Computational Aspects of Prediction Markets

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- Given n agents, their combined information $\hat{\pi}$ is the coarsest common refinement of the partitions $\pi_1, \pi_2 ..., \pi_n$.
- We assume a common prior distribution $\mathcal{P} \in \Delta(\Omega)$ shared by all agents before receiving any information.

Model of an information market

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- Agents are rational, truthful, risk neutral and myopic.

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- S^r denotes the set of commonly known possible states after round r.
- S_i^r denotes the set of states agent i consider possible after round r.
- After each round, each agent i can update their S_i^r by eliminating all x's that would result a different price than what was anounced.

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- $S_i^r = \{y \in S^r | y_i = x_i\}$
- $S_i^r \subseteq S_i^{r-1}$ and $S^r \subseteq S^{r-1}$
- A convergence is reached after a finite number of steps. We denote: S^∞, S_i^∞ and p^∞.

Example: OR function

Example: Carry bit function

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- The pair (x_1, y_1) takes on the values $(0,0), (0,1), (1,0), (1,1)$ uniformly.
- (x_2, y_2) has a distribution conditional on (x_1, y_1). If $(x_1, y_1) = (1,1)$, then (x_2, y_2) takes the values $(0,0), (0,1), (1,0), (1,1)$ with probabilities $\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$ respectively. Otherwise with probabilities $\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2}$

Rational expectations

Rational expectations equilibrium

A *rational expectations equilibrium* is a mapping $P^* : \Omega \rightarrow \Re$ s.t in every state ω, if every trader conditions her demand (or supply) on her private information π_i and the price $P^*(\omega)$, the market will clear at the price of exactly $P^*(\omega)$.

Equilibrium price characterization

Stochastically monotone function

A function $g : \Re^n \to \Re$ is called *Stochastically monotone* if it can be written in the form $g(x) = \sum_i g_i(x_i)$ where each $g_i : \Re \to \Re$ is strictly increasing.

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Stochastically regular function

A function $g : \Re^n \to \Re$ is called *Stochastically regular* if it can be written in the form $g = h \circ g'$ where g' is stochastically monotone and h is invertible on the range of g'.

Nielsen et al. 1990

Shared knowledge at equilibrium

Suppose that, at equilibrium, the n agents have a common prior, but a possibly different information about the value of F. For all i, let $p_i^\infty = E(F|x \in S_i^\infty)$. if g is stochastically monotone and $g(p_1^\infty, p_2^\infty ... p_n^\infty)$ is common knowledge then:

$$p_1^\infty = p_2^\infty = ... = p_n^\infty = E(F|x \in S_i^\infty) = p^\infty$$

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This means that at equilibrium all agents must have exactly the same expectation of the value of the security and that this mus agree with the expectation of an uninformed observer. Is equilibrium enough for the purpose of information aggregation?
No. We also want that $p^\infty = f(x)$
Example: XOR with a uniform distribution.

Characterizing computable aggregates

weighted threshold function

A function $f : \{0,1\}^n \rightarrow \{0,1\}$ is a *weighted threshold function* iff there are $w_1, w_2 \ldots w_n \in \Re$ s.t

$$f(x) = 1 \text{ iff } \sum_{i=1}^{n} w_i x_i \geq 1$$

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Equilibrium of weighted threshold functions

If f is a weighted threshold function, then for any prior probability distribution, the equilibrium price of F is equal to $f(x)$.

Proof:

$p^\infty = E(F|x \in S_i^\infty)$ thus, if $p^\infty = 0$ or 1 then $f(x) = p^\infty$

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Assume towards contradiction that $0 < p^\infty < 1$. from Nielsen et al. we get:

$$P(f(y) = 1|y \in S^\infty) = p^\infty \tag{1}$$
$$\forall_i P(f(y) = 1|y \in S_i^\infty) = p^\infty \tag{2}$$

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since $S_i^\infty = \{y \in S^\infty | y_i = x_i\}$ Equation (2) can be written as:

$$\forall_i P(f(y) = 1|y \in S^\infty, y_i = x_i) = p^\infty \tag{3}$$

Proof cont.

Define:

$$J_i^+ = P(y_i = 1 | y \in S^\infty, f(y) = 1)$$
$$J_i^- = P(y_i = 1 | y \in S^\infty, f(y) = 0)$$
$$J^+ = \sum_{i=1}^{n} w_i J_i^+$$
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Since $p^\infty \neq 0, 1$ both J_i^+, J_I^- are well defined for all i.

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Proof: Assume $x_i = 1$, we then have:

$$\frac{P(f(y) = 1 | y \in S^\infty) \cdot J_i^+}{P(f(y) = 1 | y \in S^\infty) \cdot J_i^+ + P(f(y) = 0 | y \in S^\infty) \cdot J_i^-}$$

$$= P(f(y) = 1 | y_i = 1, y \in S^\infty) \text{ (Bayes' law)}$$

$$\frac{p^\infty \cdot J_i^+}{p^\infty \cdot J_i^+ + (1 - p^\infty) J_i^-} = p^\infty \text{ by Eqs. (1) and (3)}$$

$$\Rightarrow J_i^+ = p^\infty J_i^+ + (1 - p^\infty) J_i^-$$

$$\Rightarrow J_i^+ = J_i^-$$

The proof is symmetrical for $x_i = 0$

Proof cont.

Using linearity of expectation we can write:

$$J_i^+ = E\left(\left[\sum_{i=1}^{n} w_i y_i\right] \middle| y \in S^\infty, f(y) = 1\right)$$

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Since f is a weighted threshold function, $f(y) = 1)$ iff $\sum_{i=1}^{n} w_i y_i \geq 1$ and thus $J_i^+ \geq 1$, similarly $J_i^- < 1$ which implies $J_i^+ \neq J_i^-$
And so we have a contradiction. $\square$

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Lemma

If $S^r \neq S^{r-1}$, then $dim(S^r) < dim(S^{r-1})$.

Proof

Let $k = dim(S^{r-1})$. Consider the bids in round r:

$$b_i^r = E(f(y) = 1 | y \in S^{r-1}, y_i = x_i)$$

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Denote $d_i = h_i^{(0)} - h_i^{(1)}$ and we get:

$$p^r = \frac{1}{n} \sum_{i=1}^{n} (h_i^{(0)} + d_i x_i) \tag{4}$$

proof cont.

Equation (4) defines a hyperplane the intersection of which with S^{r-1} is the resulting common knowledge S^r.

This is because all agents already know all $h_i^{(0)}$ and d_i and so they use the linear equation (4) to rule out any possibility that would not have resulted the price p^r.

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This is because all agents already know all $h_i^{(0)}$ and d_i and so they use the linear equation (4) to rule out any possibility that would not have resulted the price p^r.
If $S^r \neq S^{r-1}$, this intersection defines a linear subspace of dimension (k-1) that contains S^r and so S^r has a dimension of at most $(k-1)$.

proof cont.

Upper bound on the number of iterations

Let f be a weighted threshold function, and $\mathcal{P}$ an arbitrary distribution. Then after at most n rounds of trading, the price reaches its equilibrium value $p^{\infty} = f(x)$

This follows directly from the Lemma as $dim(S^0) = n$ and once we have a round r s.t $S^r = S^{r-1}$ then no trader has gained any new knowledge and thus $p^{\infty} = f(x)$