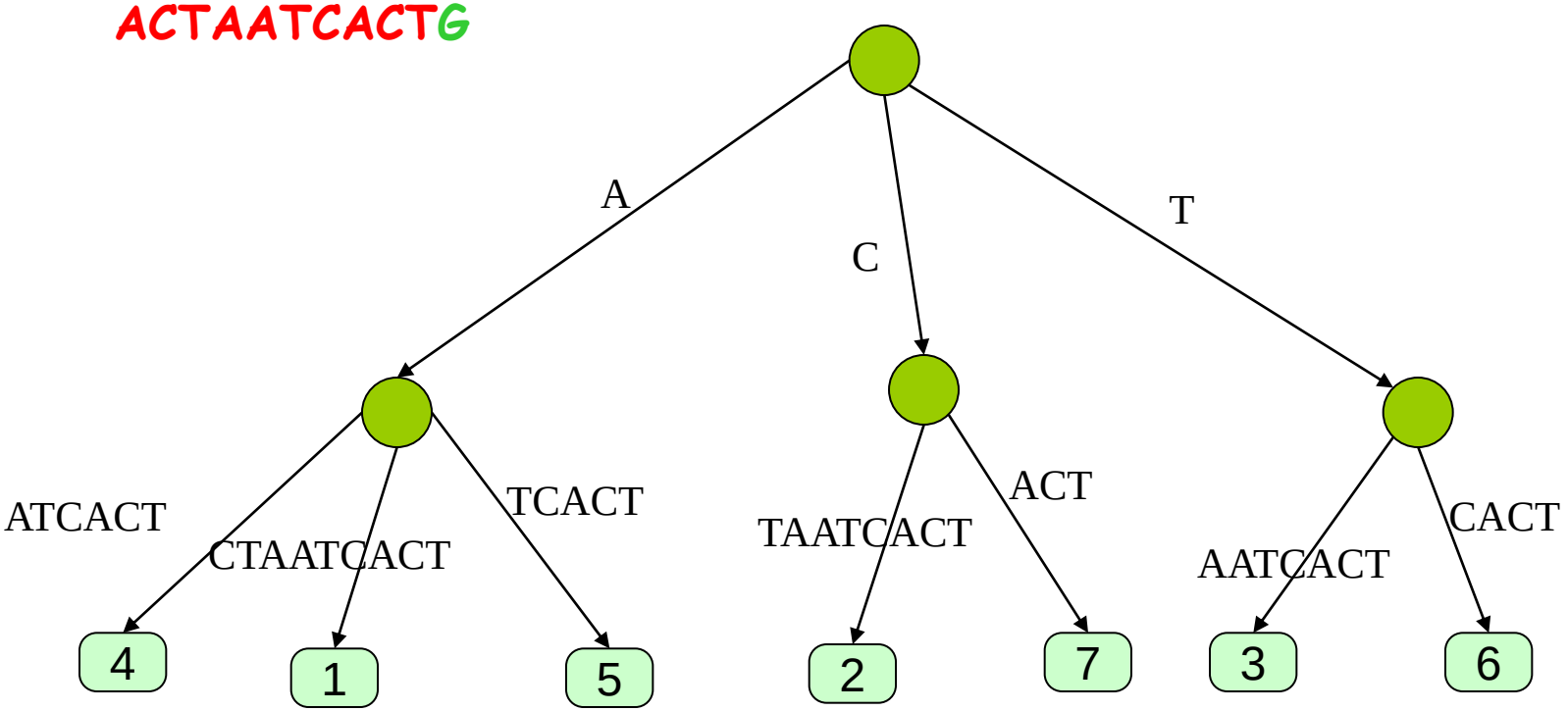
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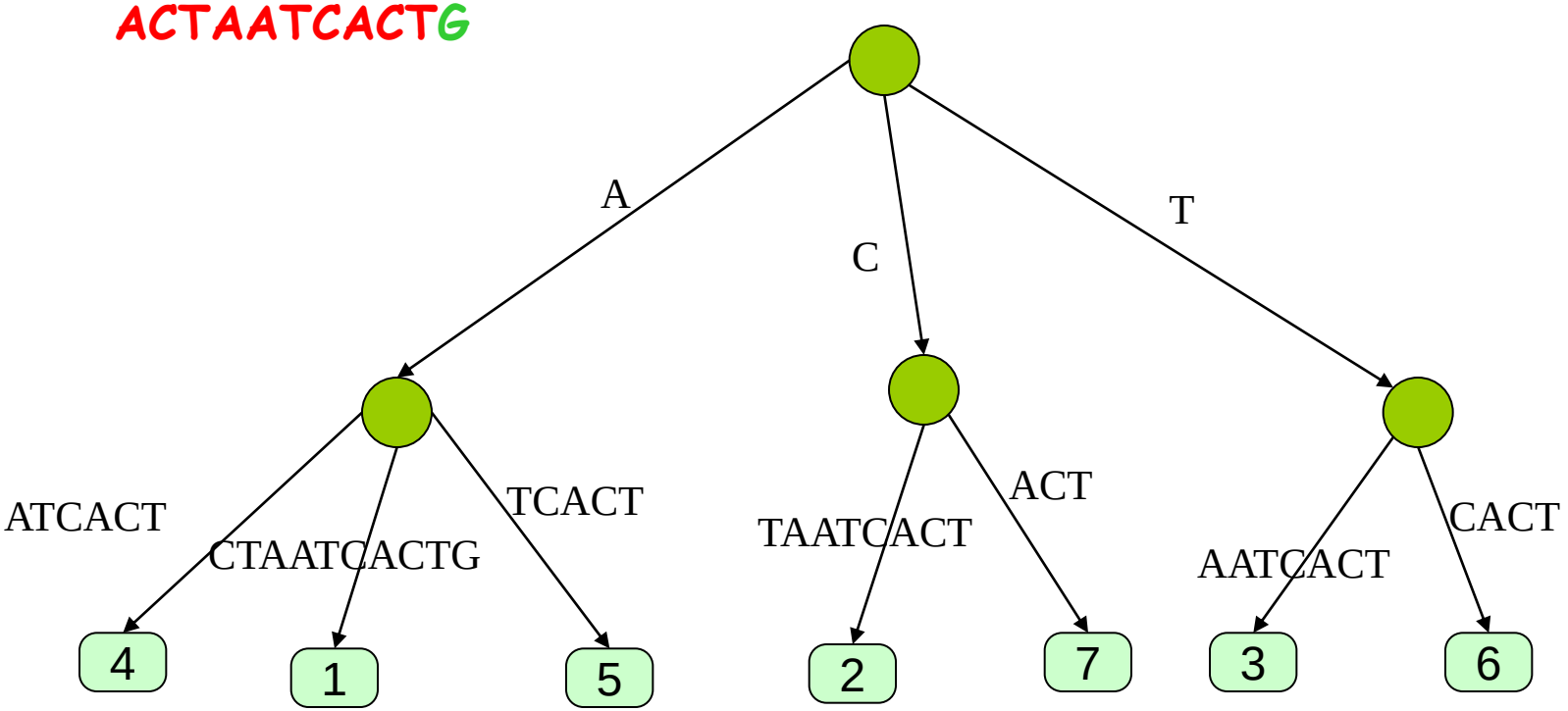
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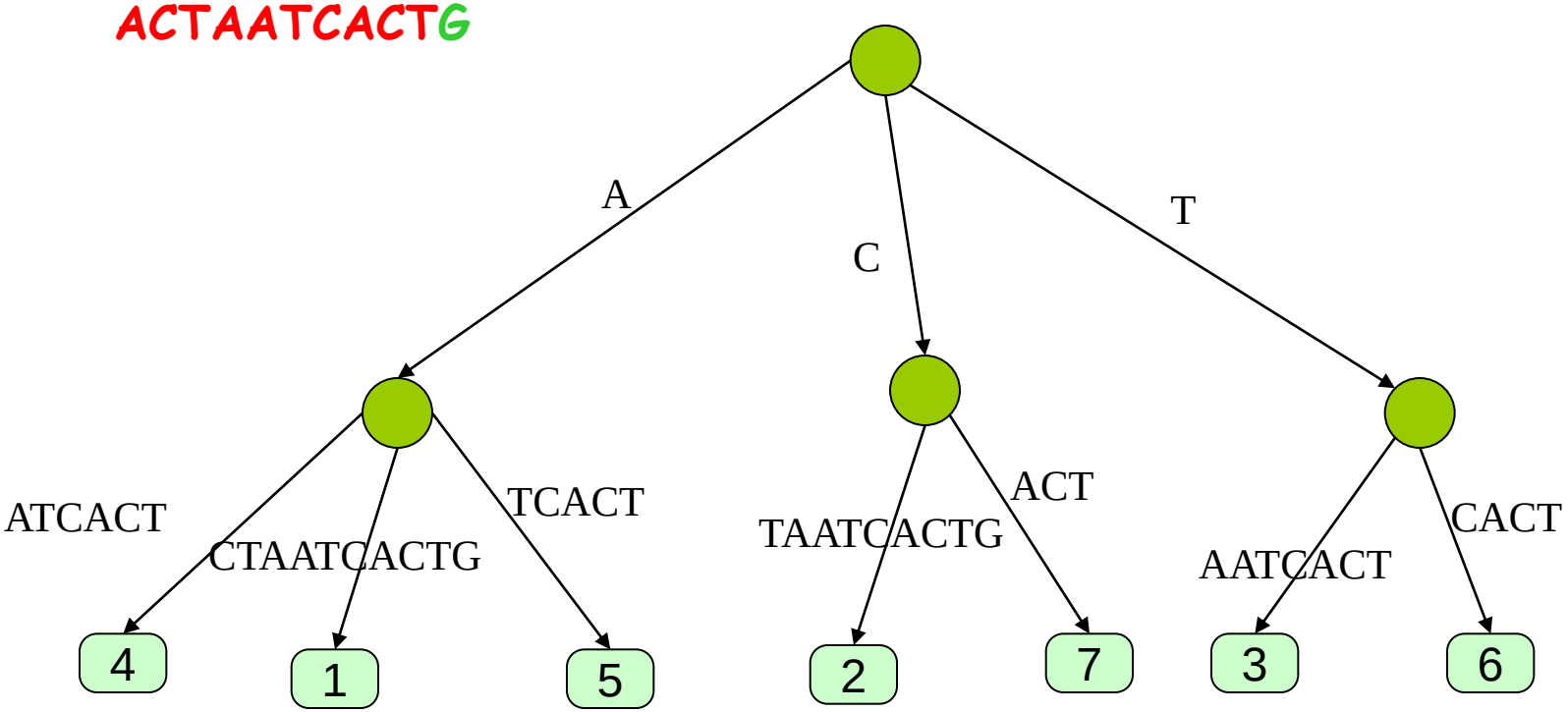
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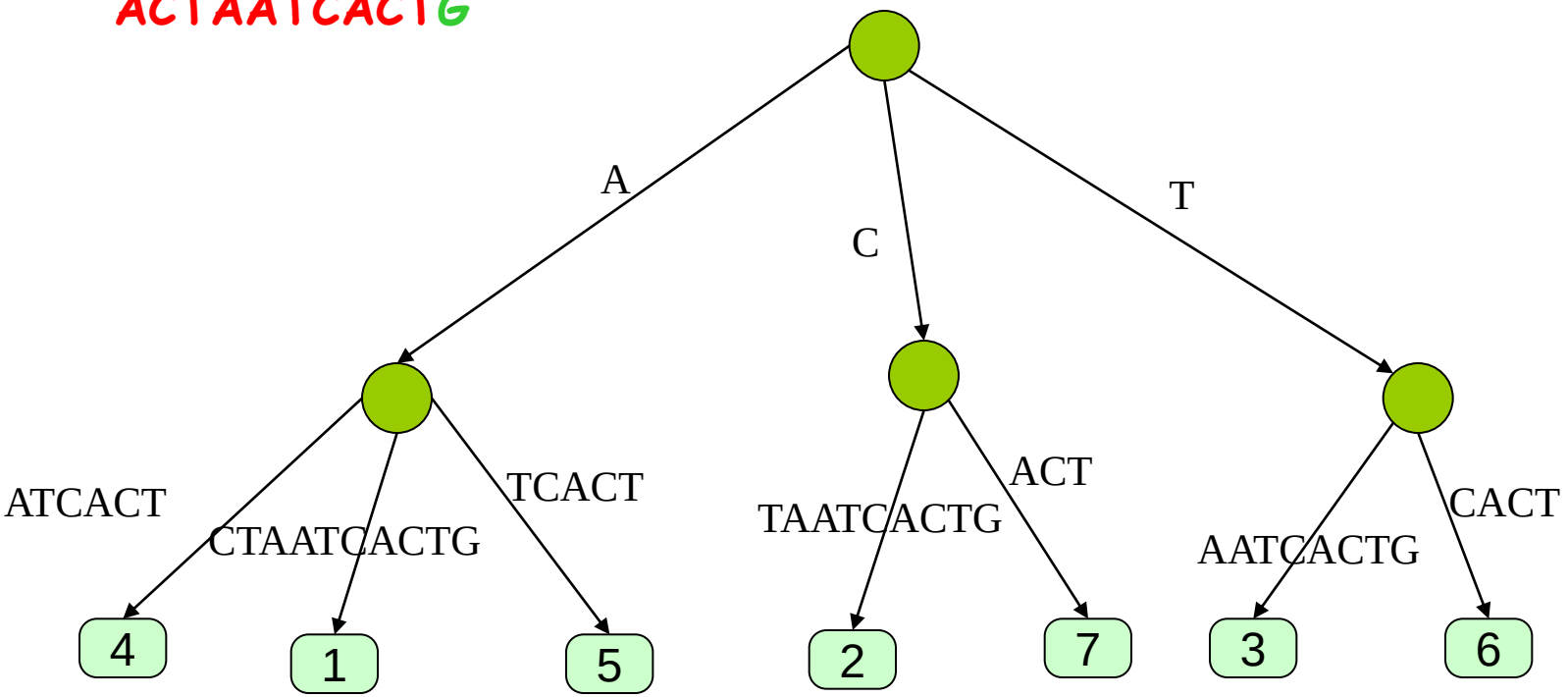

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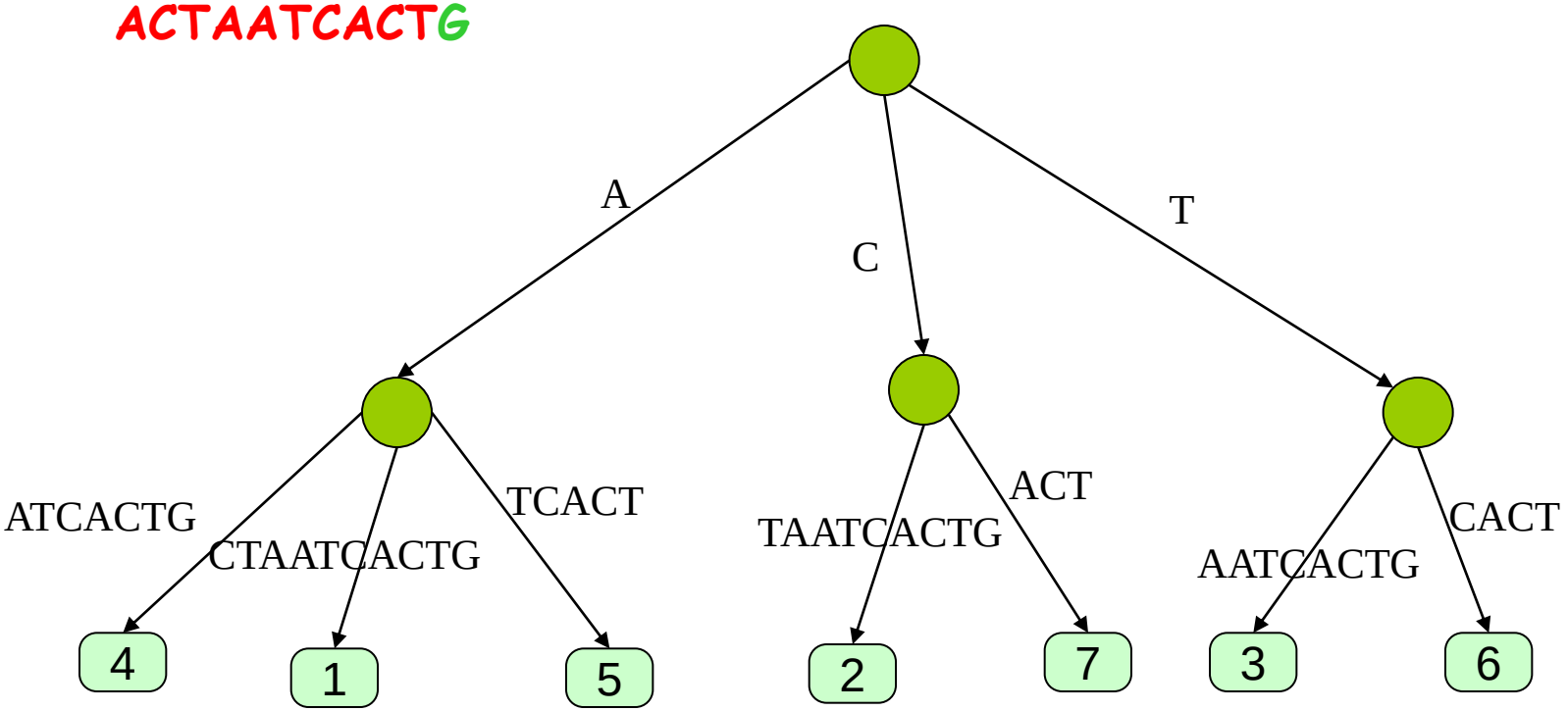

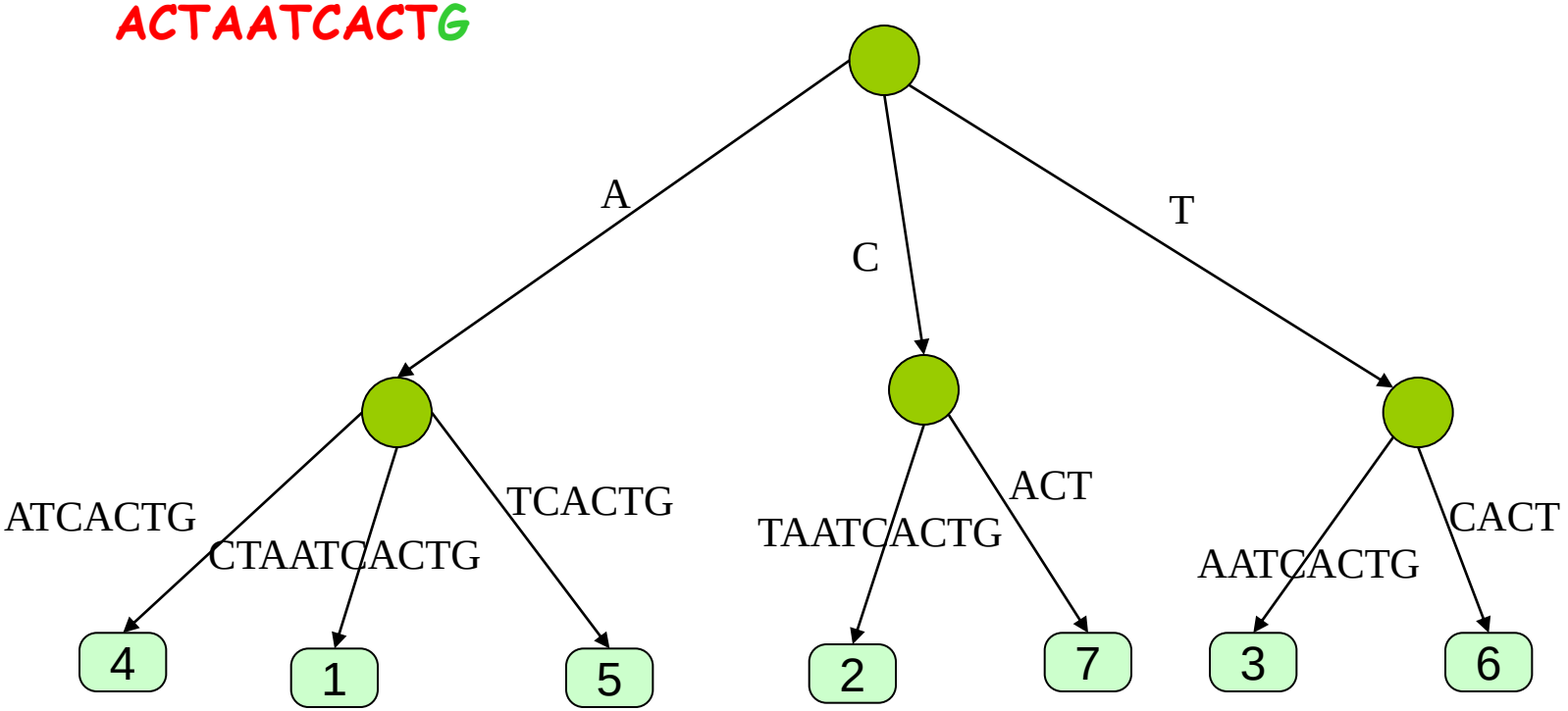
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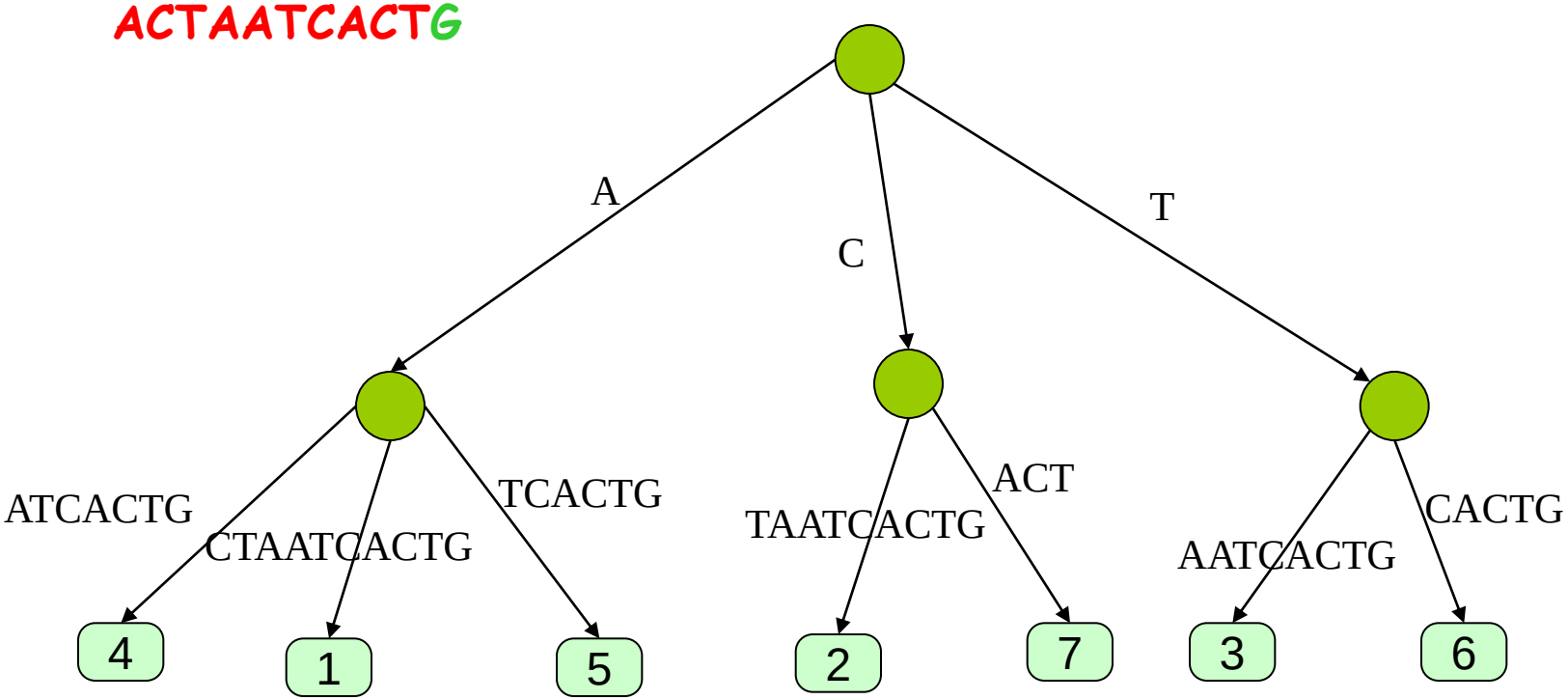
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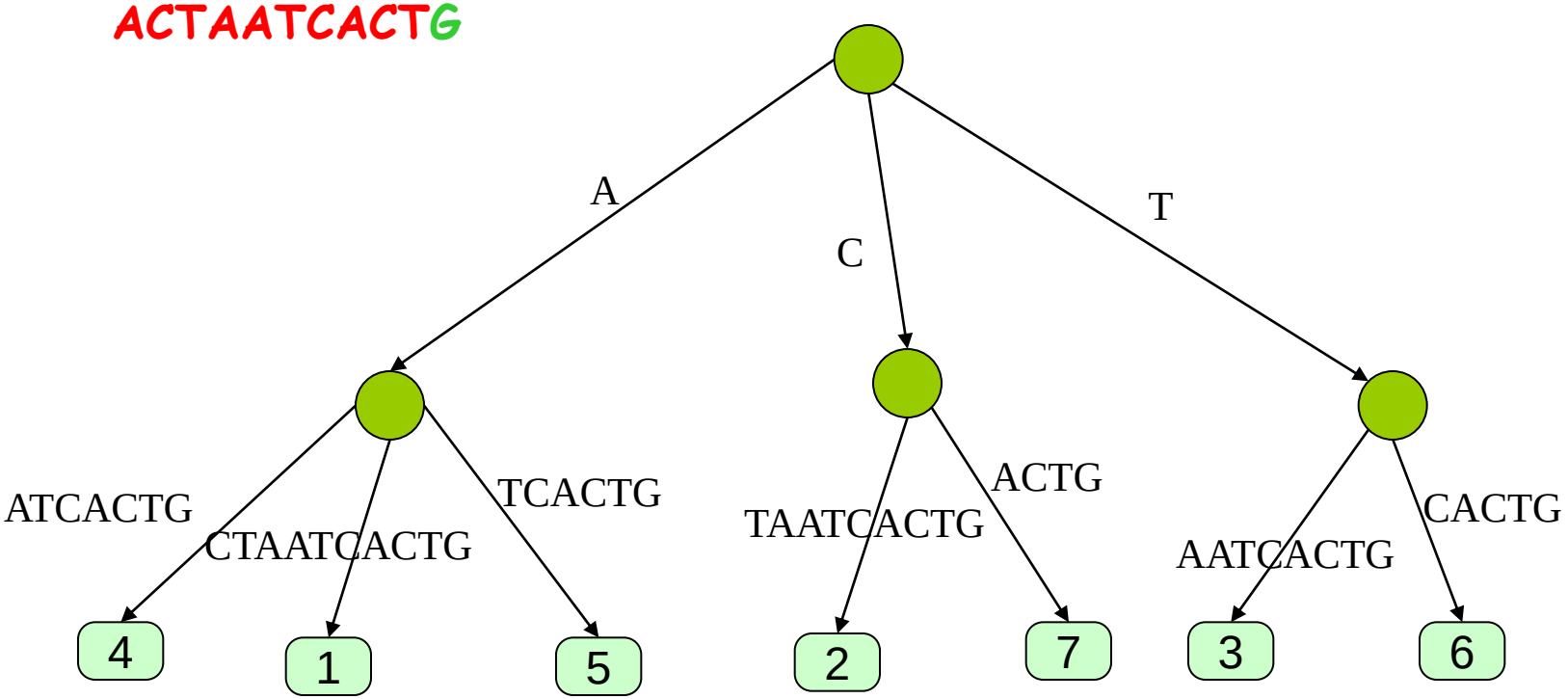

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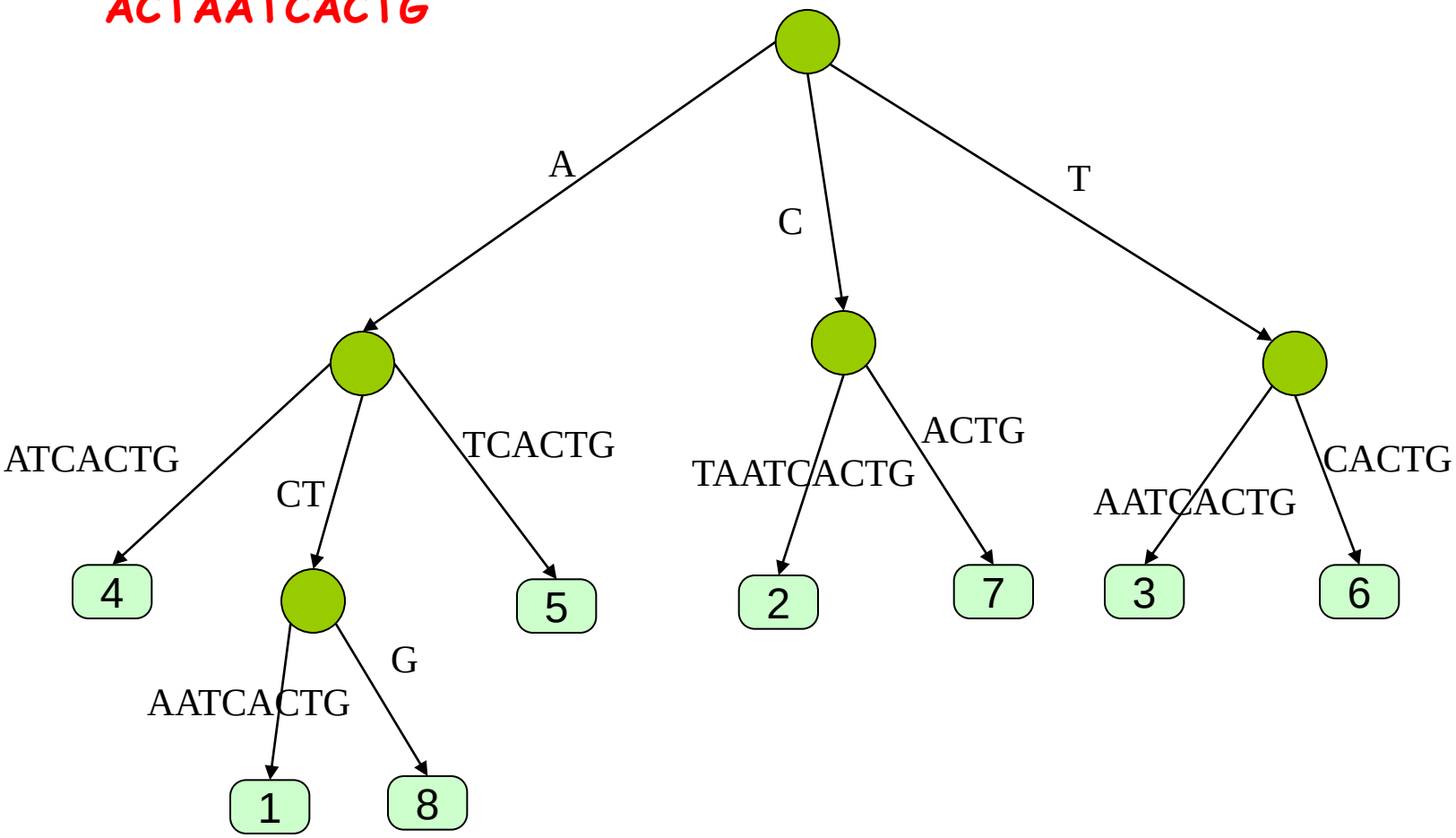

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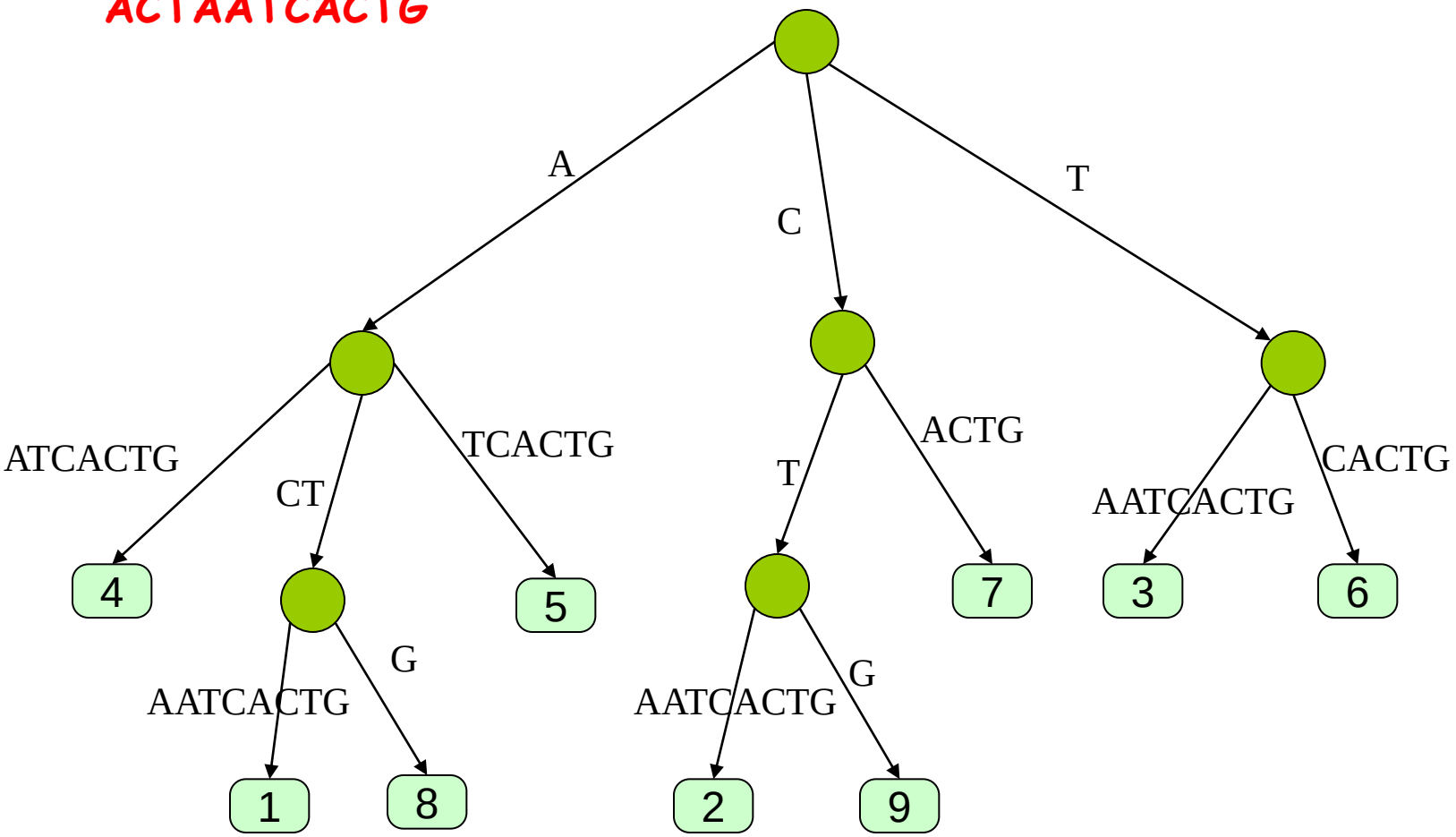

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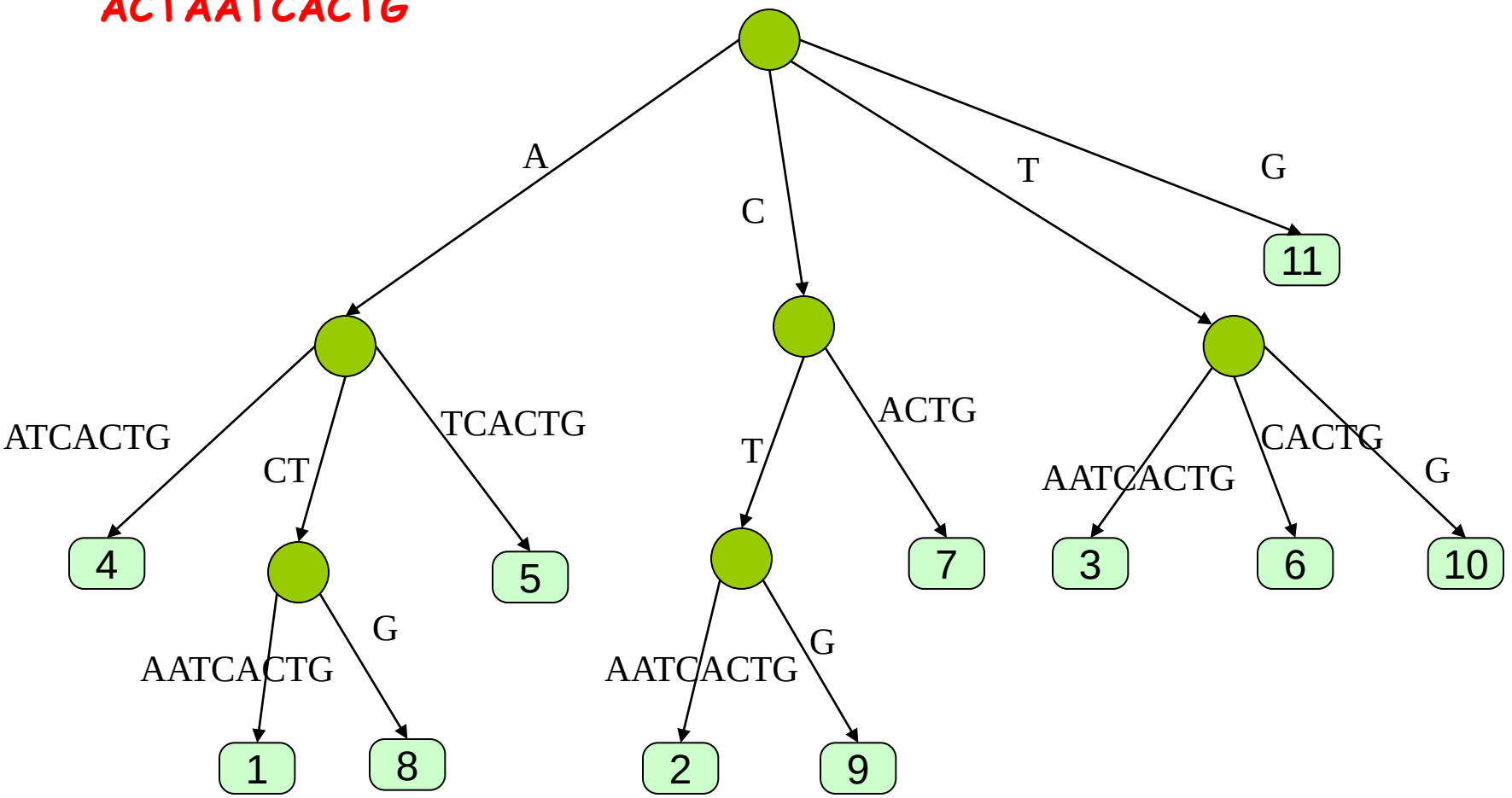
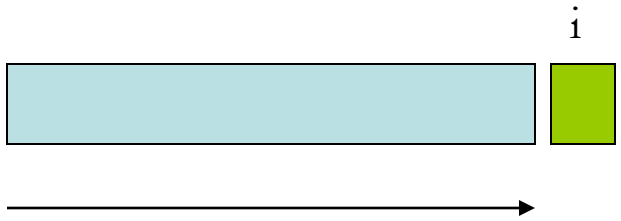
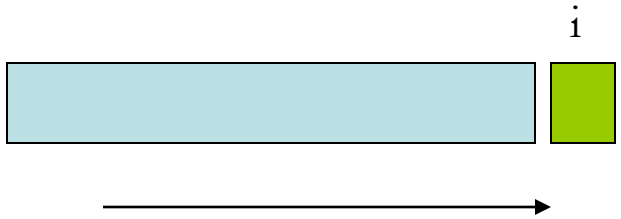


Observations

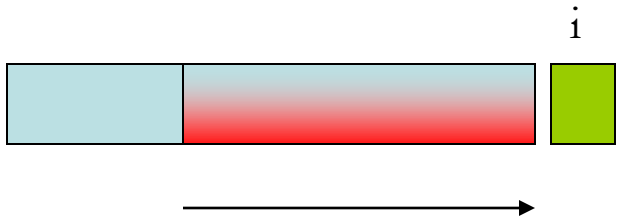


At the first extension we must end at a leaf because no longer suffix exists (rule 1)



At the second extension we still most likely to end at a leaf.

We will not end at a leaf only if the second suffix is a prefix of the first



Say at some extension we do not end at a leaf

Then this suffix is a prefix of some other suffix (suffixes)

We will not end at a leaf in subsequent extensions

Is there a way to continue using the i^{th} character ?

(Is it a prefix of a suffix where the next character is the i^{th} character ?)

Rule 3

Rule 2

Rule 3

Rule 2

If we apply rule 3 then in all subsequent extensions we will apply rule 3

Otherwise we keep applying rule 2 until in some subsequent extension we will apply rule 3

Rule 3

In terms of the rules that we apply a phase looks like:

We have nothing to do when applying rule 3, so once rule 3 happens we can stop

We don't really do anything significant when we apply rule 1 (the structure of the tree does not change)

Representation

- We do not really store a substring with each edge, but rather pointers into the starting position and ending position of the substring in the text

- With this representaion we do not really have to do anything when rule 1 applies

How do phases relate to each other

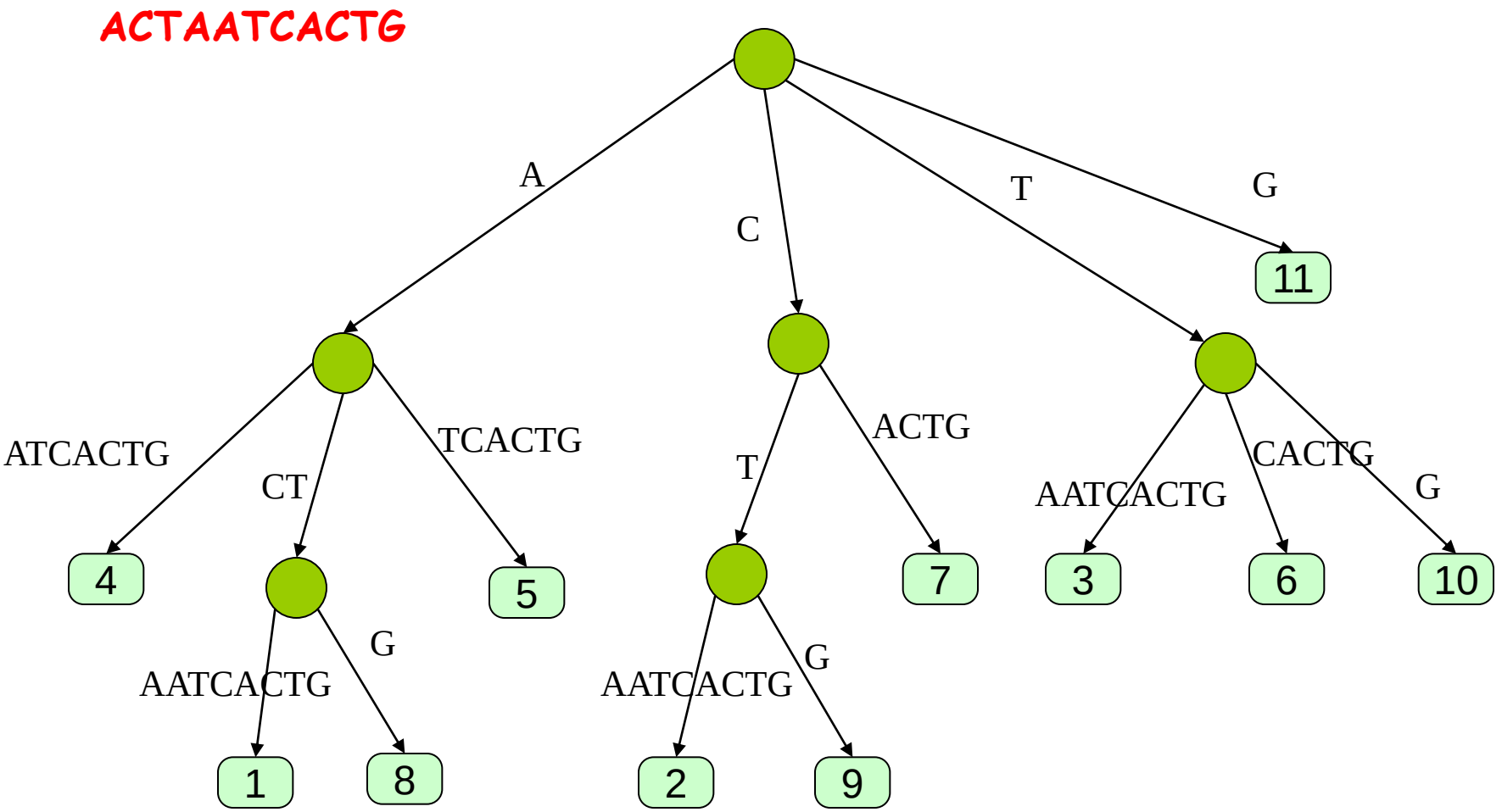
1 1 1 1 1 1 2 2 2 2 3 3 3 3

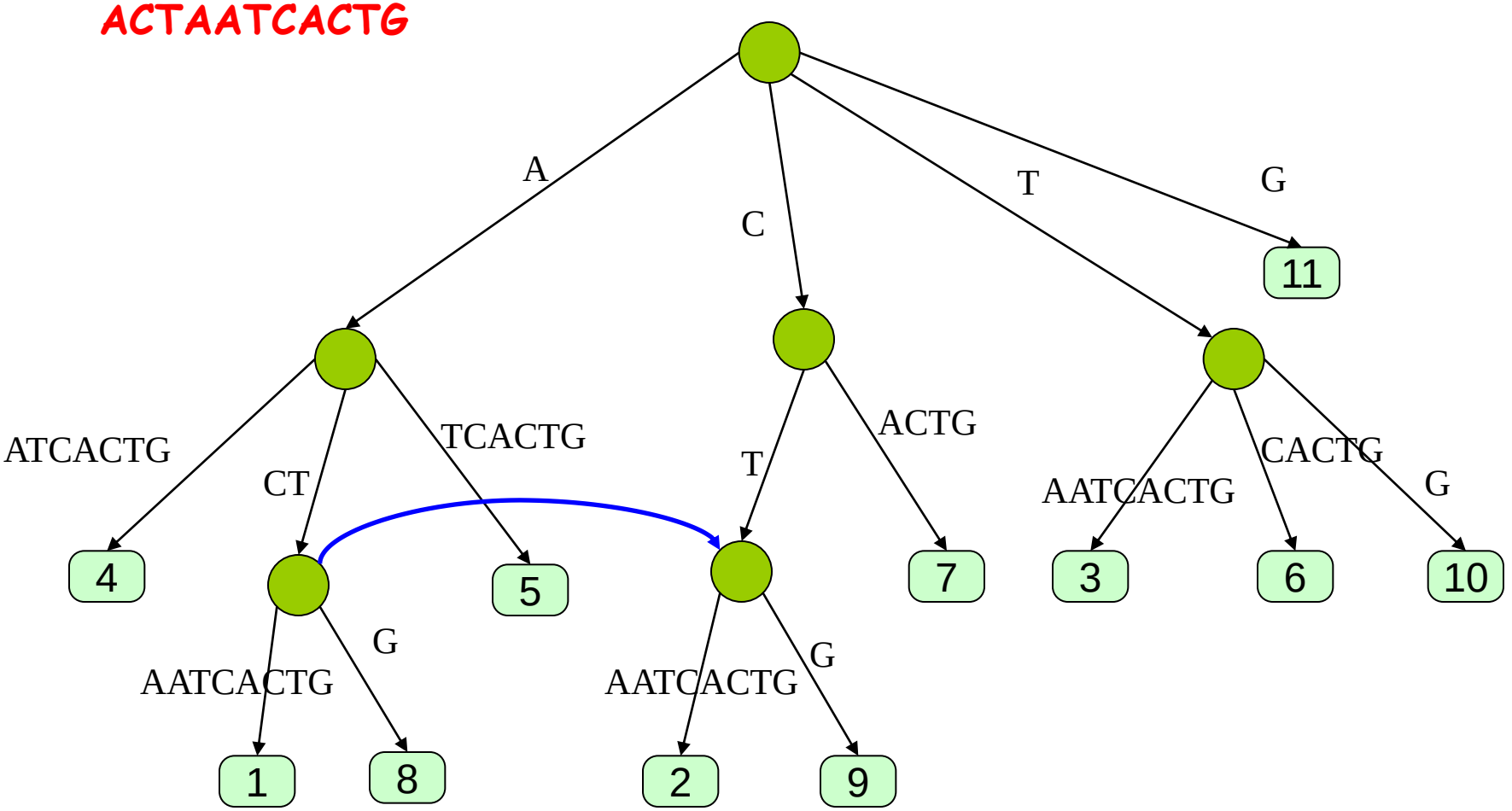
The next phase we must have:

1 1 1 1 1 1 1 1 1 1 1 2/3

So we start the phase with the extension that was the first in which we applied rule 3 in the previous phase

Suffix Links

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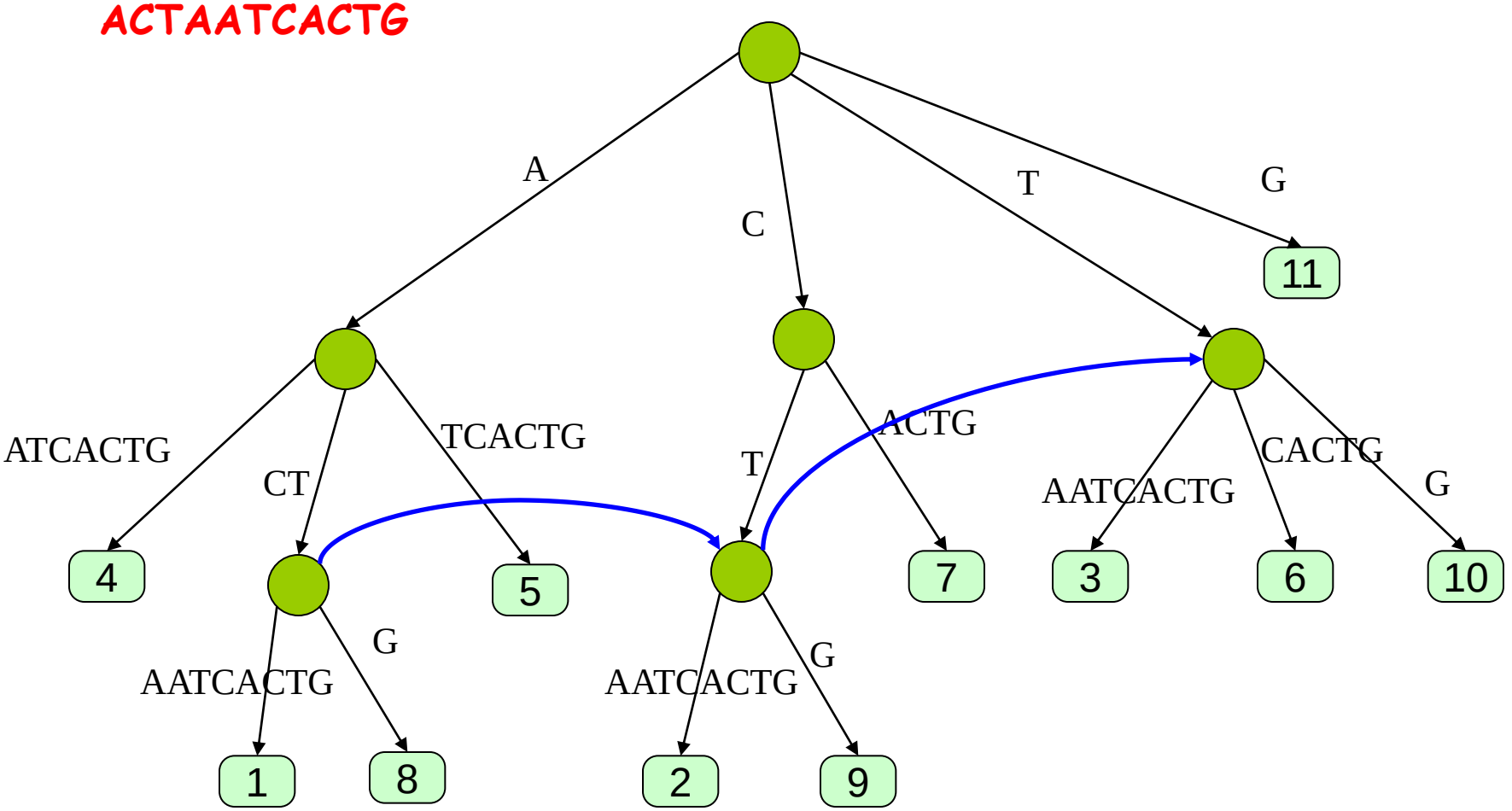
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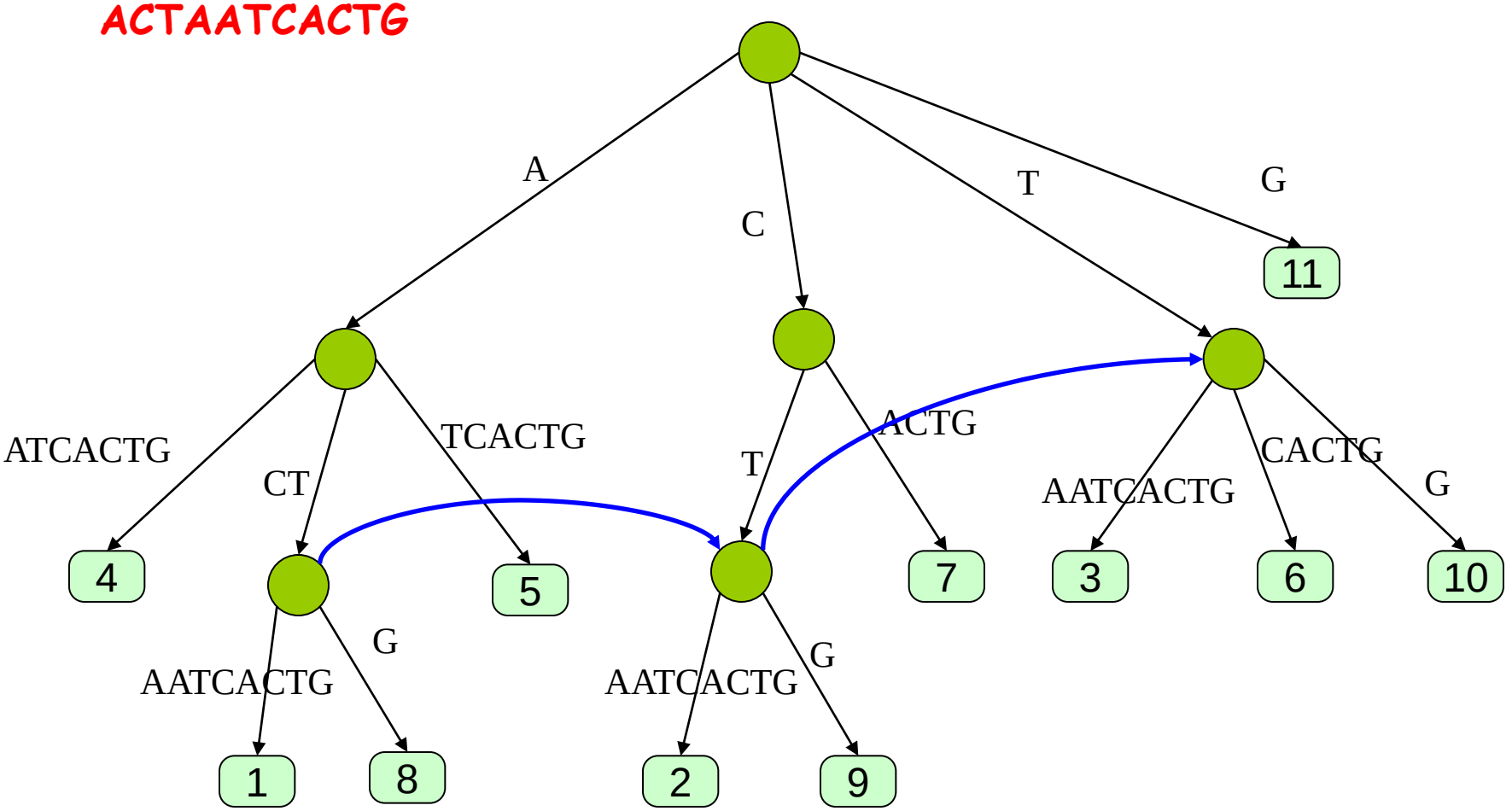
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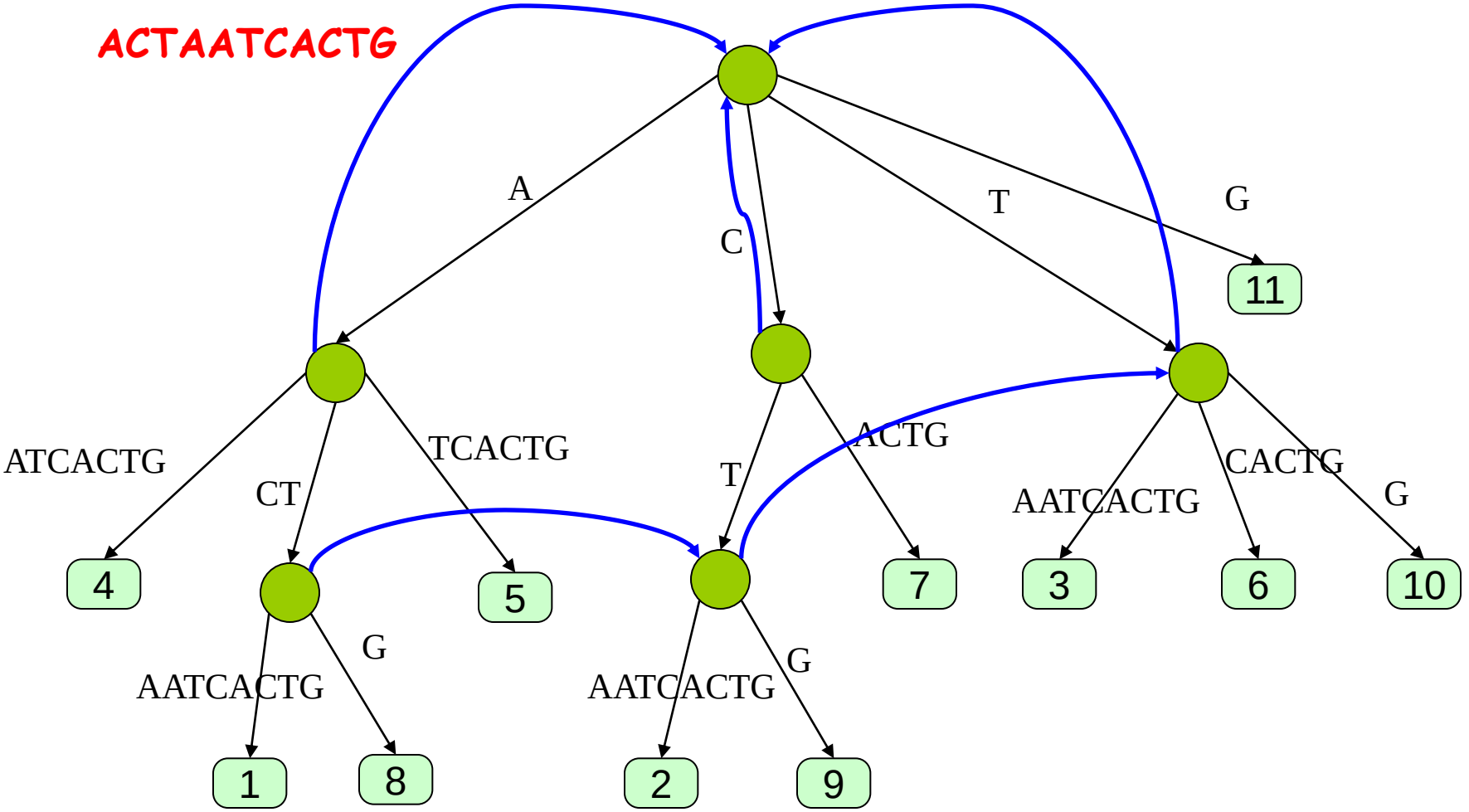
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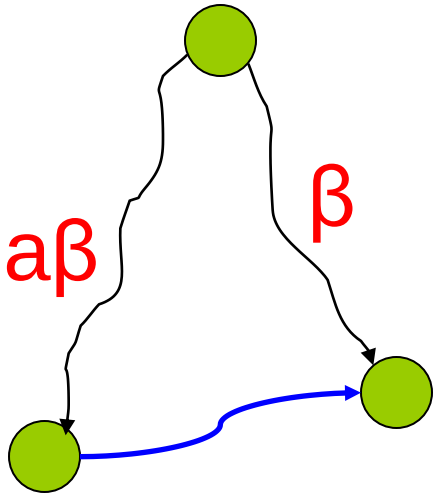
Suffix Links

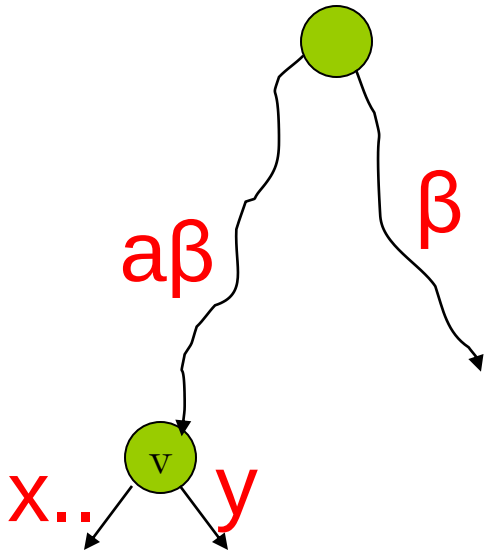
- From an internal node that corresponds to the string aβ to the internal node that corresponds to β (if there is such node)



- Is there such a node ?

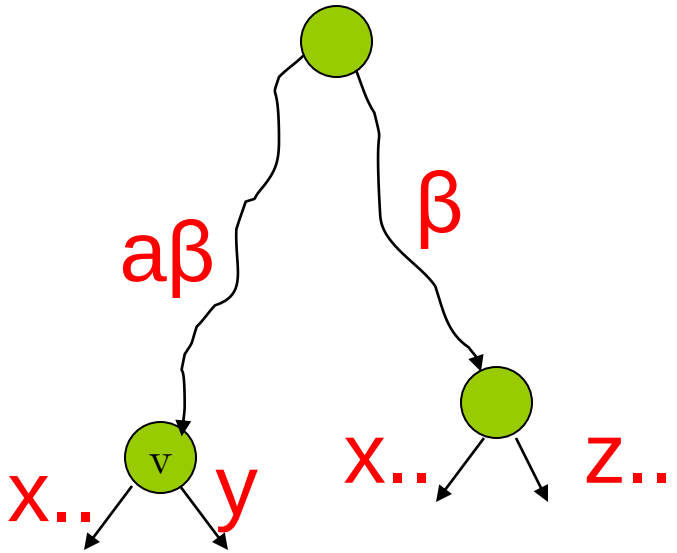
Suppose we create v applying rule 2. Then there was a suffix αβx… and now we add αβy

So there was a suffix βx…



- Is there such a node ?

Suppose we create v applying rule 2. Then there was a suffix αβx… and now we add αβy



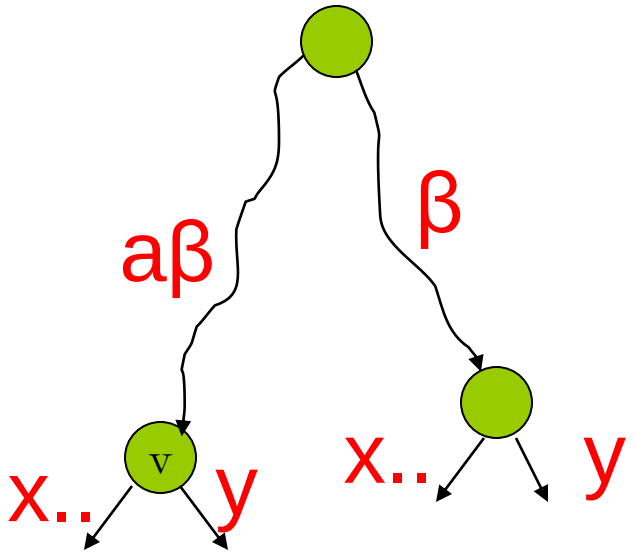
So there was a suffix βx…

If there was also a suffix βz…

Then a node corresponding to β is there

- Is there such a node ?

Suppose we create v applying rule 2. Then there was a suffix aβx… and now we add aβy



So there was a suffix βx…

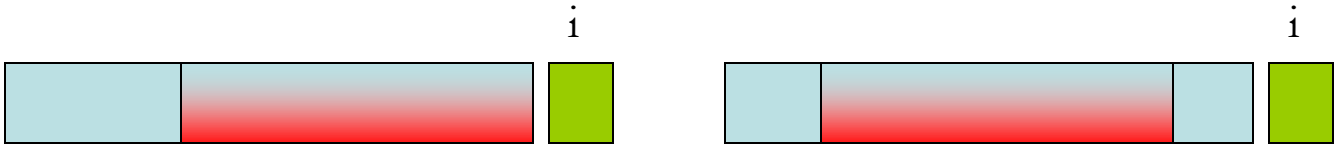
If there was also a suffix βz…

Then a node corresponding to β is there

Otherwise it will be created in the next extension when we add βy

Inv: All suffix links are there except (possibly) of the last internal node added

You are at the (internal) node corresponding to the last extension



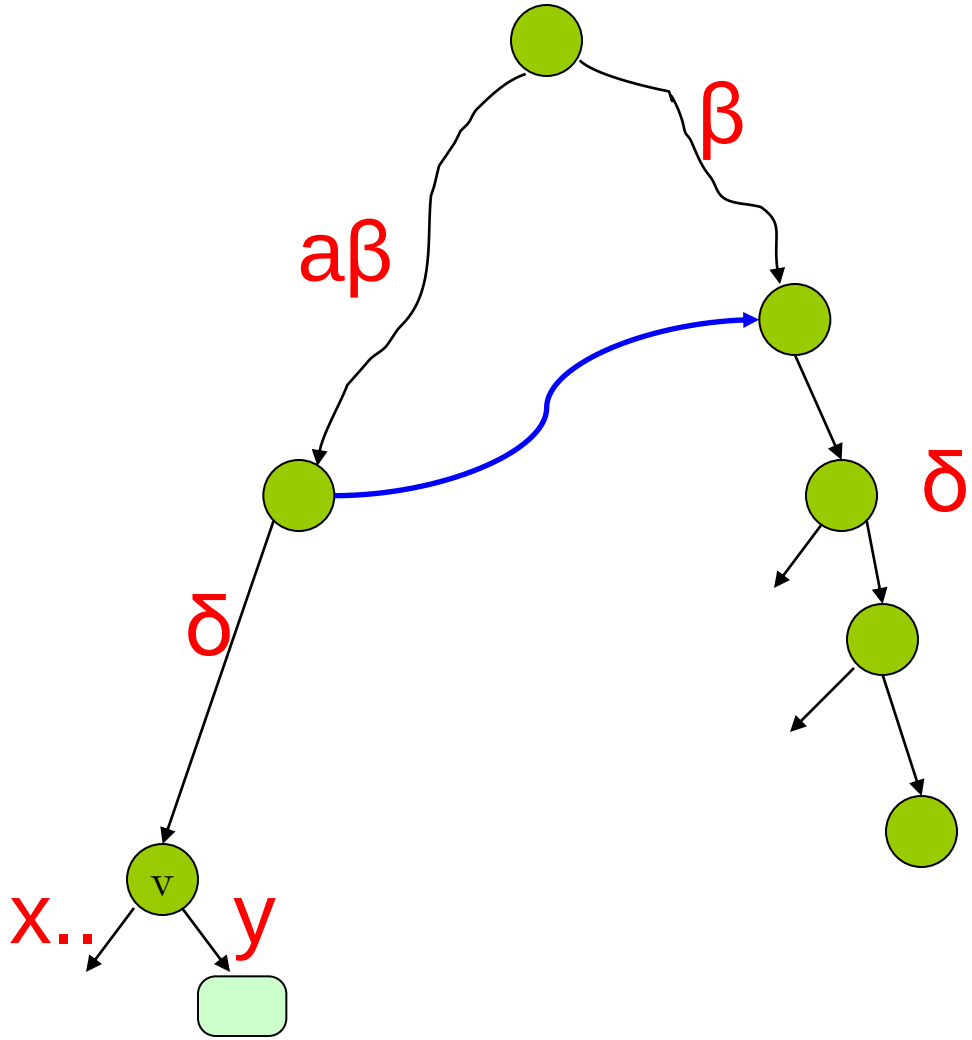
Remember: we apply rule 2

You start a phase at the last internal node of the first extension in which you applied rule 3 in the previous iteration

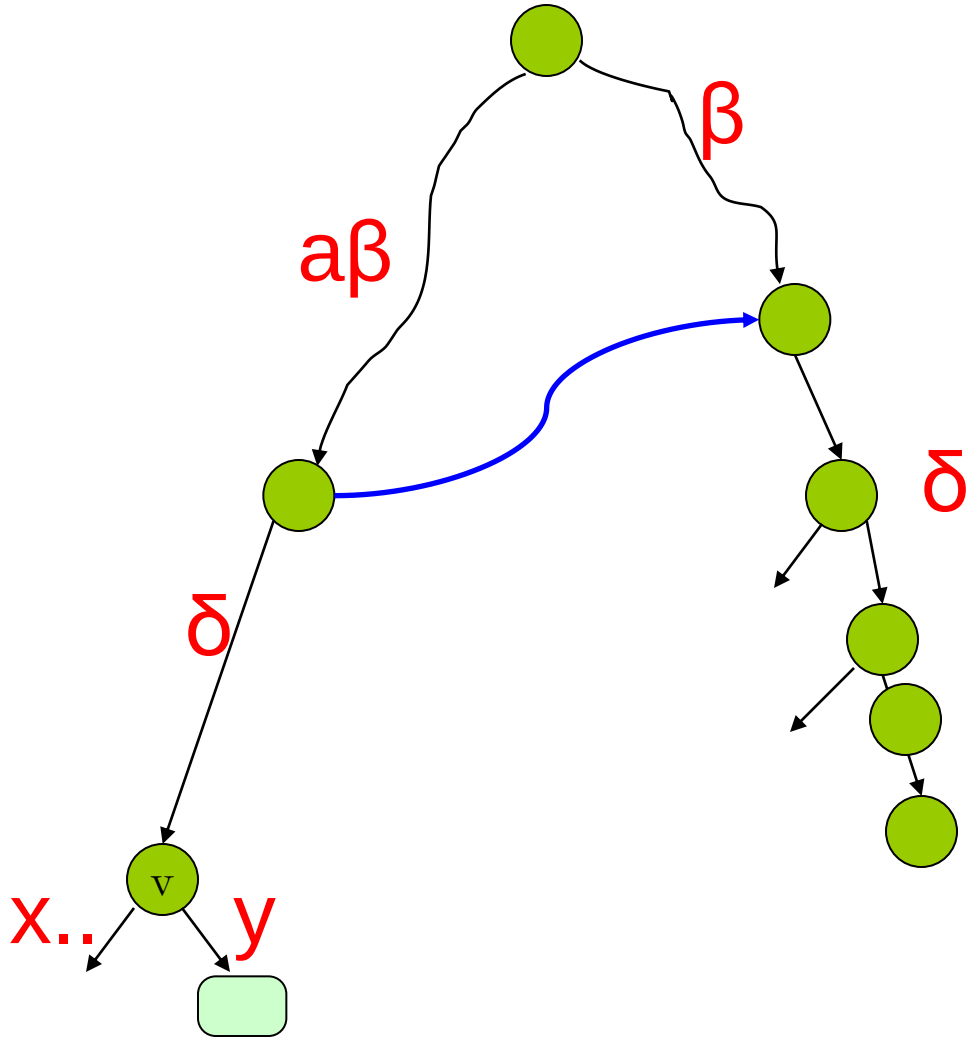
1) Go up one node (if needed) to find a suffix link

2) Traverse the suffix link

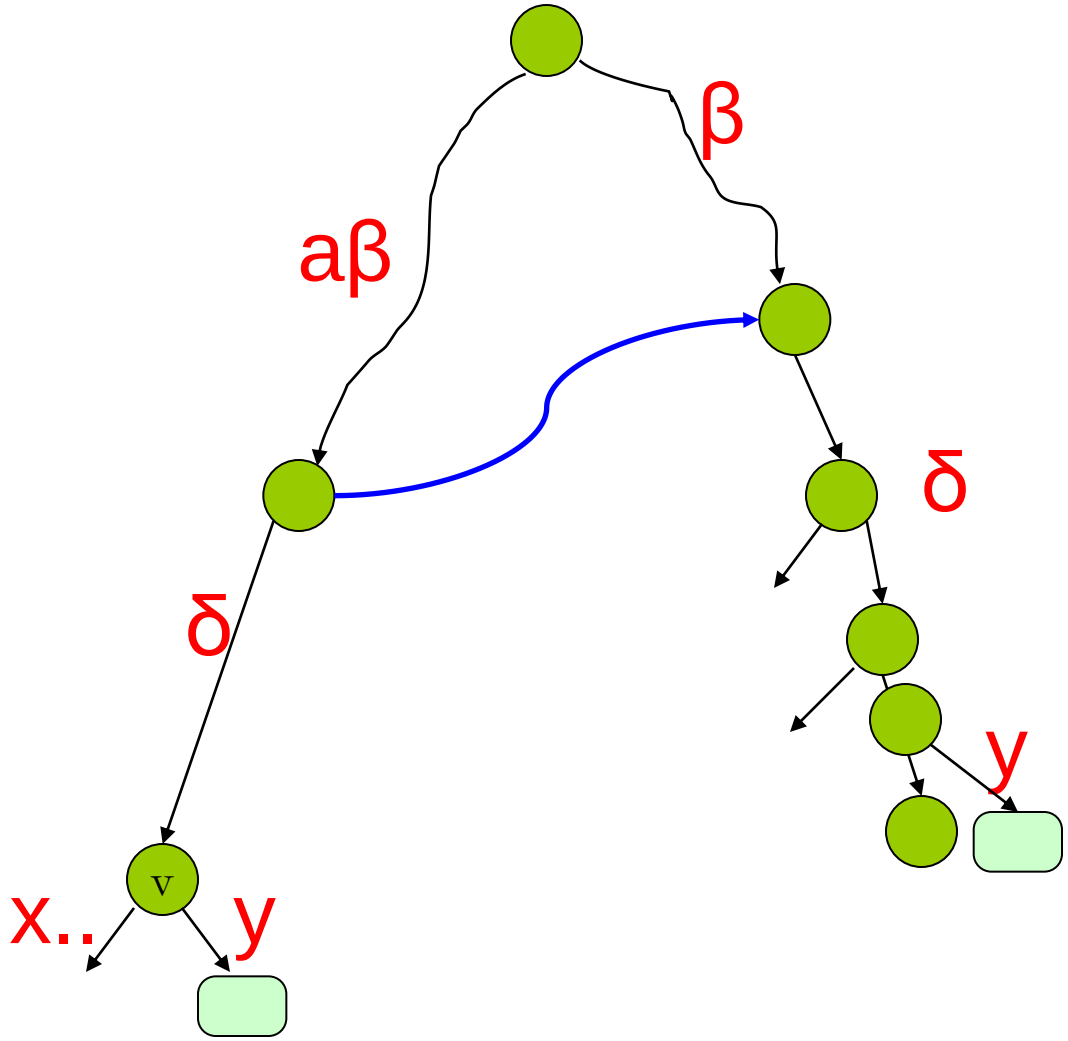
3) If you went up in step 1 along an edge that was labeled δ then go down consuming a string δ



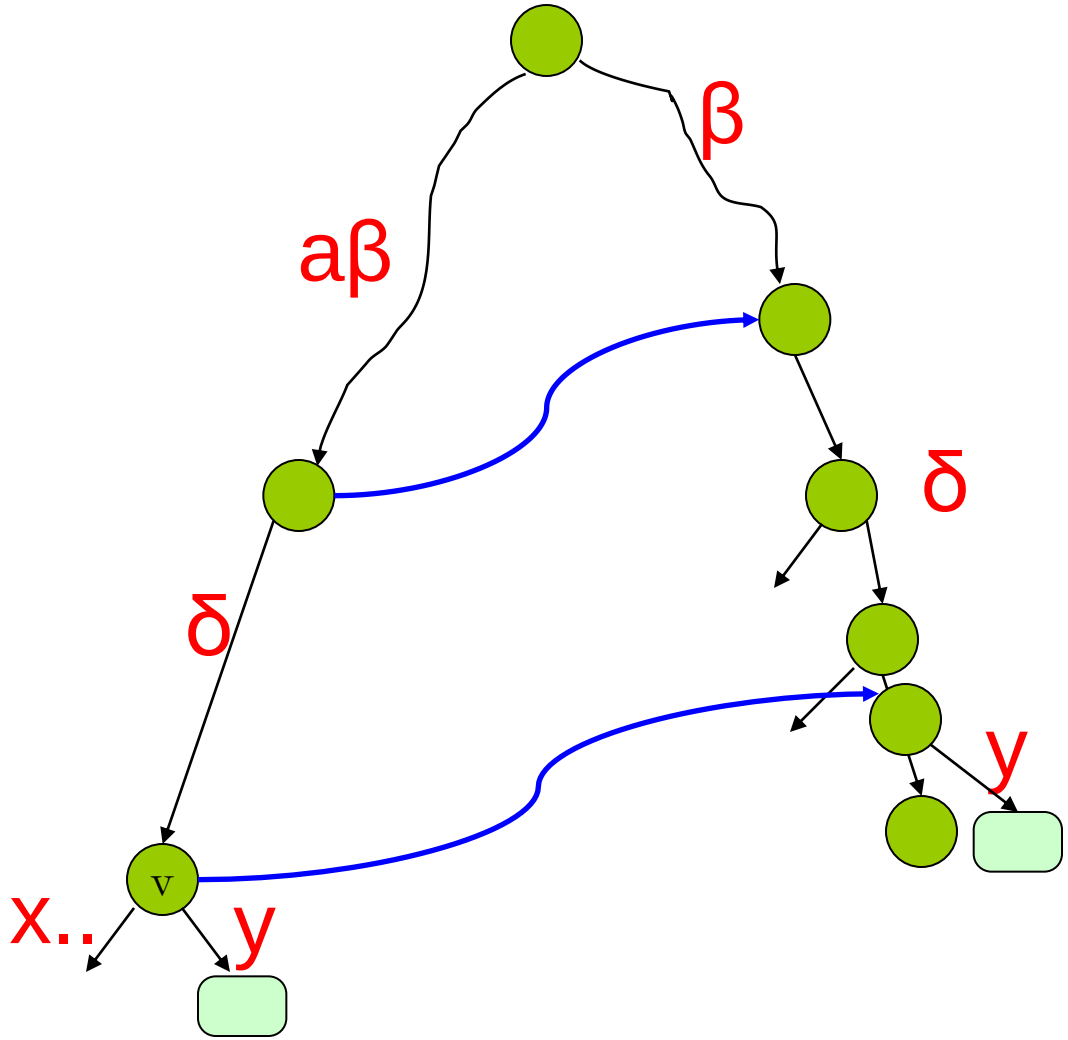
Create the new internal node if necessary



Create the new internal node if necessary



Create the new internal node if necessary, add the suffix



Create the new internal node if necessary, add the suffix and install a suffix link if necessary

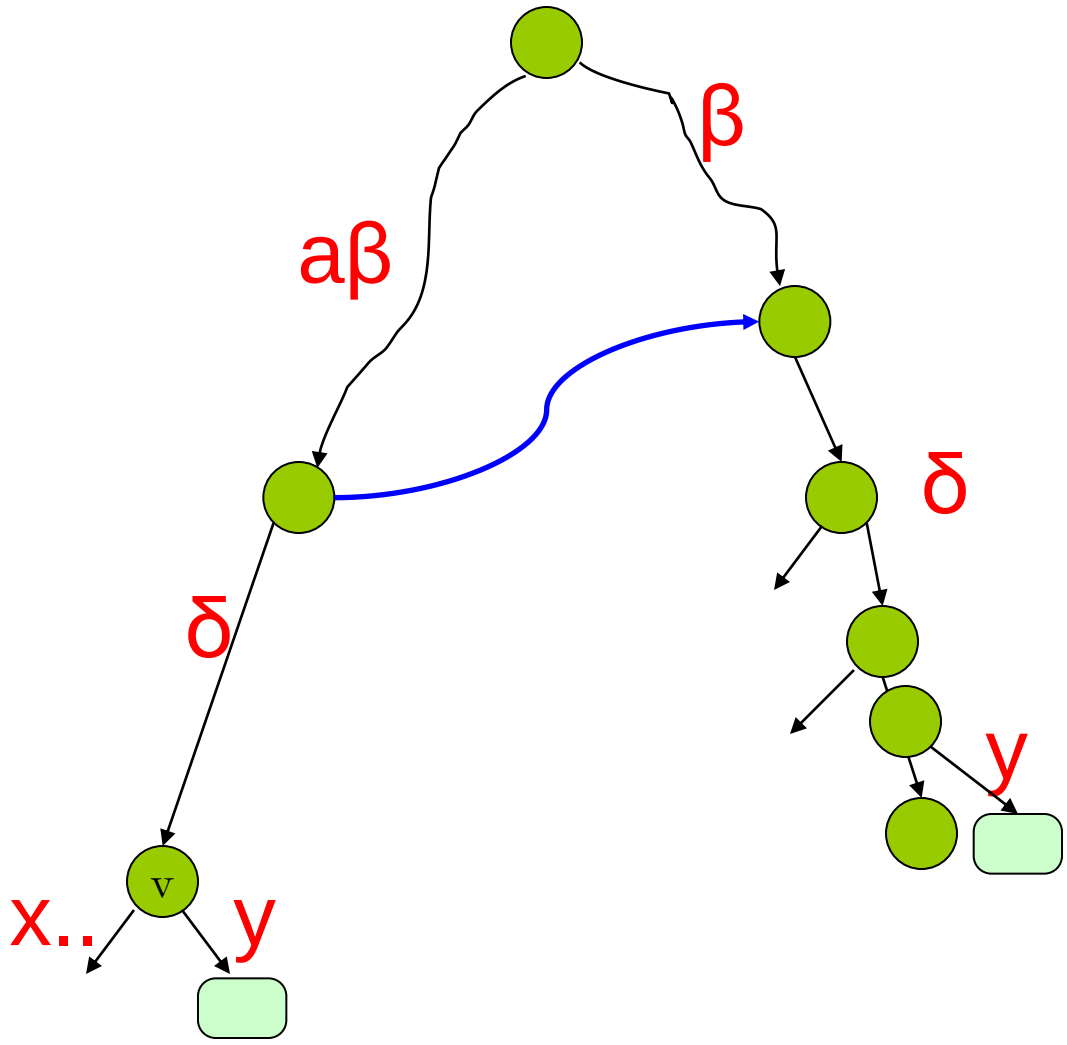
Analysis

How many times do we carry out rule 2/3 in all phases ?

O(n)

Does each application of rule 2/3 takes constant time ?

No ! (going up and traversing the suffix link takes constant time, but then we go down possibly on many edges..)


β
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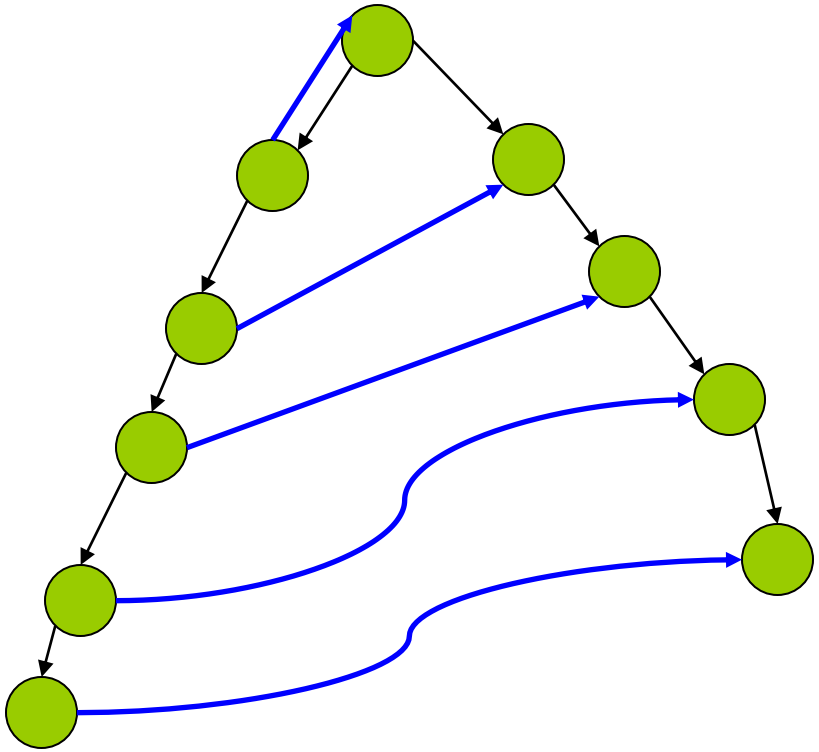
So why is it a linear time algorithm ?

How much can the depth change when we traverse a suffix link ?



It can decrease by at most 1

Punch line

Each time we go up or traverse a suffix link the depth decreases by at most 1

When starting the depth is 0, final depth is at most n

So during all applications of rule 2 together we cannot go down more than 3n times

THM: The running time of Ukkonen's algorithm is O(n)

Another application

- Suppose we have a pattern P and a text T and we want to find for each position of T the length of the longest substring of P that matches there.

- How would you do that in O(n) time ?

Drawbacks of suffix trees

- Suffix trees consume a lot of space
- It is $O(n)$ but the constant is quite big
- Notice that if we indeed want to traverse an edge in $O(1)$ time then we need an array of ptrs. of size $|\Sigma|$ in each node

Suffix array

- We loose some of the functionality but we save space.

Let s = abab

Sort the suffixes lexicographically:
ab, abab, b, bab

The suffix array gives the indices of the suffixes in sorted order

3	1	4	2

How do we build it ?

- Build a suffix tree
- Traverse the tree in DFS, lexicographically picking edges outgoing from each node and fill the suffix array.

- O(n) time

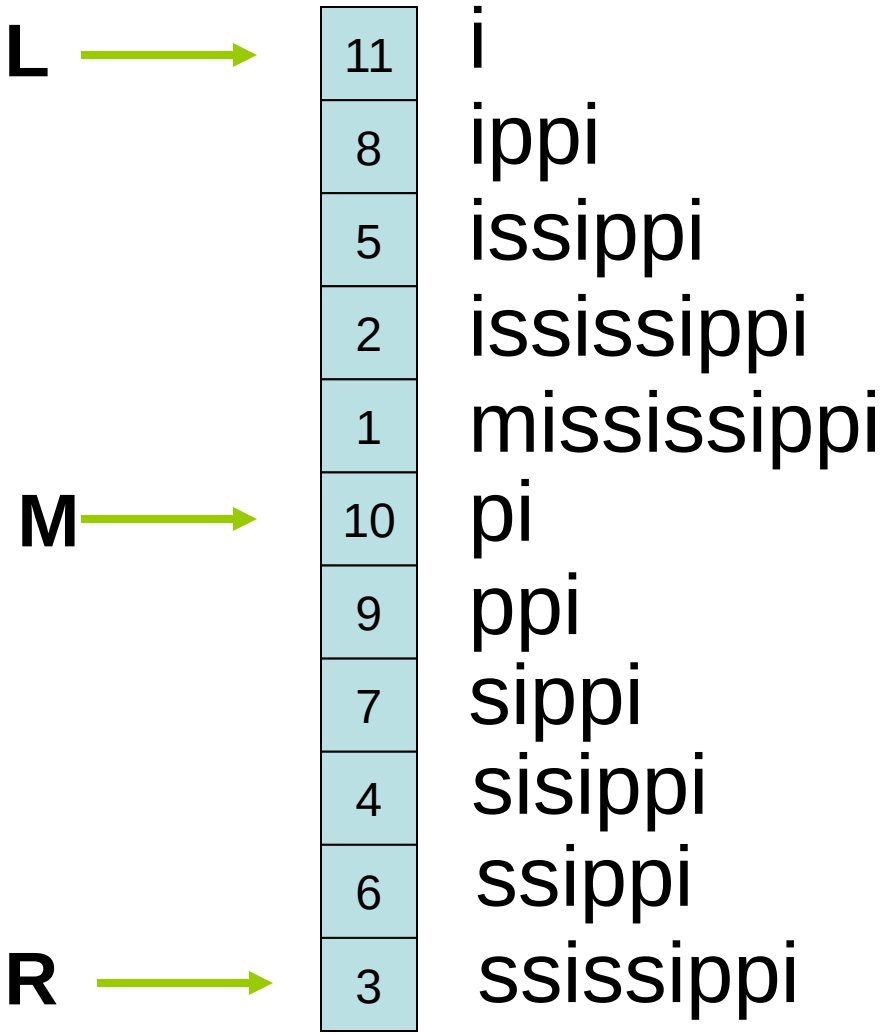
How do we search for a pattern ?

- If P occurs in T then all its occurrences are consecutive in the suffix array.
- Do a binary search on the suffix array
- Takes O(mlogn) time

Example

Let S = mississippi

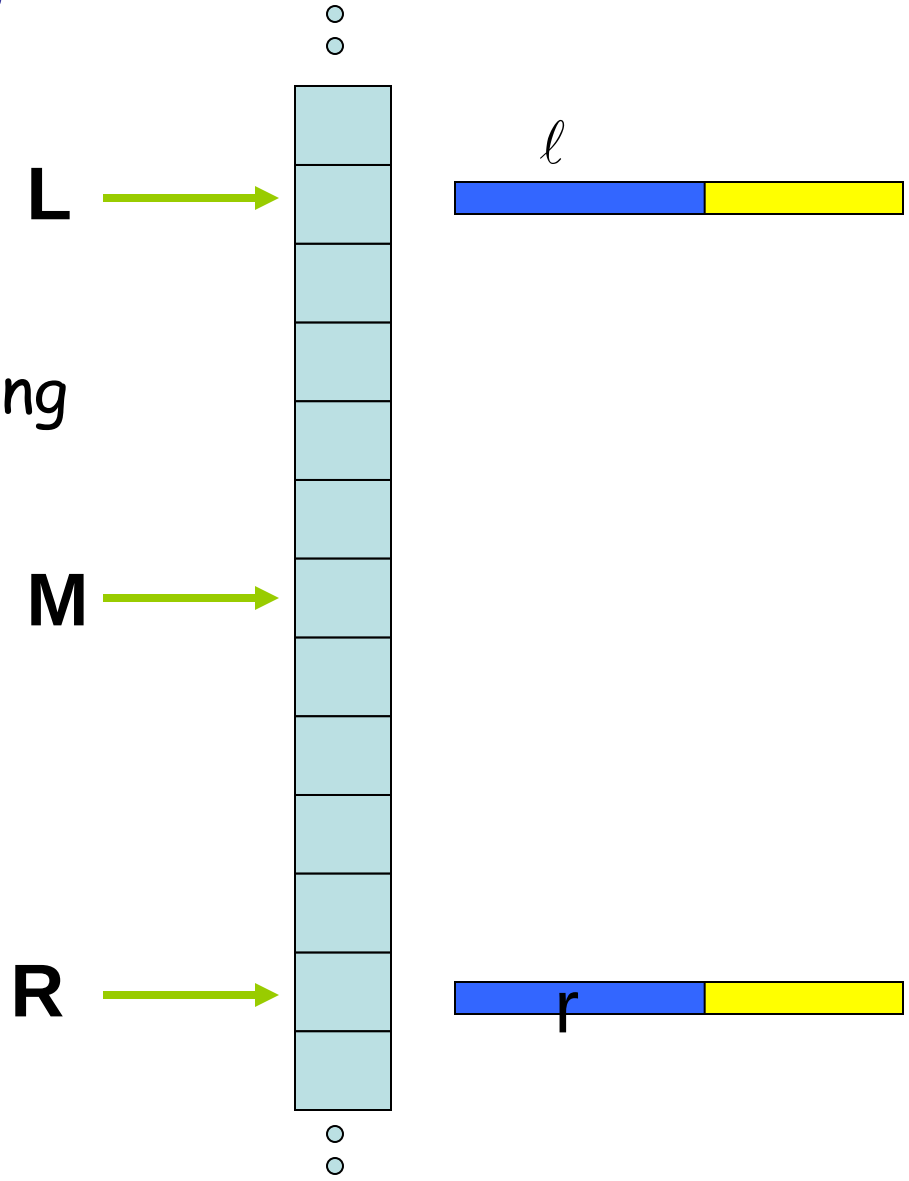
Let P = issa



How do we accelerate the search ?

Maintain l = LCP(P,L)
Maintain r = LCP(P,R)

If l = r then start comparing M to P at l + 1



How do we accelerate the search ?

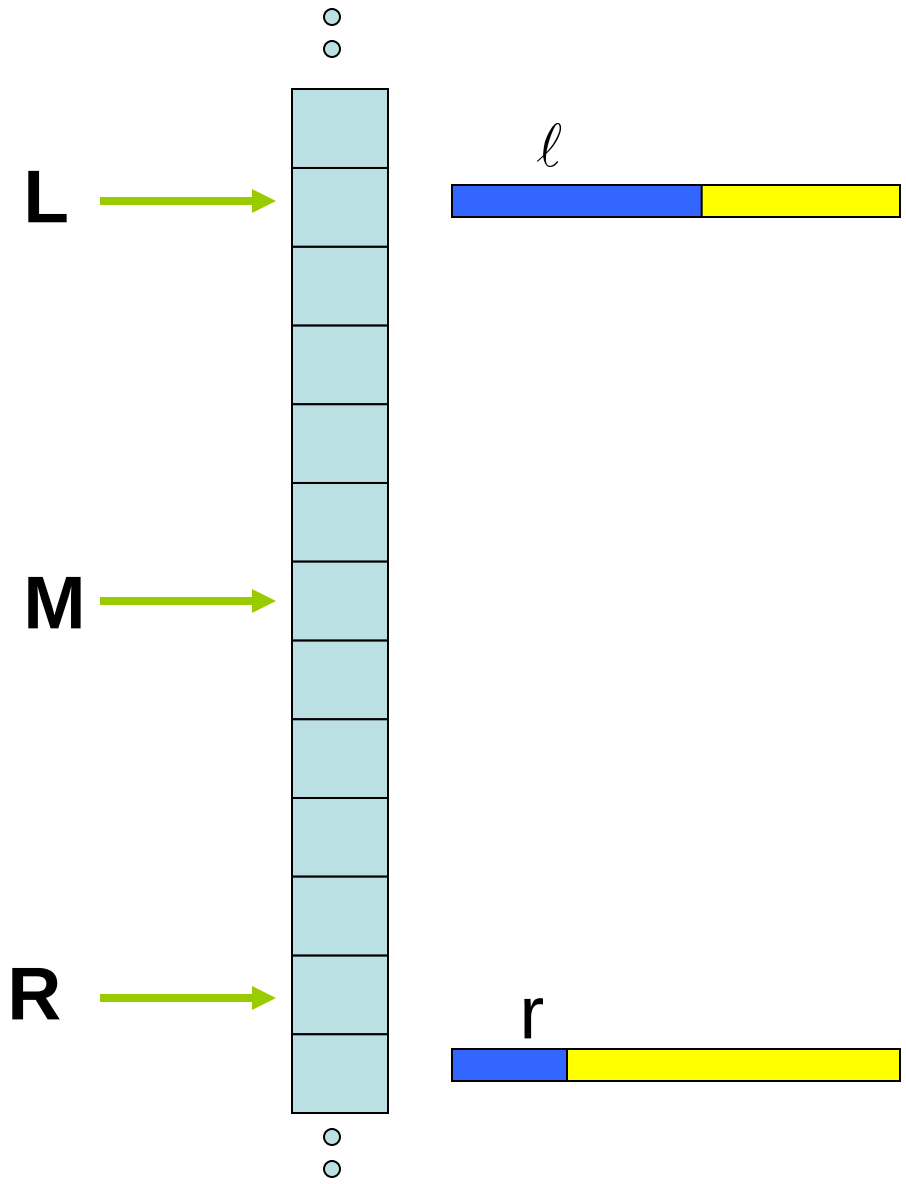
If l > r then

Suppose we know LCP(L,M)

If LCP(L,M) < l we go left

If LCP(L,M) > l we go right

If LCP(L,M) = l we start comparing at l + 1



Analysis of the acceleration

If we do more than a single comparison in an iteration then $\max(l, r)$ grows by 1 for each comparison ➔ $O(\log n + m)$ time