Lecture 6: November 27th

Lecturer: Amos Fiat *Scribe: Omri Sharabi*

6.1 the bottom line of myerson

for each agent i we are given $V_i \sim F_i$. we then look at the ironed revenue curve in quantile space, which is simply the minimum concaved function that upper bounds the revenue curve $R(q)$.we are not maximizing the social welfare, instead we are maximizing the social welfare of the virtual values in order to maximize profit. it is an optimal mechanism for maximizing the revenue. its optimality comes from the optimality of VCG, we actually simulate VCG on the virtual values $\bar{\Phi}(b)$.

6.1.1 myerson auction

1. collect bids $b = \{b_1, b_2, ..., b_n\}$.
2. compute the ironed virtual values $\bar{\Phi}(b)$ and throw away negative virtual values.
3. compute social welfare maximization (VCG) on the remaining virtual values $\bar{\Phi}(b)$
4. compute the payments from BIC payment identity.

6.1.2 examples

the three following examples are for regular distributions, hence no ironing is needed.

6.1.2.1 two bidders $V_1, V_2 \sim U[0,1]$

- $F_1, = F_2 = U[0,1]$

- $q(v) = 1 - F(v)$
- $R(v) = v \cdot q(v) = v(1-v) = v - v^2$, maximized at $\frac{1}{2}$
- $R(q) = q \cdot v(q)$
- $\Phi(v) = v - \frac{1-F(v)}{f(v)} = 2v - 1 = R'(q)$
- note that the revenue curve is concaved, no ironing needed.
- we throw away any v_i such that $2v_i - 1 < 0$, i.e $v_i < \frac{1}{2}$

now say agents i=1,2 get values V_1, V_2 then $\Phi_1(v) = 2v_1 - 1$ and $\Phi_2(v) = 2v_2 - 1$, in order to maximize revenue run VCG for these values. in that case, since $F_1, = F_2 = U[0,1]$ the higher value also has higher virtual value.

6.1.2.2 n i.i.d bidders $V_1, V_2 ..., V_n \sim U[0,k]$

- $\Phi(v) = v - \frac{1-\frac{v}{k}}{\frac{1}{k}} = 2v - k$
- again, no ironing needed as revenue curve is concaved.
- we throw away any v_i such that $2v_i - k < 0$, i.e $v_i < \frac{k}{2}$
- run VCG on remaining values.

here we have a different threshold value then the previous example but here too, the higher value will also have higher virtual value.

6.1.2.3 2 bidders $V_1 \sim U[0,1], V_2 \sim U[0,2]$

- $\Phi_1(v) = v - \frac{1-F_1(v)}{f_1(v)} = 2v_1 - 1$
- $\Phi_2(v) = v - \frac{1-F_2(v)}{f_2(v)} = 2v_2 - 2$
- taking the same steps we took in the previous 2 examples we will ignore v_1 if $v_1 < \frac{1}{2}$ and ignore v_2 if $v_2 < 1$ and give it to the highest virtual value left.
- if both values are below the critical values we don't sell.

now consider the case where $v_2 = 1.1$ and $v_1 = 1$, their virtual values are $\Phi_1(v_1) = 1$ and $\Phi_2(v_2) = 0.2$. for this specific case we didn't maximize profit and the winner paid a lower price than the one we could charge. it happens because we must set the auction rules in advance. we use an auction that maximize the expected revenue, but it doesn't guarantee that for any set of V it maximize profit. when maximizing with respect to virtual values we actually consider the values combined with their probability to occur so that it maximizes the expected revenue.

6.2 the expected payment

6.2.1 expected payment for a regular distribution

recall that for regular distributions, the expected payment p for agent i with value v and quantile $q(v)$ is:

$$E_q[p(q)] = -E_q[R(q)x'(q)] = E_q[R'(q)x(q)] = E_q[\Phi(q)x(q)] \tag{6.1}$$

from that we concluded that the optimal expected revenue is given by:

$$argmax_x(\sum_i \Phi_i(v_i) \cdot x_i(v_i)) \tag{6.2}$$

this optimum is monotone in v_i

6.2.2 expected payment for irregular distribution

the following lemma for the expected payments considering the ironed virtual values holds for an irregular distribution:

Lemma 6.1

$$E_v(p(v)) = E_q[p(q)] \leq E_q[\bar{\Phi}(q)x(q)] \tag{6.3}$$

the equality occurs when $\forall q.\bar{R}(q) > R(q) \implies x'(q) = 0$, i.e all have the same virtual value.

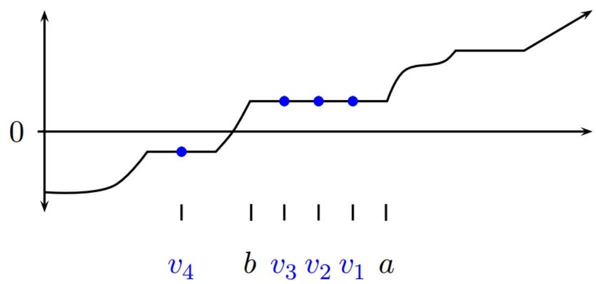

Figure 6.1: virtual value as a function of the real value, here v_1, v_2, v_3 have the same virtual value

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in order to prove the lemma, lets look at the following statement:

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\begin{aligned}
&E_q[p(q)] \\
&= E_q[R'(q)x(q)] + E_q[\bar{R}'(q)x(q)] - E_q[\bar{R}'(q)x(q)] \\
&= E_q[\bar{R}'(q)x(q)] - E_q[(\bar{R}'(q) - R'(q))x(q)] \\
&= E_q[\bar{R}'(q)x(q)] + E_q[(\bar{R}(q) - R(q))x'(q)]
\end{aligned}
\tag{6.4}
$$

the last equality follows from integrating by parts the integral form of the expectation. note that the derivative of the allocation rule is non-positive, since the the allocation rule is monotone non-decreasing in value (non-increasing in quantile), also $(\bar{R}(q) - R(q))$ is never negative, because the ironed revenue curve upper bounds the original revenue curve, combining the two observations gives that $E_q[(\bar{R}(q) - R(q))x'(q)] \leq 0$ which prove the lemma.

6.2.3 approximating myerson

in the last lessons we have shown an optimal mechanism to maximize revenue, the myerson auction. however in some cases we want a mechanism that is simpler and more “fair”, in the next lessons we will see such mechanisms that are simpler and yet approximate myerson's auction up to a constant factor.