Computational Game Theory	Fall Semester, 2012/2013
Lecture 8: December 18	
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8.1 Reminder - The Settings

Consider the following auction: single item, different and non-regular distributions and different thresholds for each agent (i.e., not $\overline{\phi}_i^{-1}(0)$ – the monopoly reserve prices).

Choose a single threshold t, and specific thresholds, t_i, for each agent, that meets the following conditions:

- $\forall i \neq j\colon t = \overline{\phi}_i(t_i) = \overline{\phi}_j(t_j)$.
- $\prod_i (F_i(t_i)) = 1/2$, where $v_i \sim F_i$ (i.e., v_i is chosen according to distribution F_i).

8.2 Prophet Inequality

What does setting the above thresholds grantees? Consider the following scenario: a gambler plays a series of n games in a casino, where at the end of each game he gets a payoff. In order to play in the next game, he must give back to the casino the payoff he won so far.

There is an optimal strategy to play in this scenario: after playing game n the gambler takes the payoff. Say the gambler has played game $n-1$, and has some payoff. If the expected payoff of playing game n is larger than the payoff the gambler has now, he should play game n. Applying the same strategy for games $1, \ldots, n-2$ results in the optimal strategy.

Computing the optimal strategy might be very difficult. What other strategies might grantee? Consider the following strategy:

Threshold Strategy A strategy $S(t)$ for the gambler is the following:

- After playing game i, if payoff$(i+1) < t$, then stop playing.

- Otherwise, continue to game $i+1$.

$\mathrm{E}[S(t)]$ denotes the expected profit of a gambler playing according to $S(t)$.

Theorem 8.1 (Prophet Inequality Theorem) *$\exists t$ such that $E[S(t)] \geq REF/2$.*

8.2.1 Relation to Auctions

How does this gambler story relates to auctions? for every t there is some probability that the gambler will quit after the i'th game. Take the t guarantees to exists from Theorem 8.1 and calculate t_i's accordingly. Consider the agents bidding as the games; they come in one after the other, and the mechanism ignores agent i if its value is less than t_i.

Theorem 8.1 guarantees that this mechanism is in fact a 2-approximation to the optimal mechanism (Mayerson's mechanism)

8.2.2 Proof of Prophet Inequality Theorem

In the following we let $(x-y)^+ = \max\{x-y, 0\}$.

Set t' such that the probability that the gambler will leave the casino with nothing is $1/2$, and set $t = \max\{t', 0\}$. Let x be the probability that the gambler will leave with nothing when playing according to $S(t)$ (since $t \geq t'$, then $x \geq 1/2$). For every t, it holds that

$$\begin{aligned}\mathrm{REF} &\leq t + \mathrm{E}[\max_i\{(p_i - t)^+\}] \\ &\leq t + \sum_i \mathrm{E}[(p_i - t)^+].\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{E}[S(t)] &\geq (1-x)\cdot t + \sum_i \mathrm{E}[(p_i-t)^+ \mid p_j < t, j \neq i]\cdot \Pr[p_j < t, j \neq i] \\ &\geq (1-x)\cdot t + x\cdot \sum_i \mathrm{E}[(p_i-t)^+ \mid p_j < t, j \neq i] \\ &= (1-x)\cdot t + x\cdot \sum_i \mathrm{E}[(p_i-t)^+].\end{aligned}$$

If $x = 1/2$, then we immediately get $\mathrm{REF} \leq 2\cdot \mathrm{E}[S(t)]$. If $x > 1/2$, then $t \neq t'$, namely $t = 0$. In this case we get $\mathrm{REF} \leq \sum_i \mathrm{E}[(p_i - t)^+]$, and $\mathrm{E}[S(t)] \geq \sum_i \mathrm{E}[(p_i-t)^+]/2$. Hence, getting again $\mathrm{REF} \leq 2\cdot \mathrm{E}[S(t)]$.