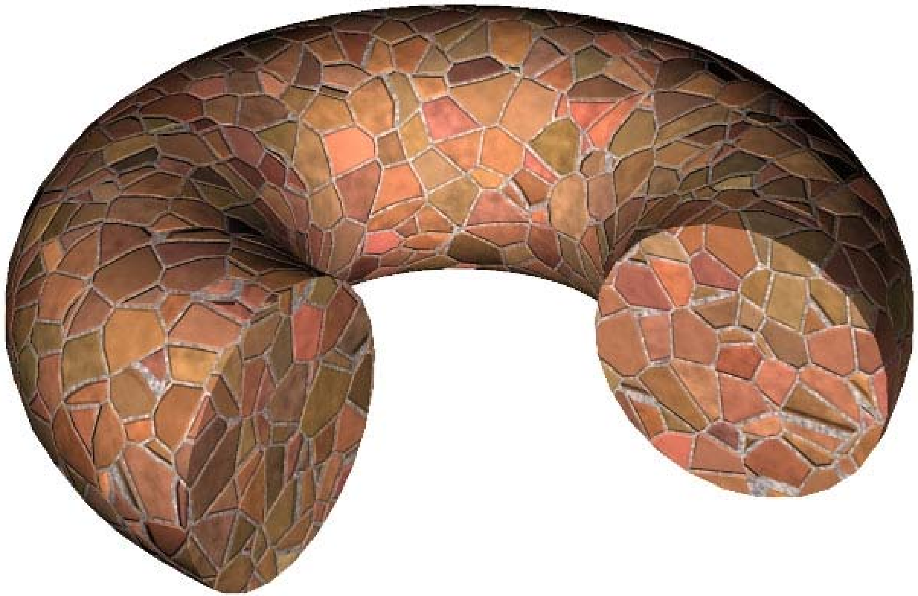
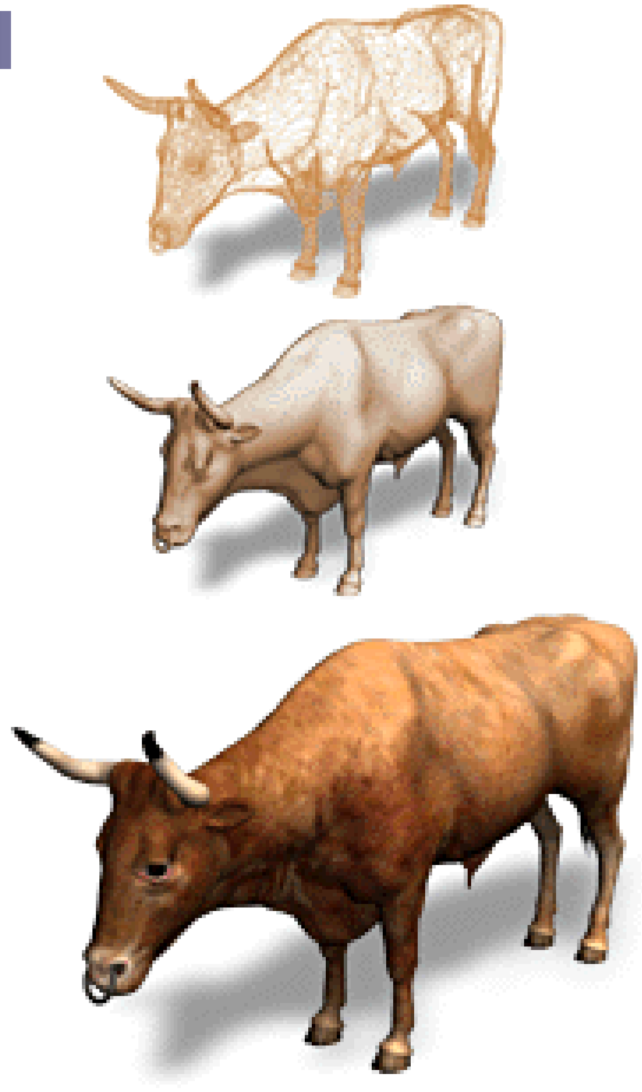
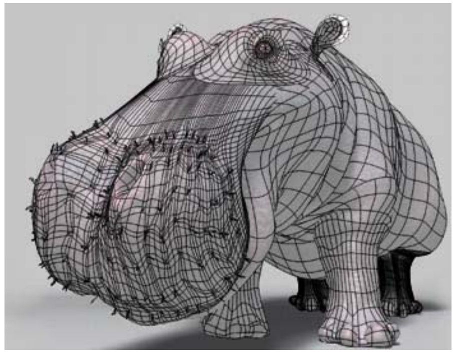
Texture Mapping

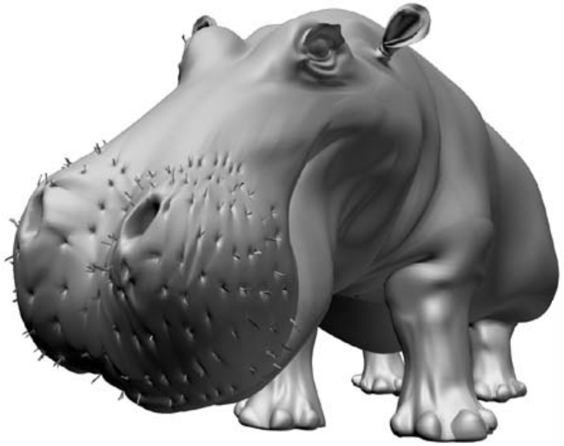


- Motivation: Add interesting and/or realistic detail to surfaces of objects.
- Problem: Fine geometric detail is difficult to model and expensive to render.
- Idea: Modify various shading parameters of the surface by mapping a function (such as a 2D image) onto the surface.



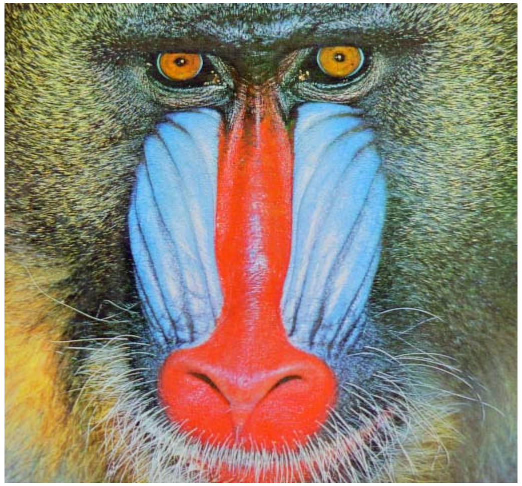
Textures and Shading

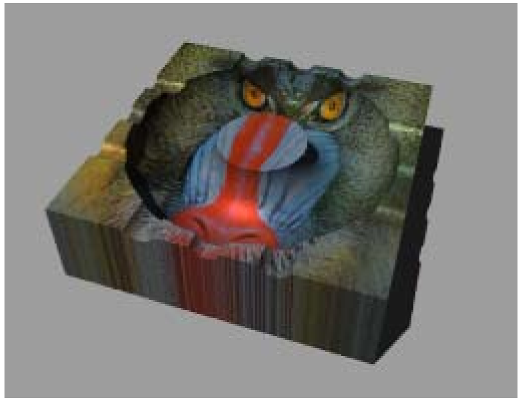


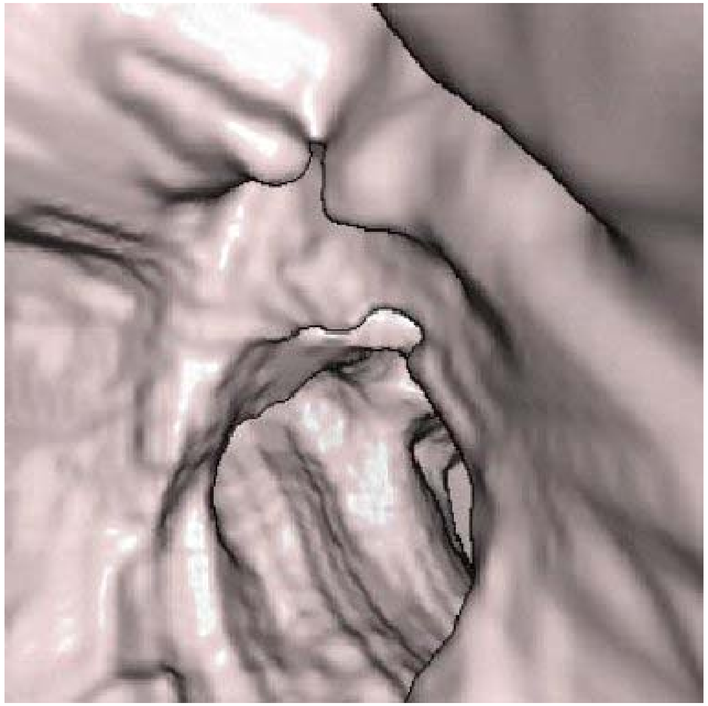


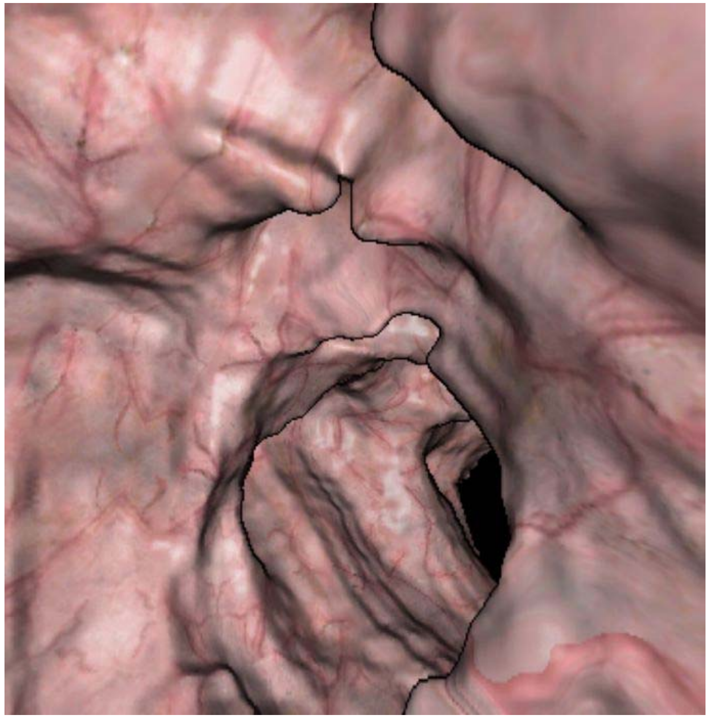
http://www.3drender.com/jbirn/hippo/hairyhipponose.html

Texture Mapping – Simple Example





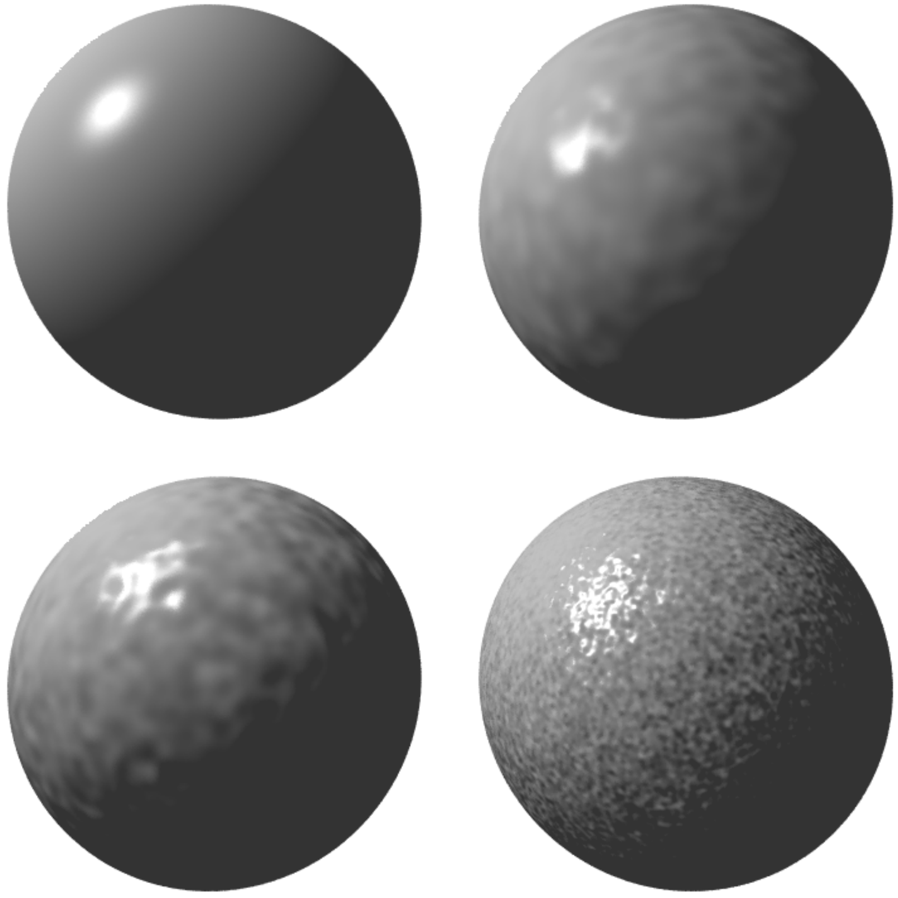




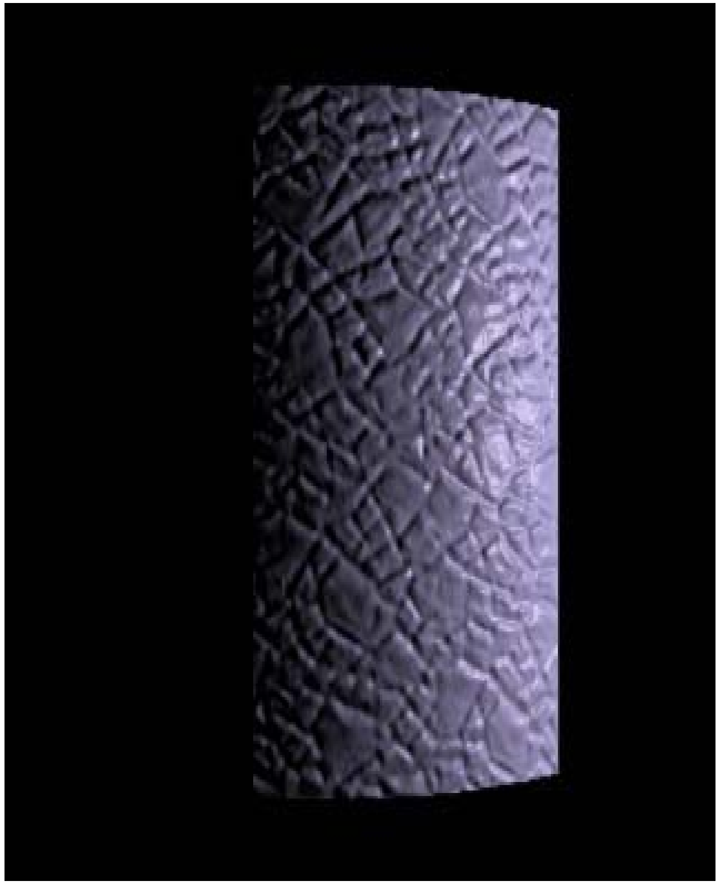
Simple parametrization

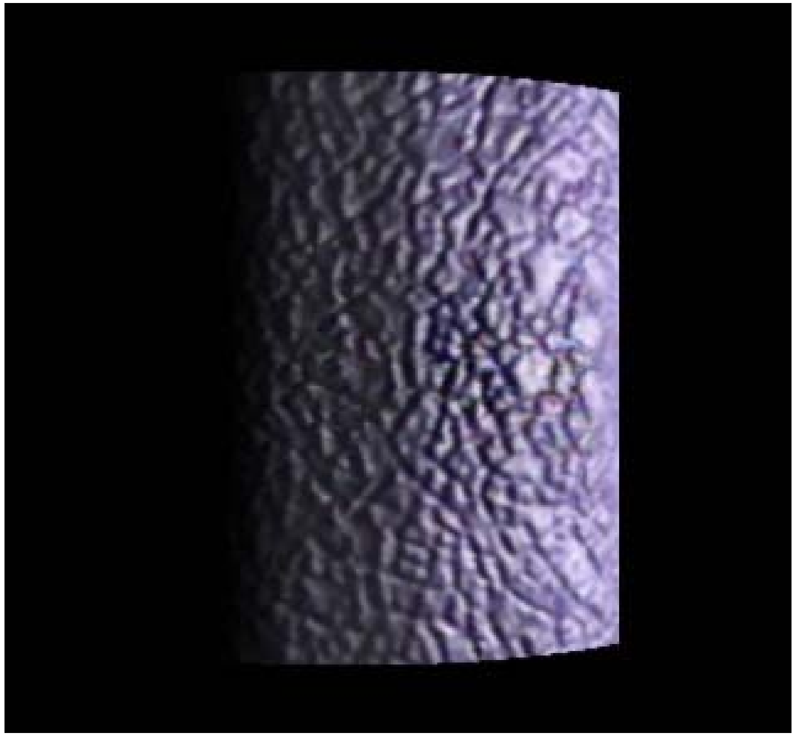
Mapping is not unique

Bump Mapping



Bump Mapping



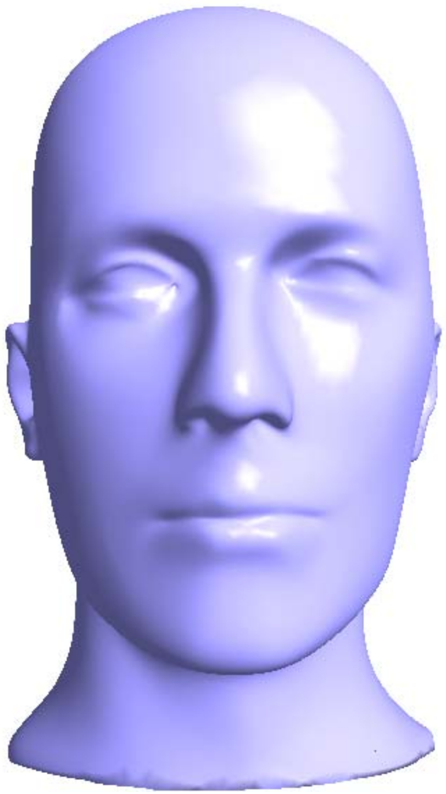


Surface Parametrization

Most slides courtesy of Pierre Alliez, Craig Gotsman, and Noam Aigerman

Triangle mesh

- Discrete surface representation
- Piecewise linear surface (made of triangles)



Triangle mesh

- Geometry:
  - Vertex coordinates

    (x_1, y_1, z_1)

    (x_2, y_2, z_2)

    . . .

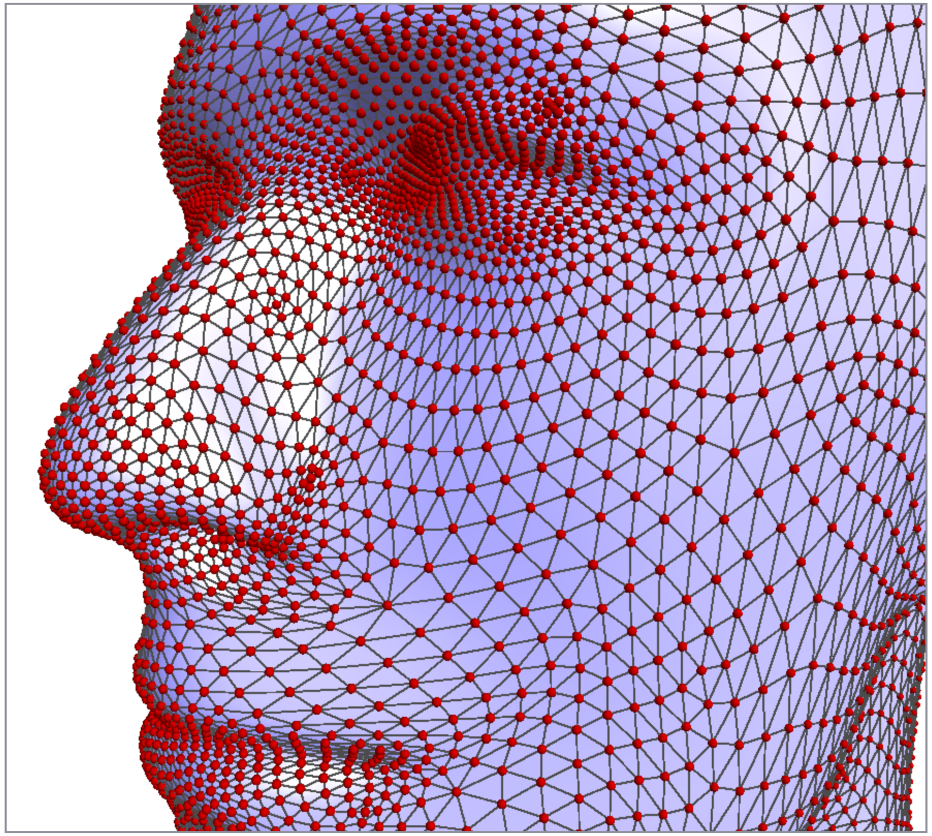
    (x_n, y_n, z_n)
- Connectivity (the graph)
  - List of triangles

    (i_1, j_1, k_1)

    (i_2, j_2, k_2)

    . . .

    (i_m, j_m, k_m)



What is a parameterization?

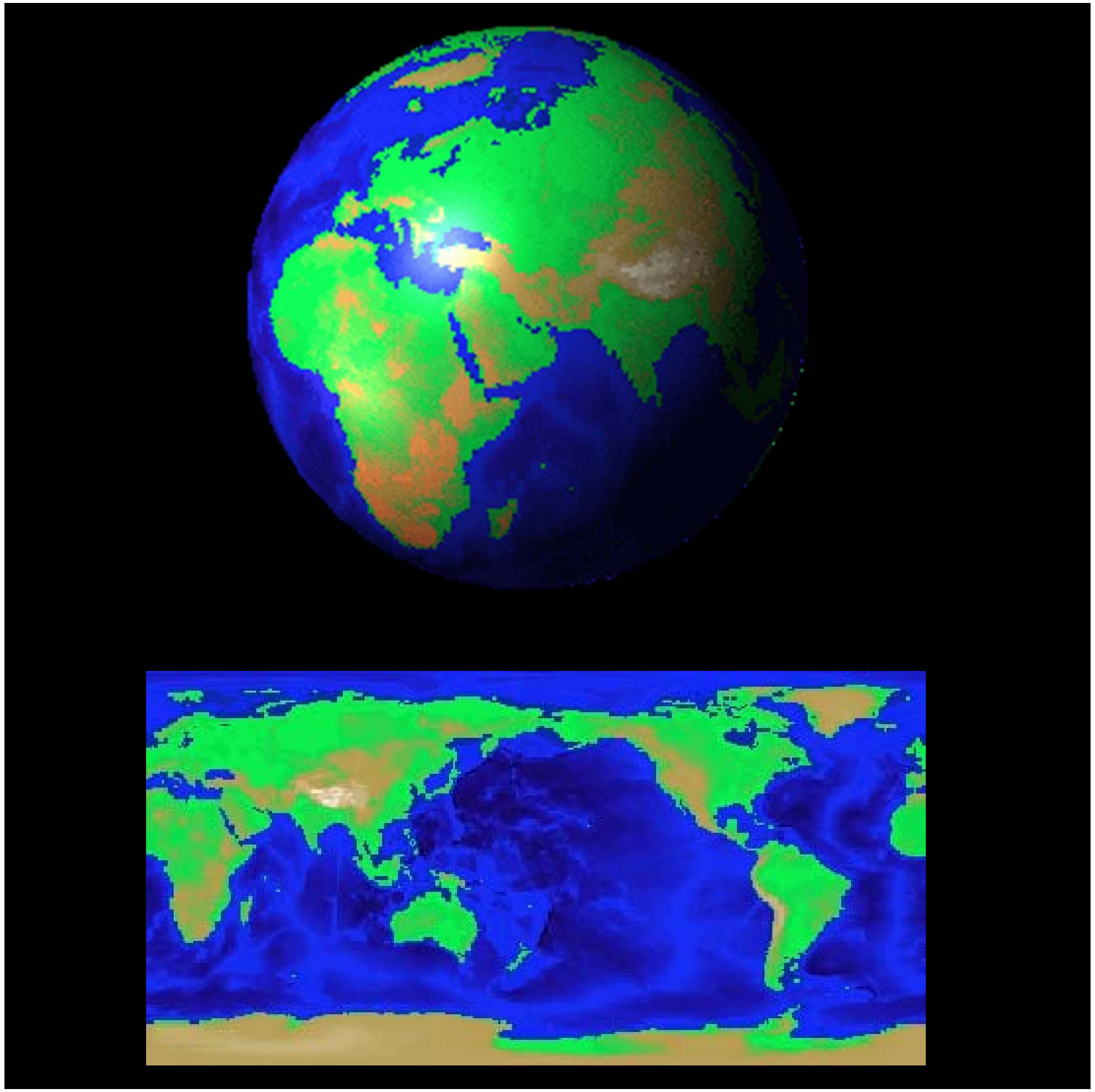
$S \subseteq R^3$ - given surface

$D \subseteq R^2$ - parameter domain

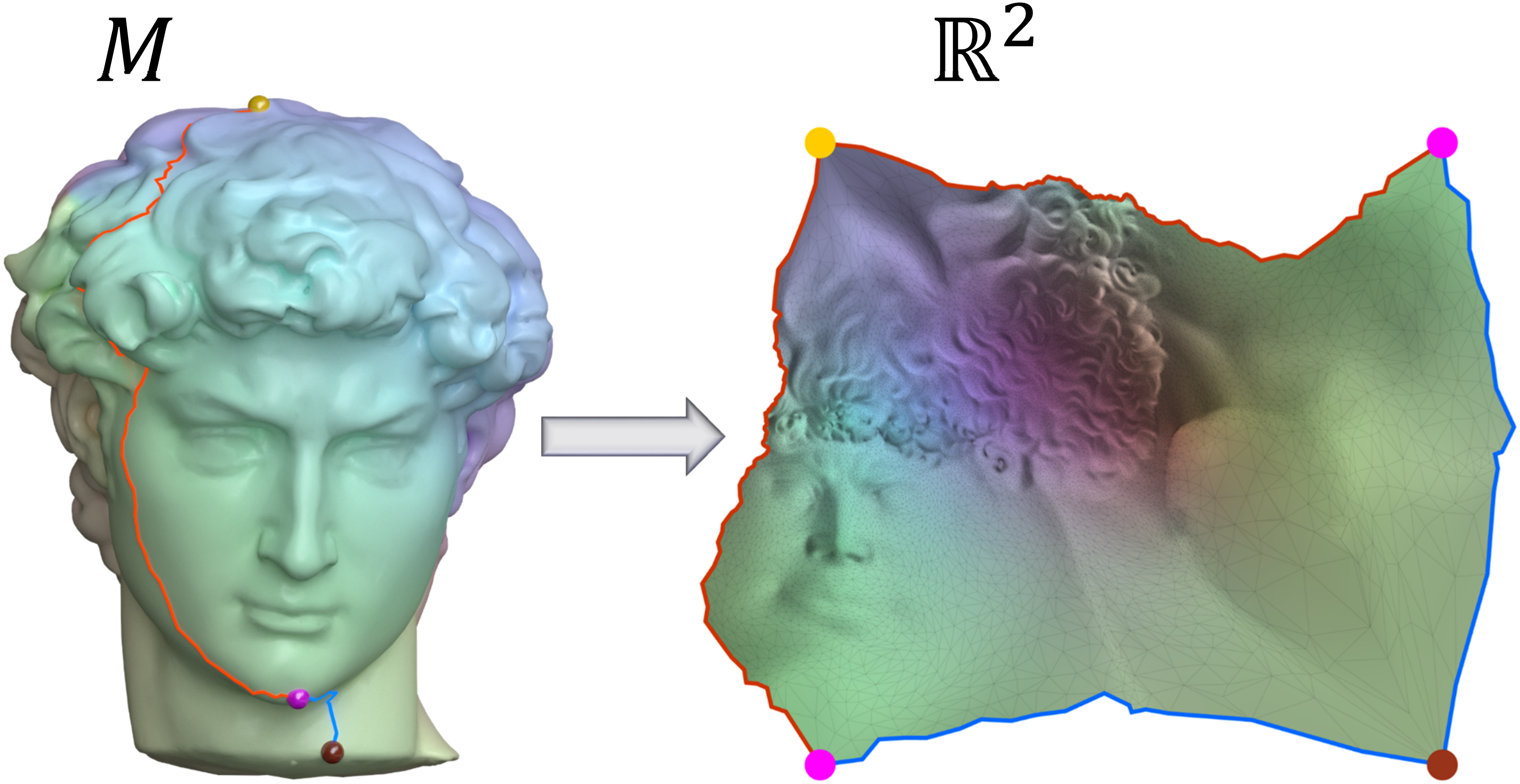
$\mathbf{s} : D \rightarrow S$ *1-1 and onto*

$$\mathbf{s}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

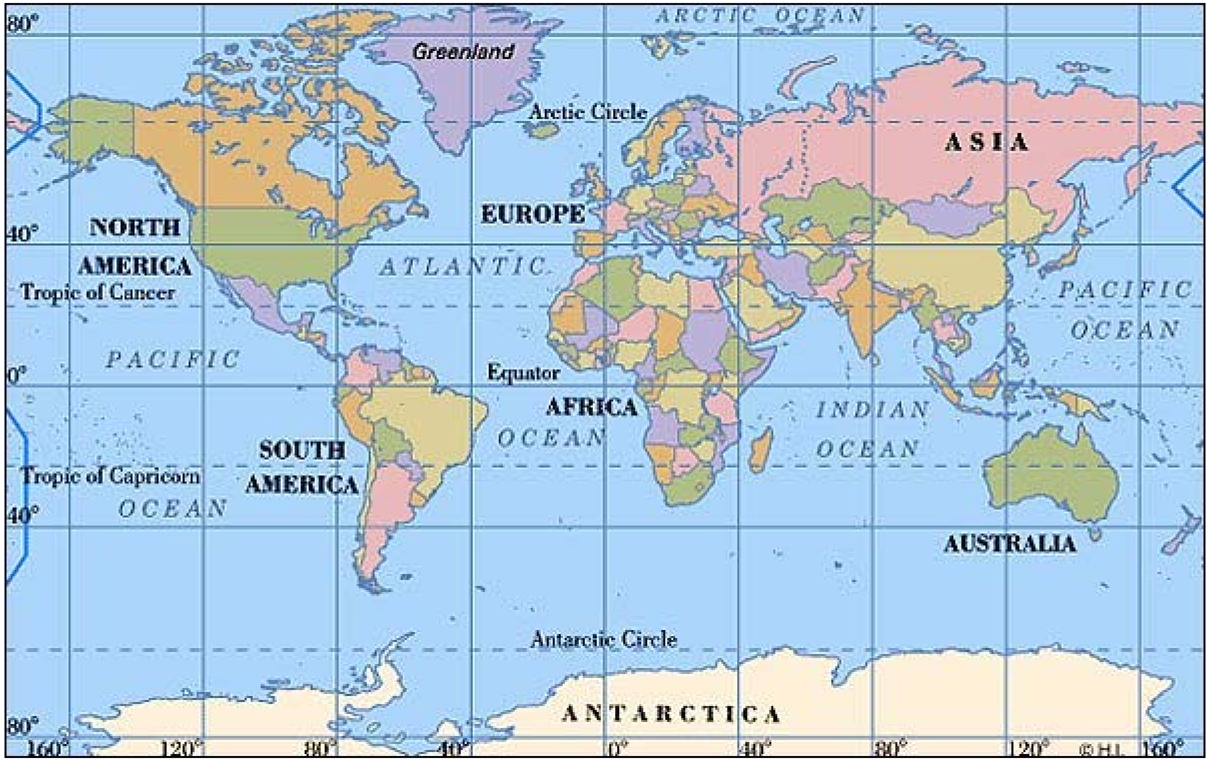
Example – flattening the earth



Mesh Parameterization

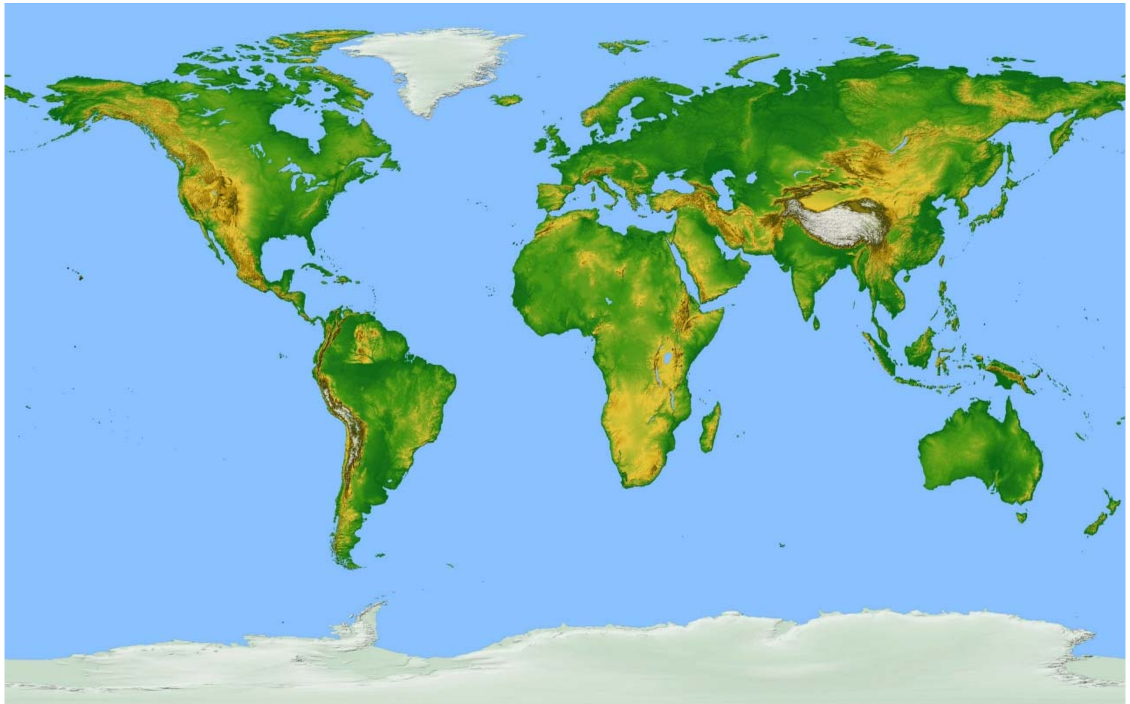


World Atlas

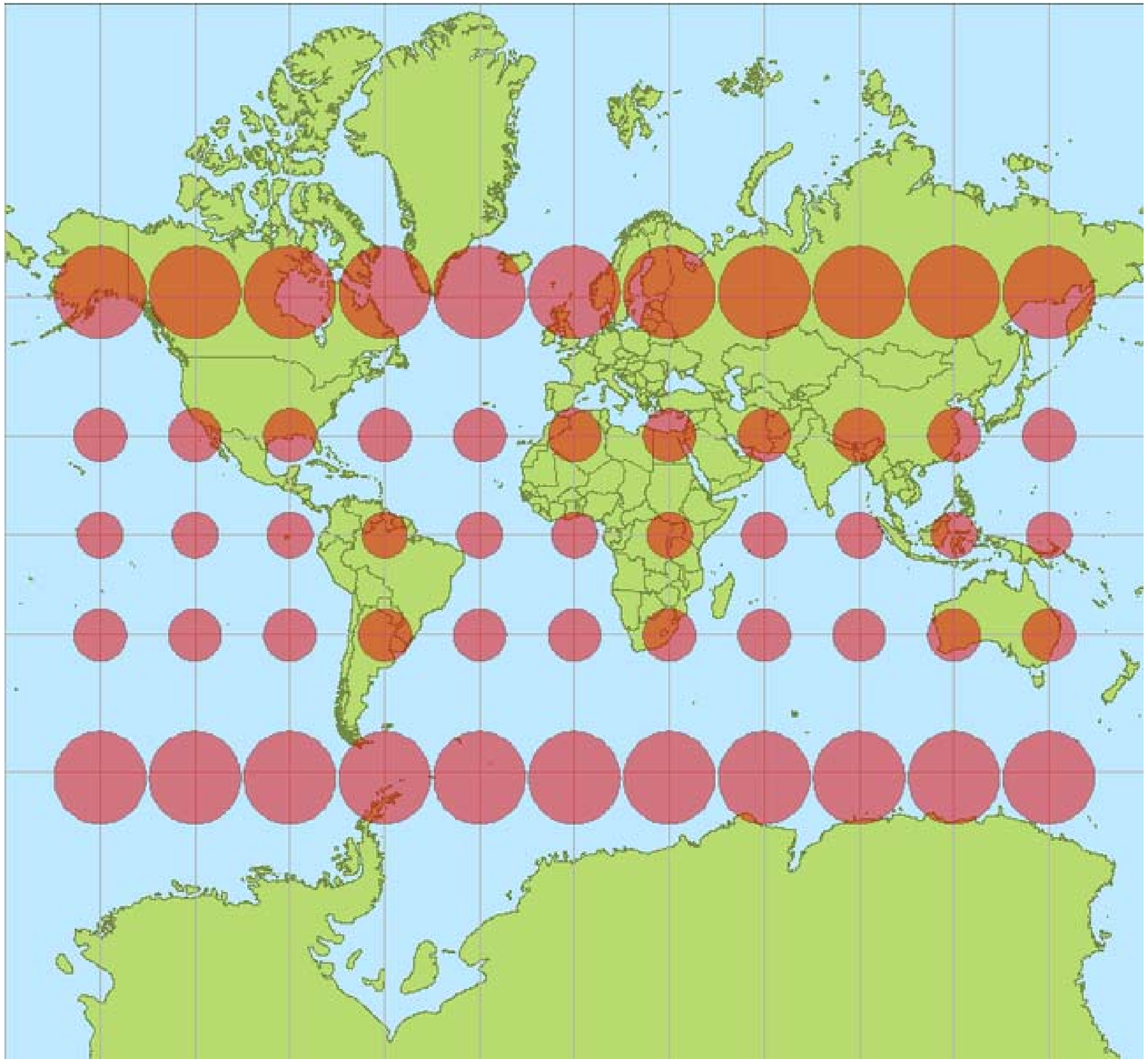


80°
ARCTIC OCEAN
Greenland
Arctic Circle
ASIA
EUROPE
40°
NORTH
AMERICA
Tropic of Cancer
ATLANTIC
PACIFIC
OCEAN
PACIFIC
0°
Equator
AFRICA
INDIAN
OCEAN
OCEAN
SOUTH
AMERICA
Tropic of Capricorn
OCEAN
40°
AUSTRALIA
Antarctic Circle
ANTARCTICA
80°
160°
120°
80°
40°
0°
40°
80°
120°
©H.L.
160°

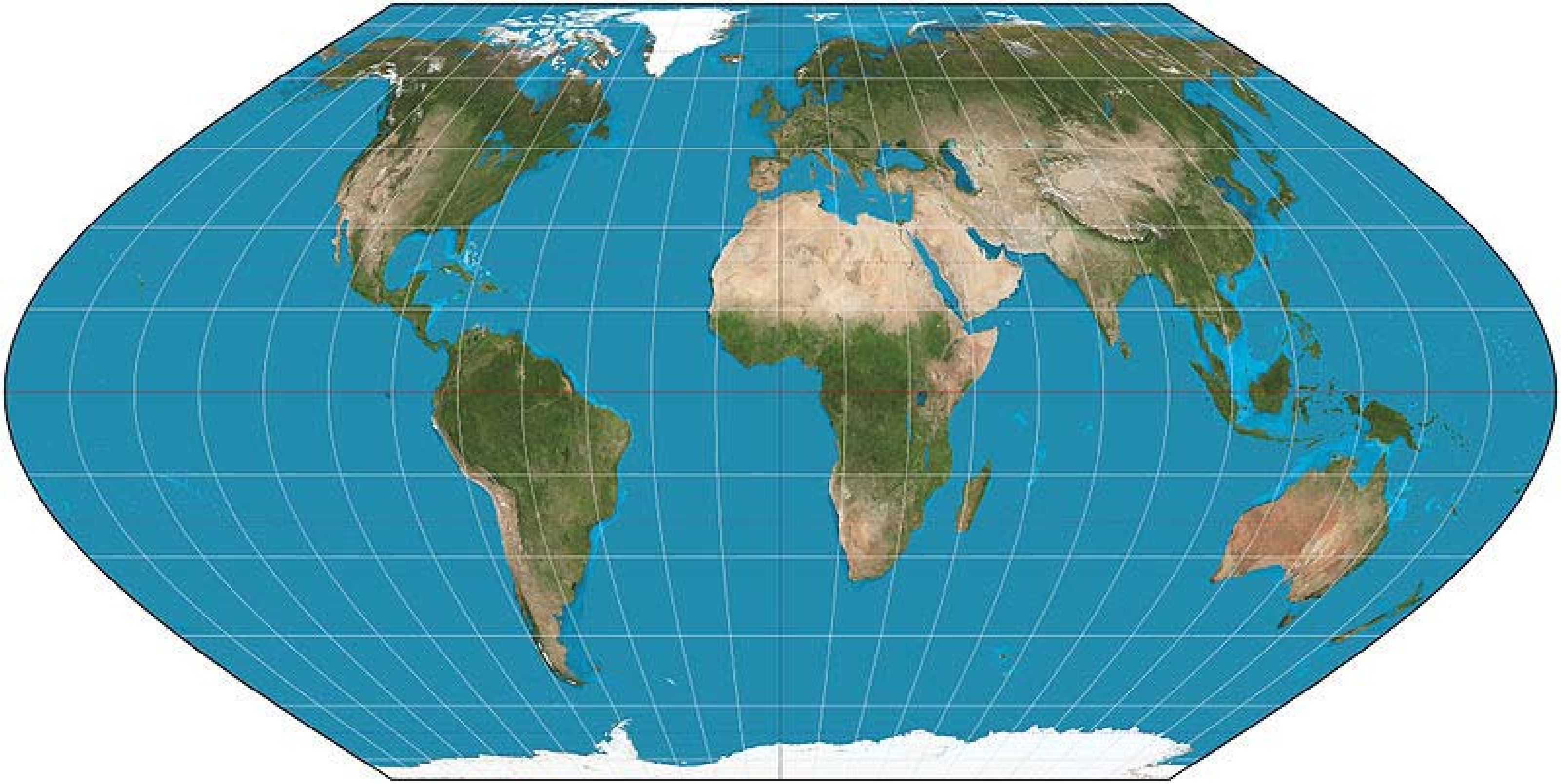
Parameterizations are **atlases**



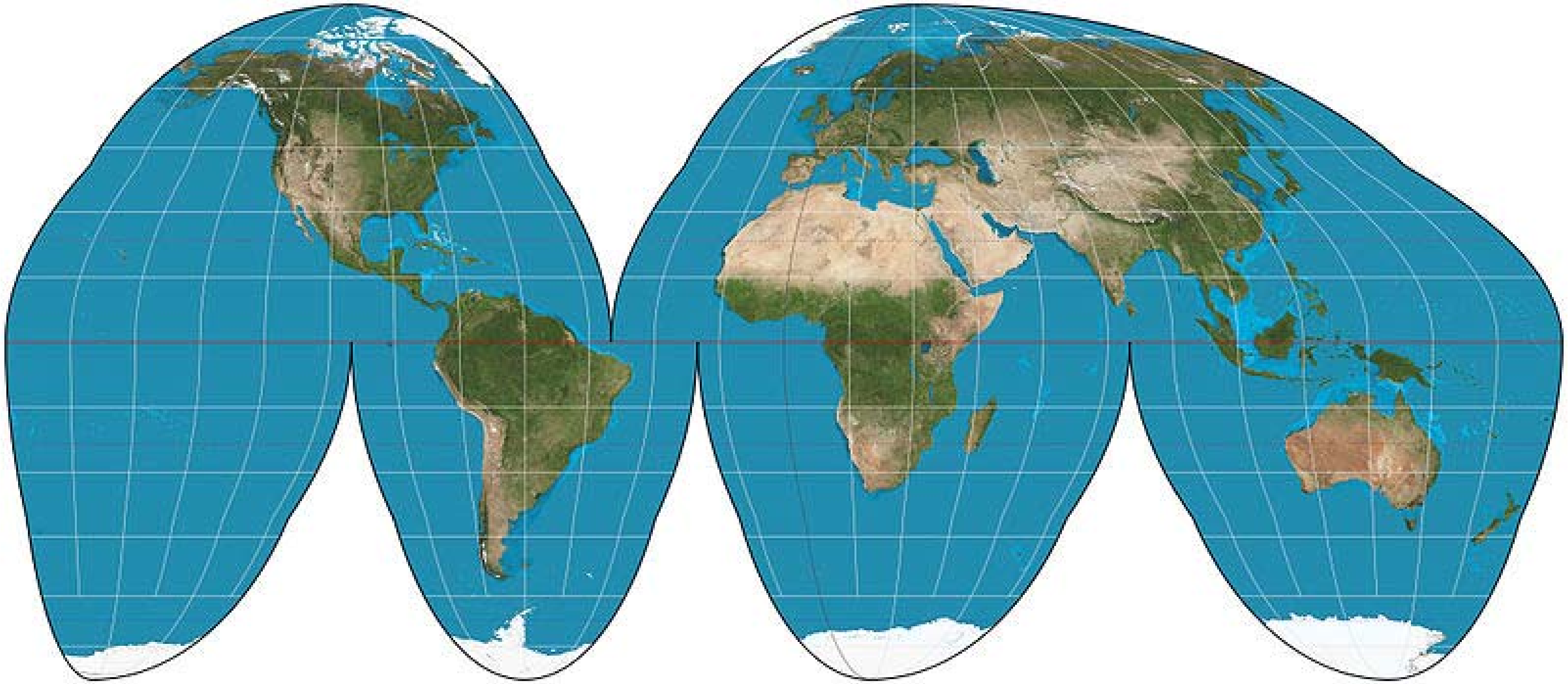
World Atlas



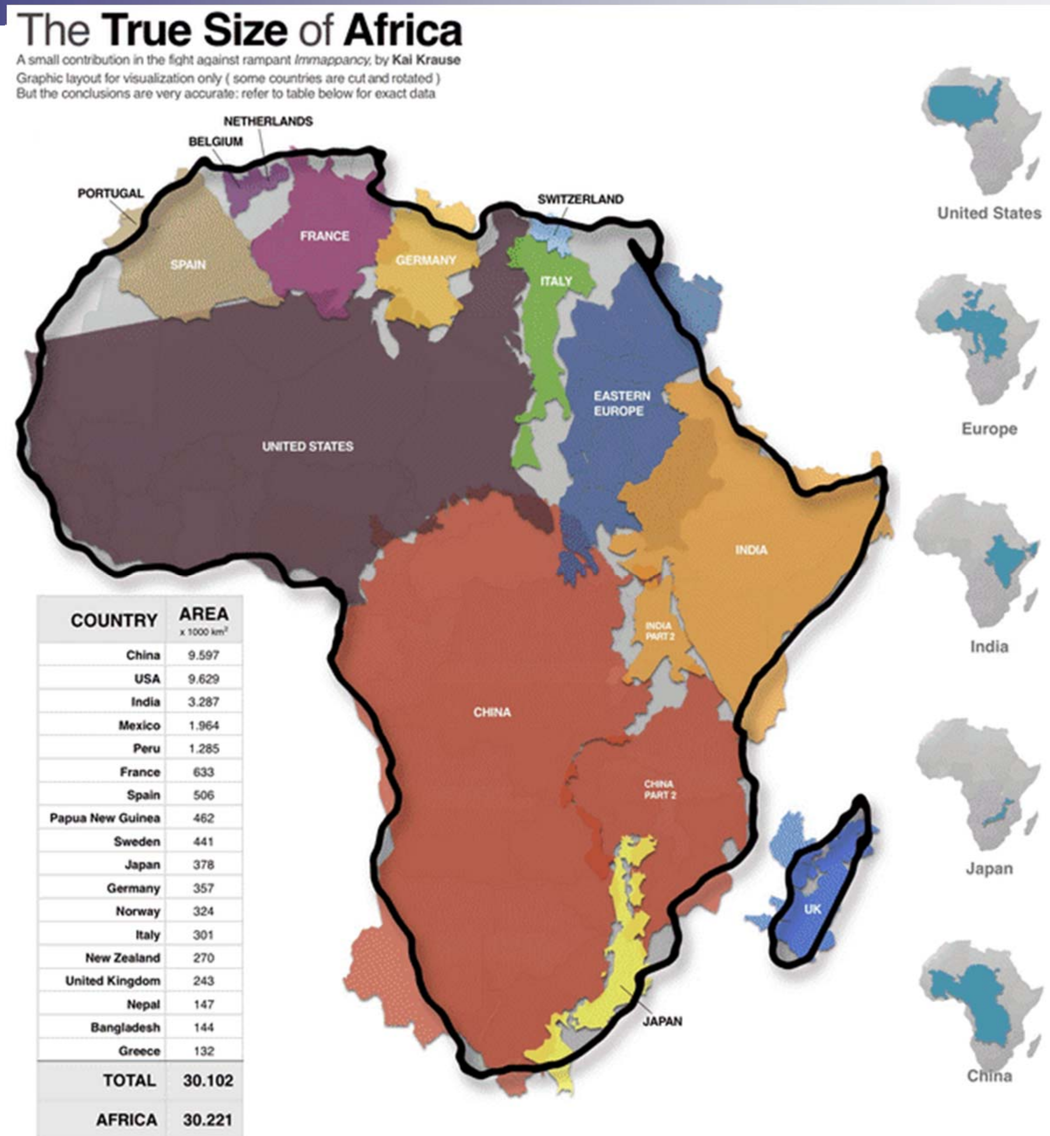
World Atlas



World Atlas



The true size of Africa

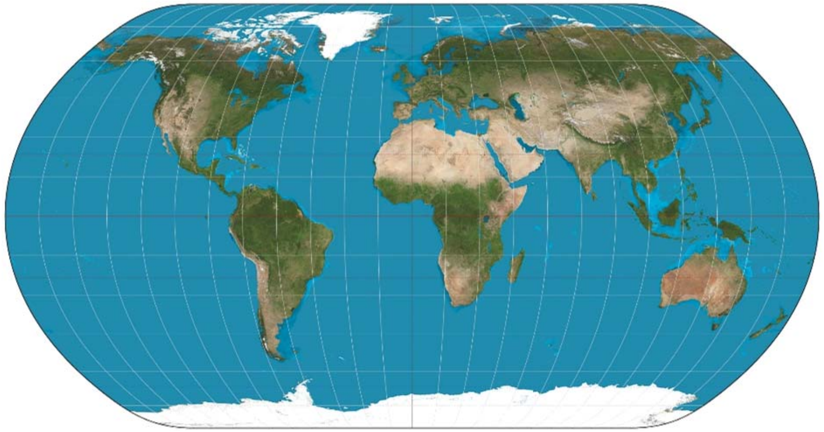


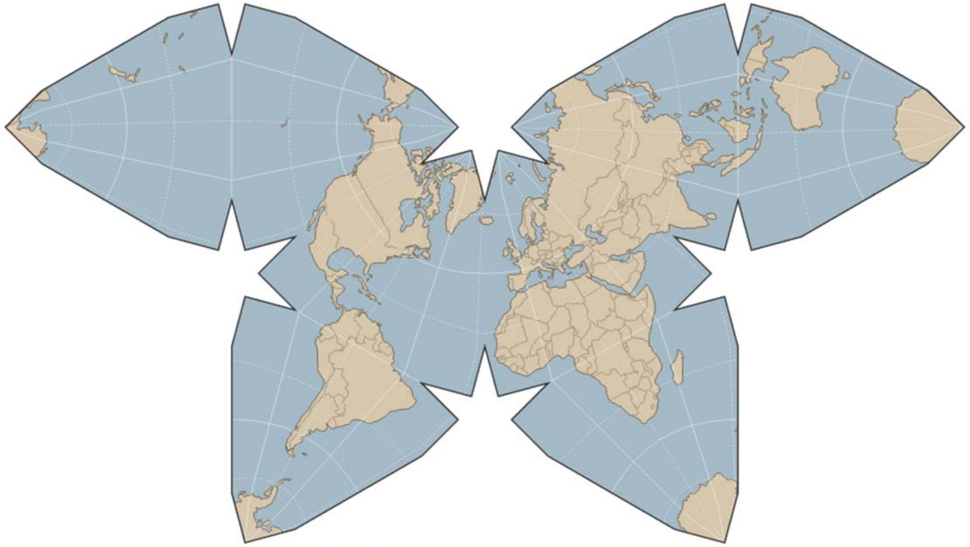
COUNTRY	AREA x 1000 km²
China	9.597
USA	9.629
India	3.287
Mexico	1.964
Peru	1.285
France	633
Spain	506
Papua New Guinea	462
Sweden	441
Japan	378
Germany	357
Norway	324
Italy	301
New Zealand	270
United Kingdom	243
Nepal	147
Bangladesh	144
Greece	132
TOTAL	30.102
AFRICA	30.221

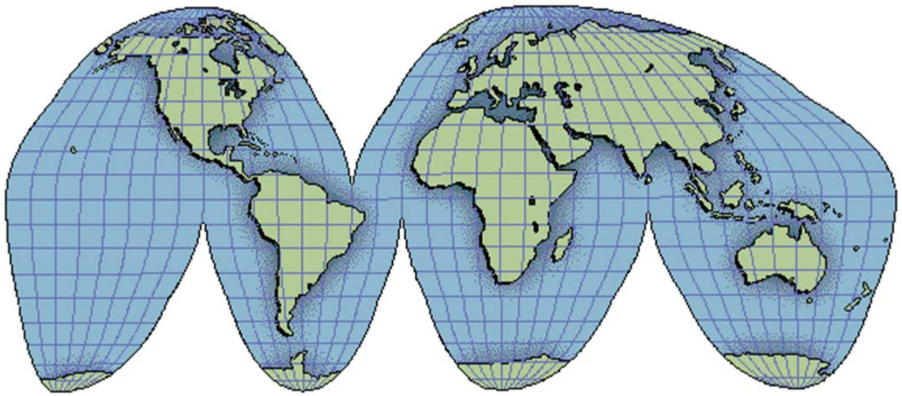
Another view of the same idea

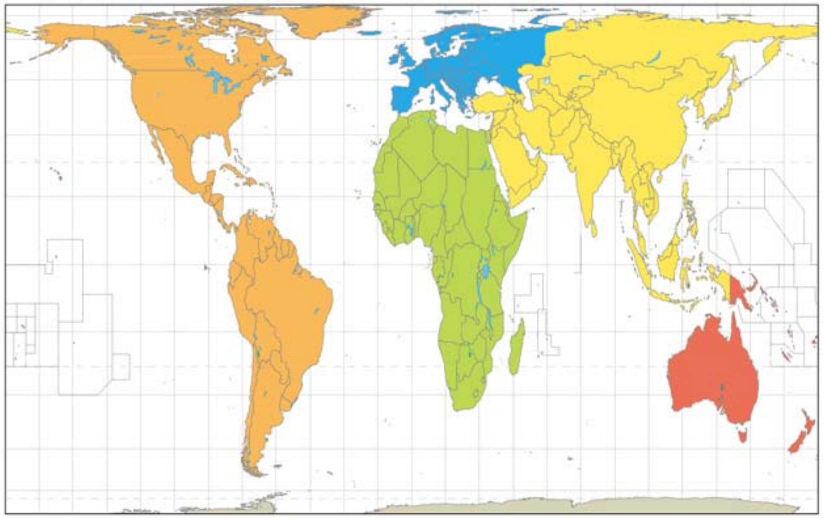
There are many possible maps

Is one of them “correct”?

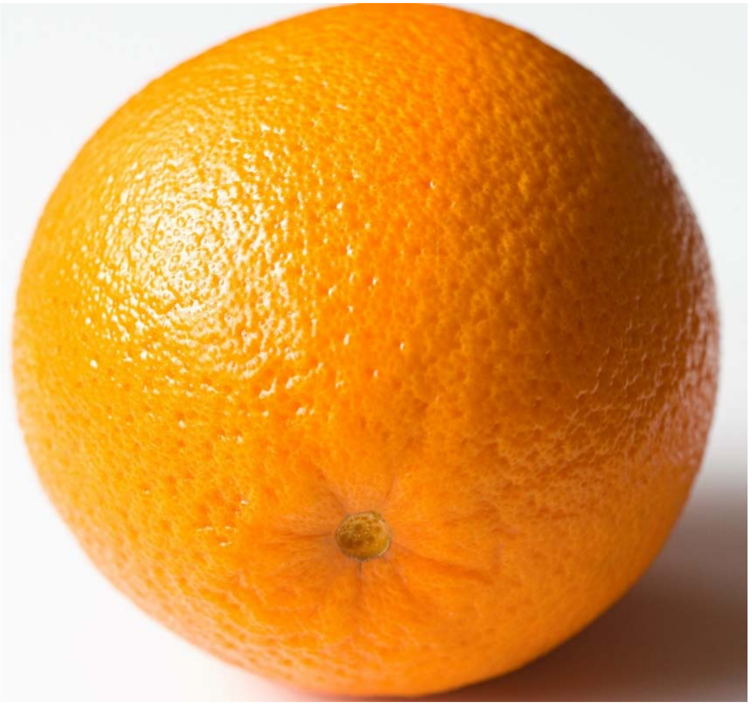


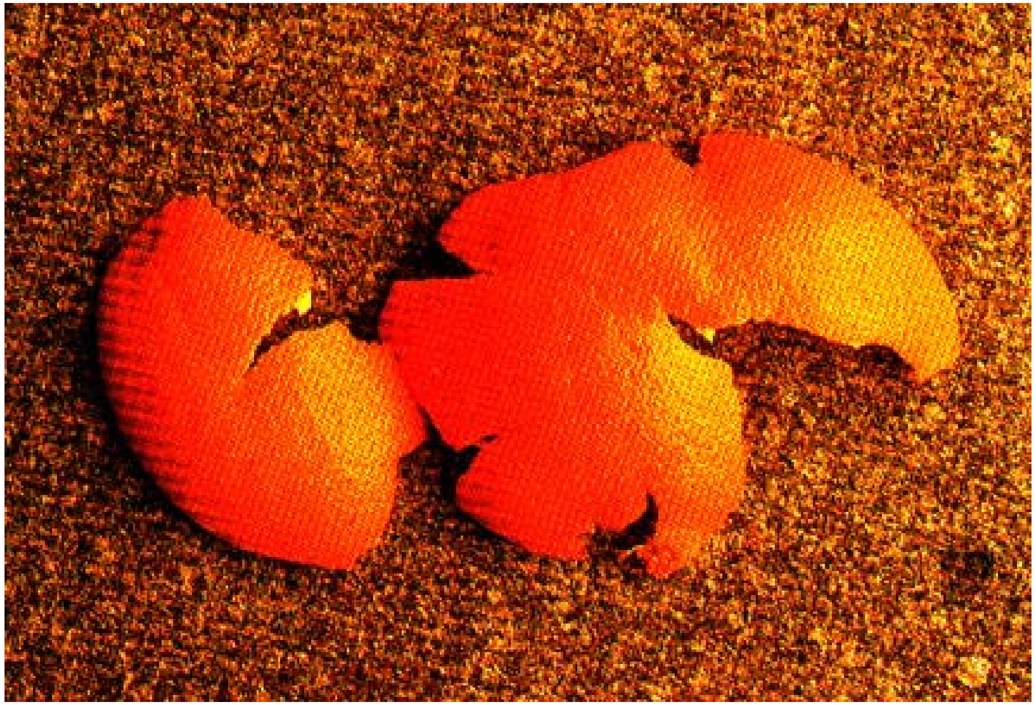




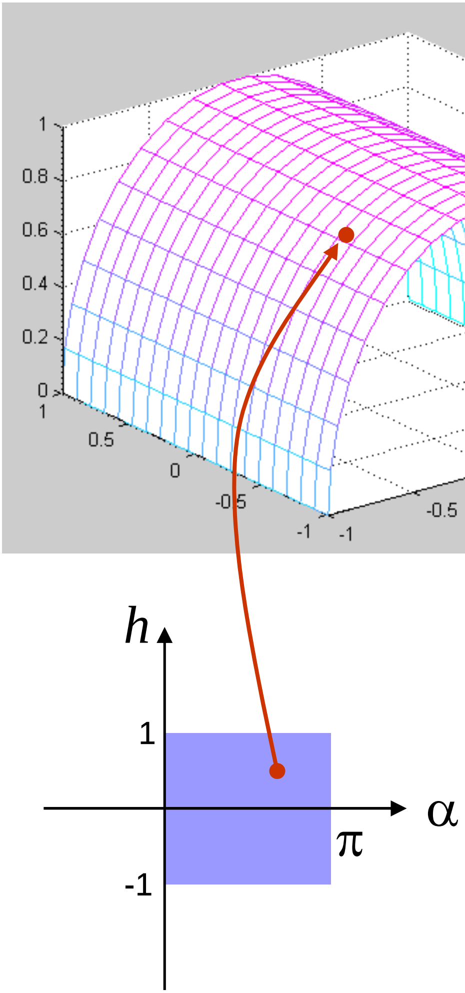


Can’t flatten without distorting





Another example:



Parameters: α, h

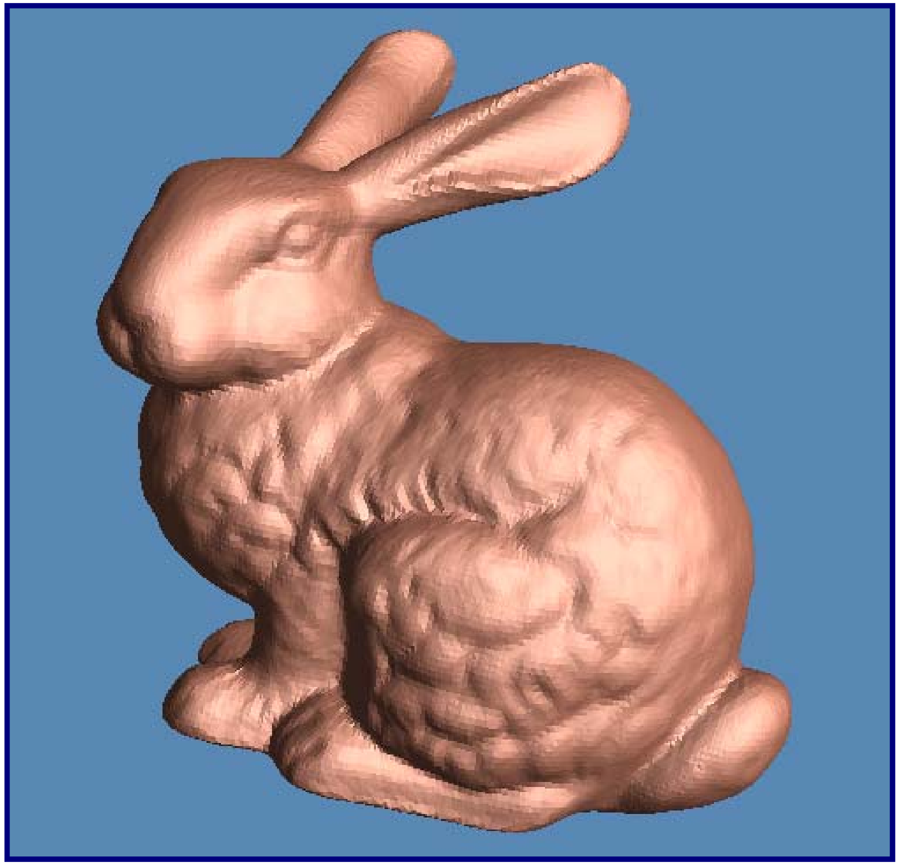
$D = [0,\pi]\times[-1,1]$

$x(\alpha, h) = cos(\alpha)$

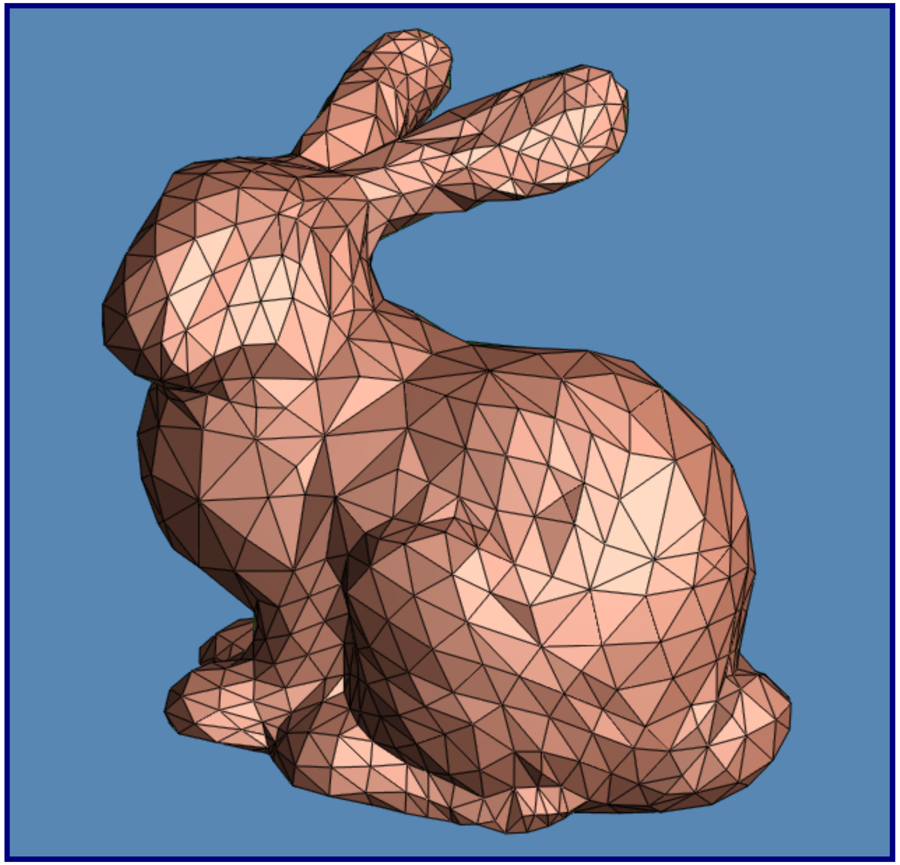
$y(\alpha, h) = h$

$z(\alpha, h) = sin(\alpha)$

Triangular Mesh



Triangular Mesh



Mesh Parameterization

- Uniquely defined by mapping mesh vertices to the parameter domain:

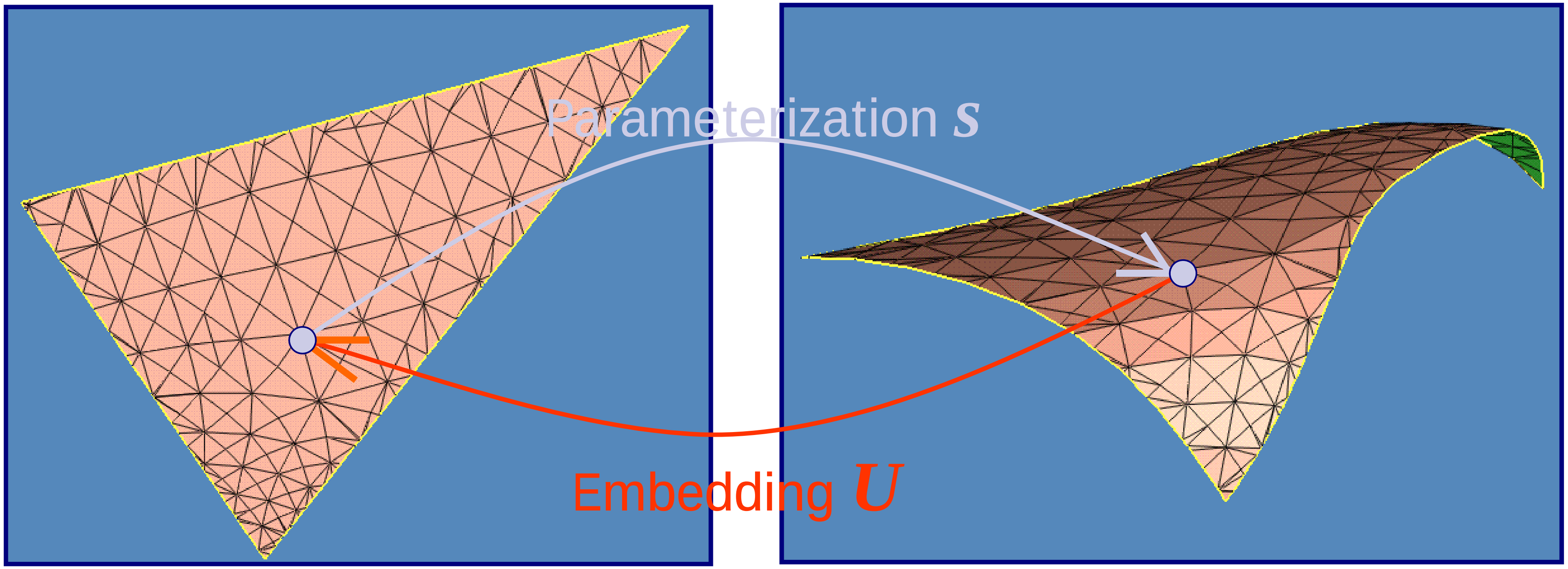
$$U : \{\mathrm{v}_1, \ldots, \mathrm{v}_\mathrm{n}\} \rightarrow D \subseteq R^2$$

$$U(\mathrm{v}_i) = (u_i, v_i)$$

- No two edges cross in the plane (in D)

Mesh parameterization $\Leftrightarrow$ mesh embedding

Mesh parameterization

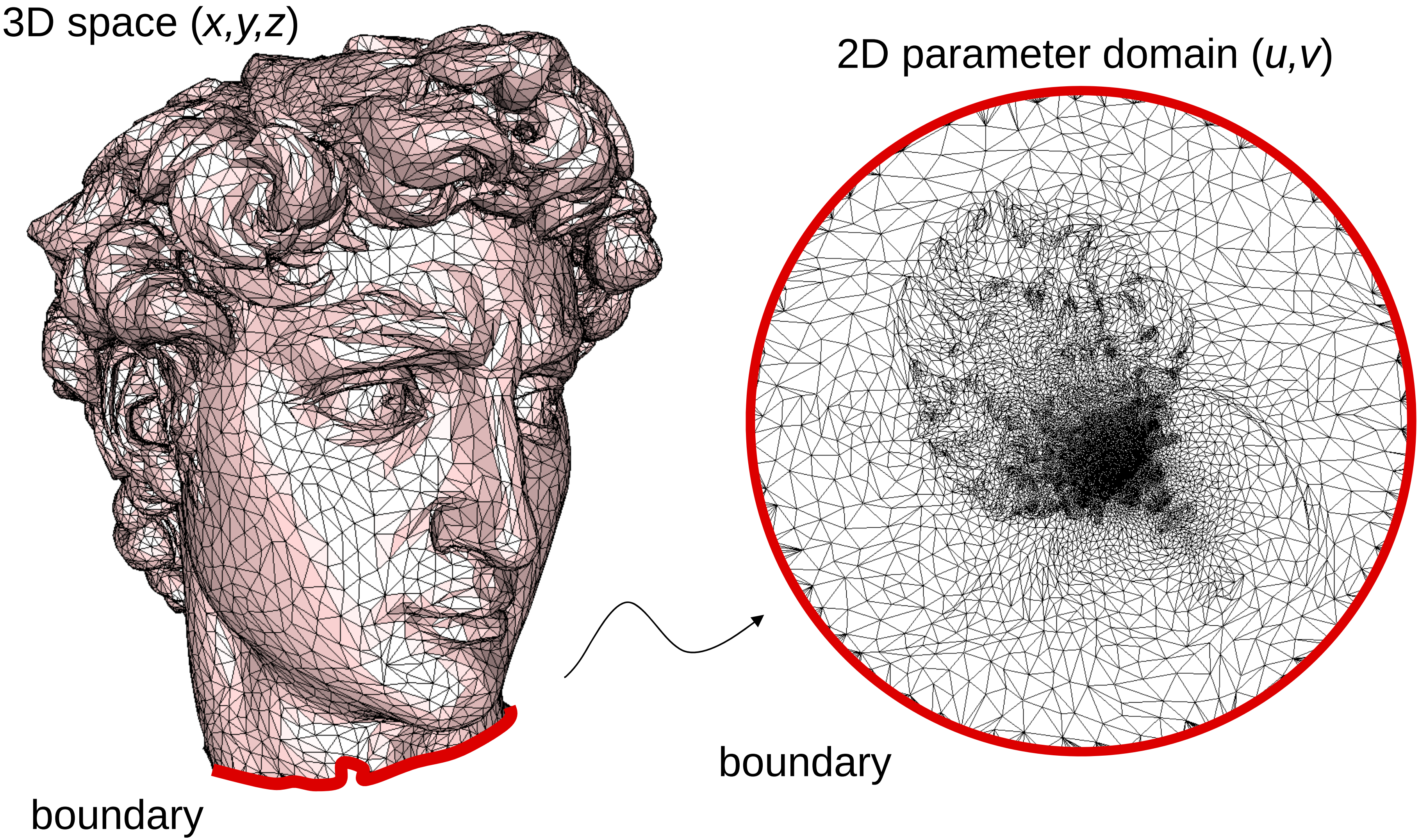


Parameter domain $D \subseteq R^2$

$$\boxed{s = U^{-1}}$$

Mesh surface $S \subseteq R^3$

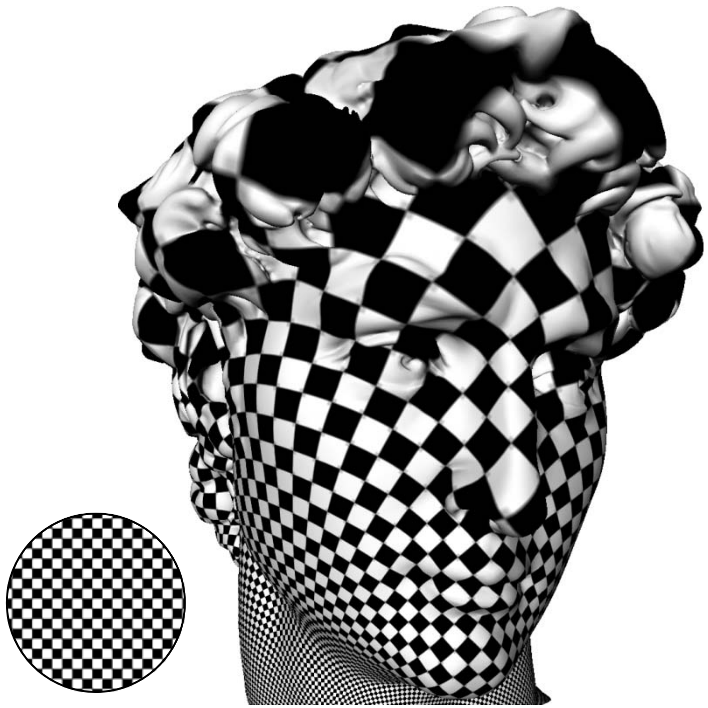
2D parameterization



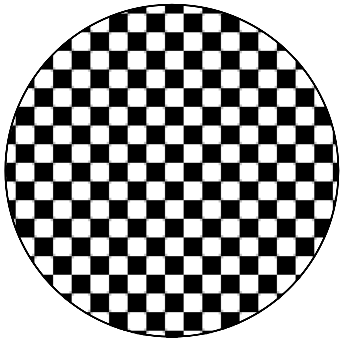
Application - Texture mapping

Requirements

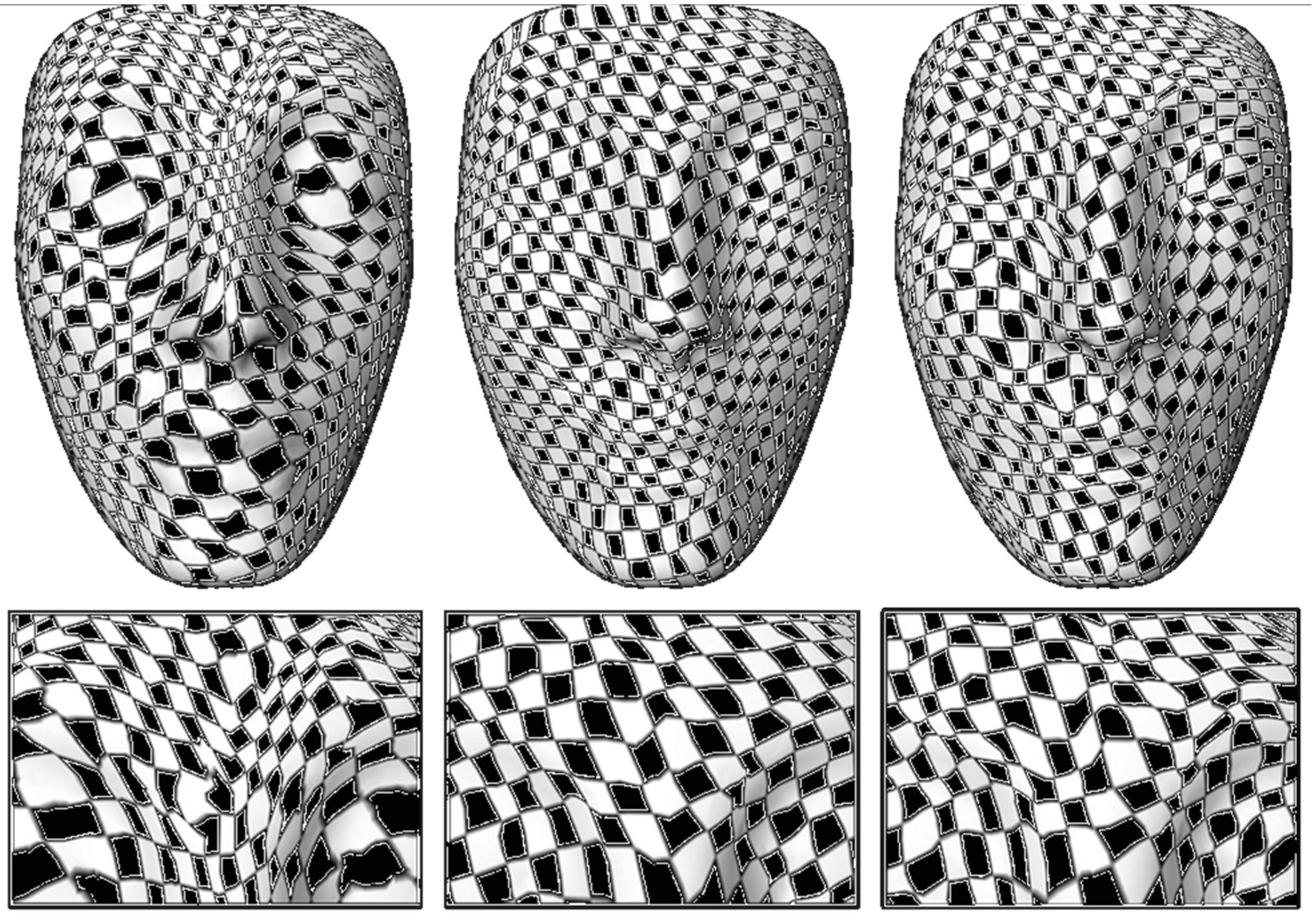
- Bijective (1-1 and onto): No triangles fold over.
- Minimal “distortion”
  - Preserve 3D angles
  - Preserve 3D distances
  - Preserve 3D areas
  - No “stretch”



Distortion minimization

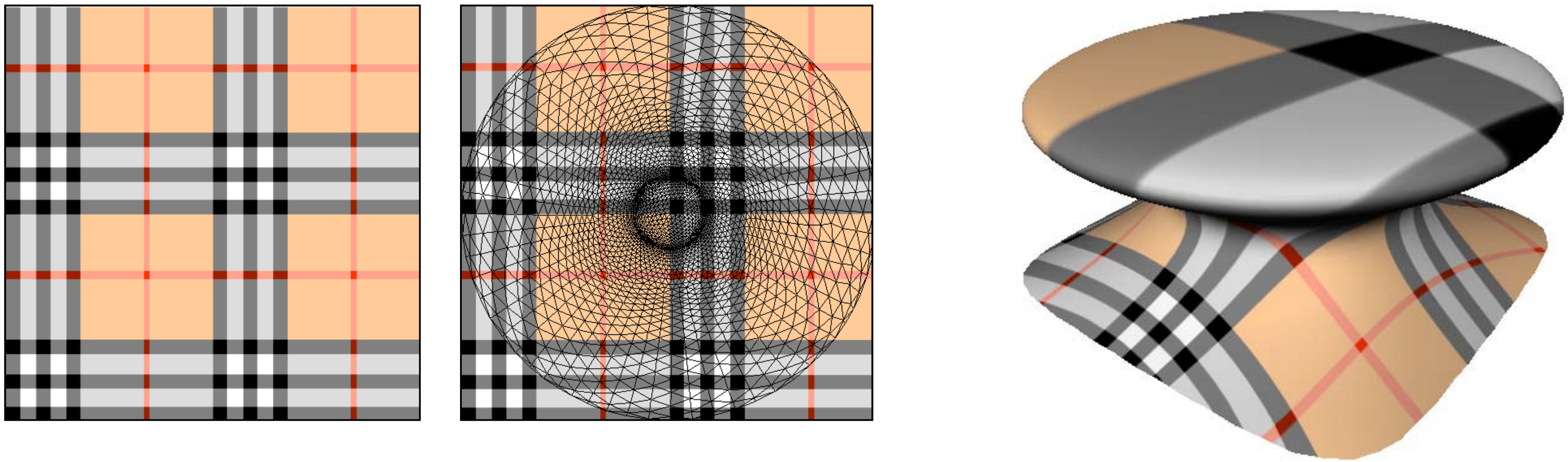


Texture map

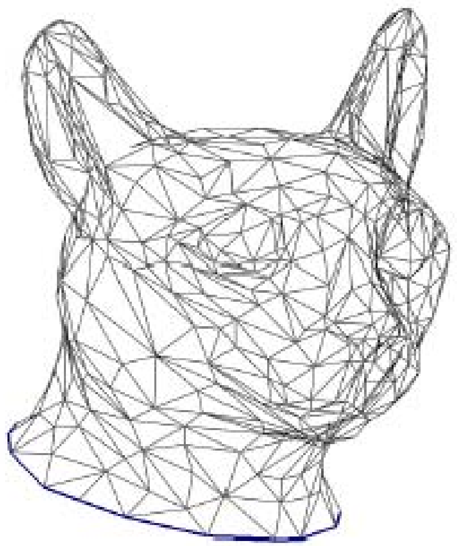


Kent et al ‘92 Floater 97 Sander et al ‘01

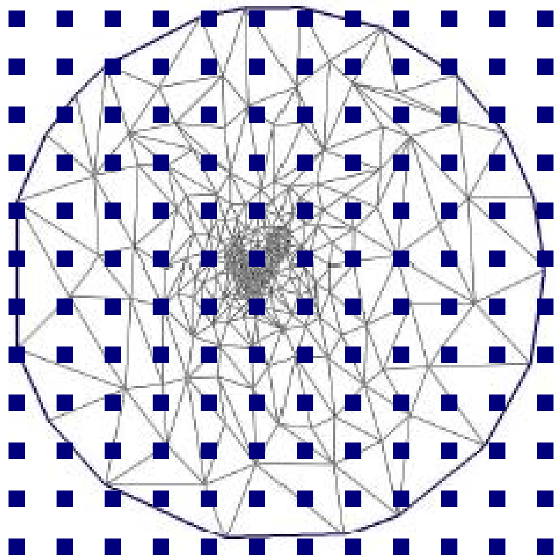
More texture mapping



Resampling problems



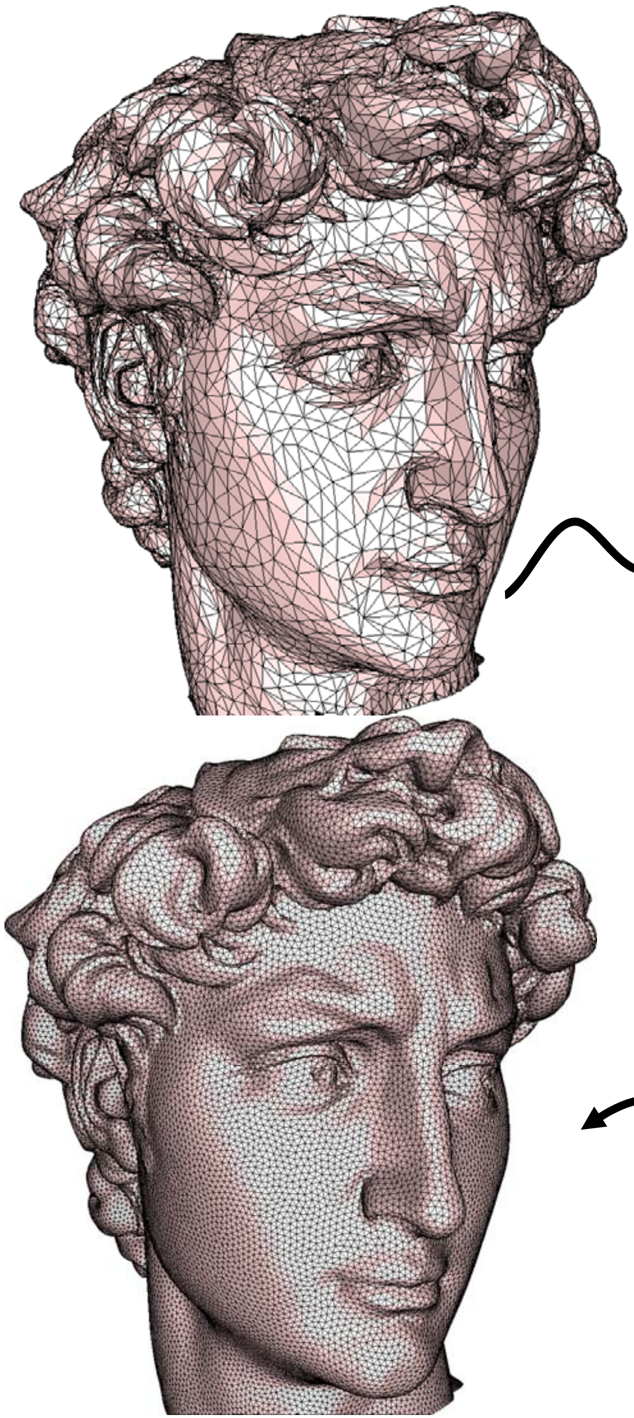
Cat mesh

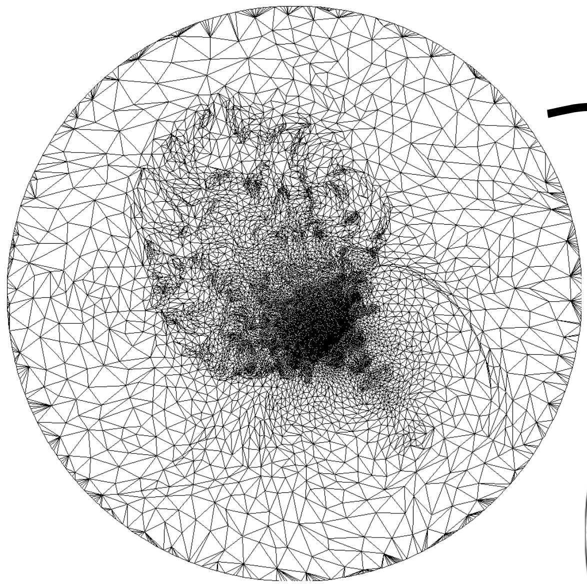


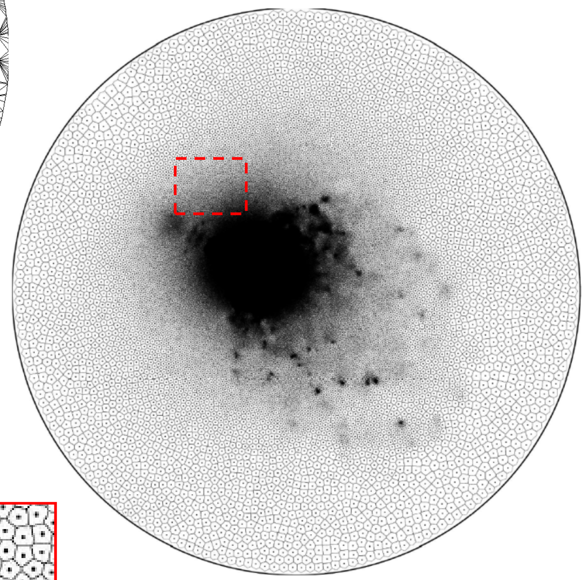
Distorting embedding

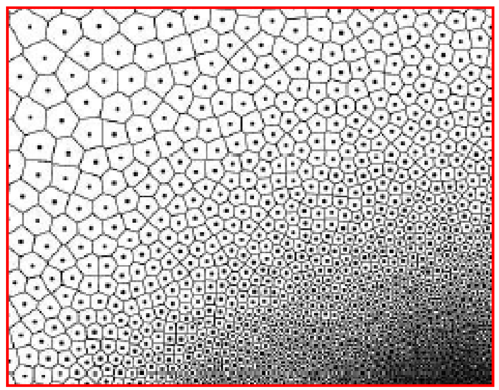
Resampling on regular grid

Remeshing

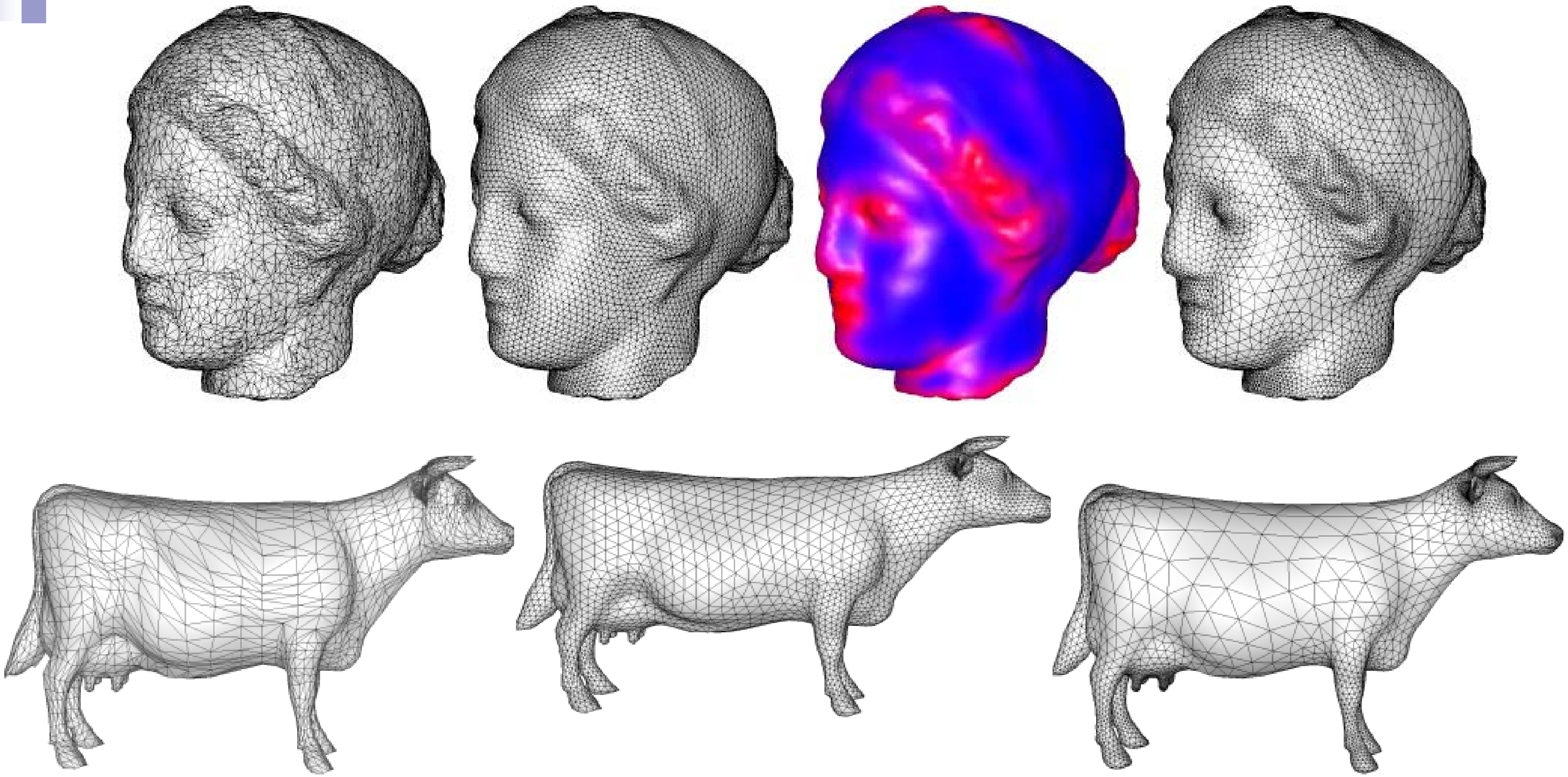




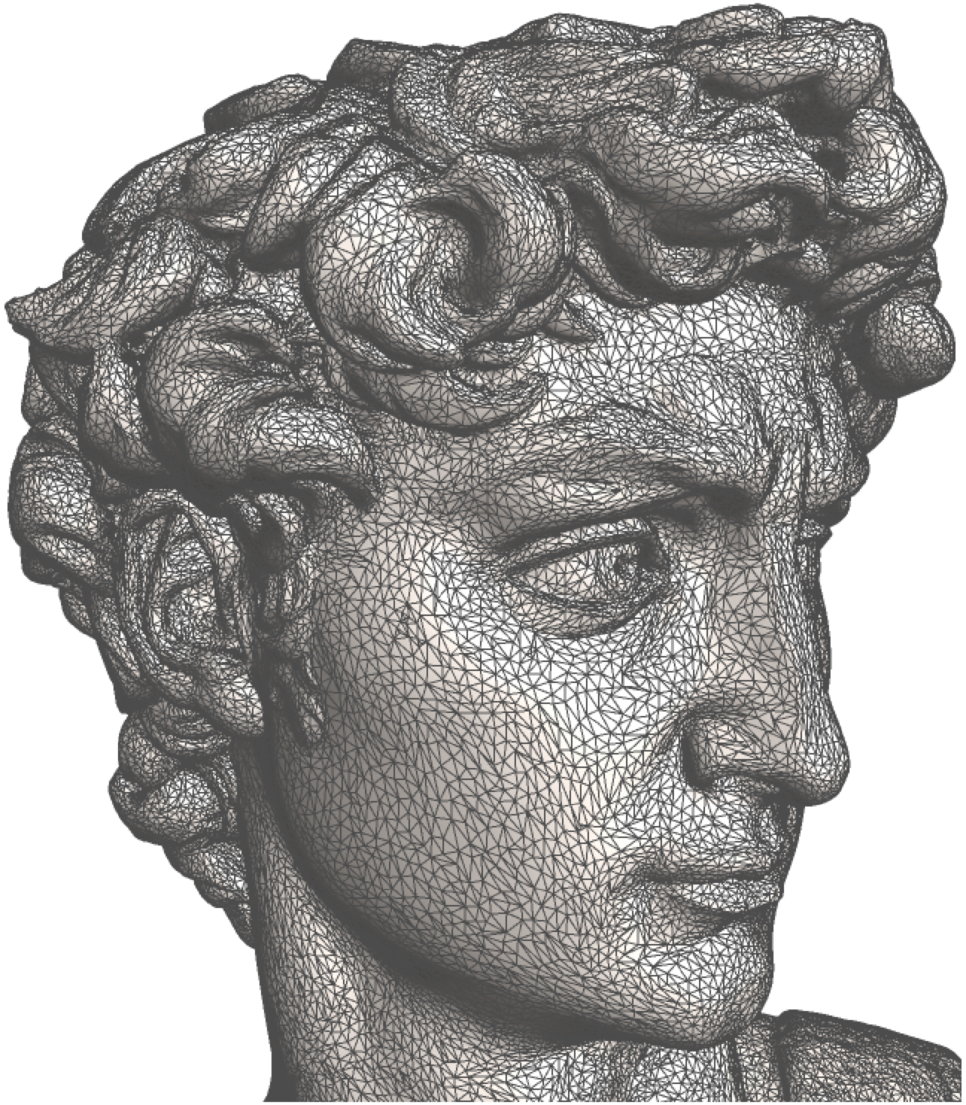




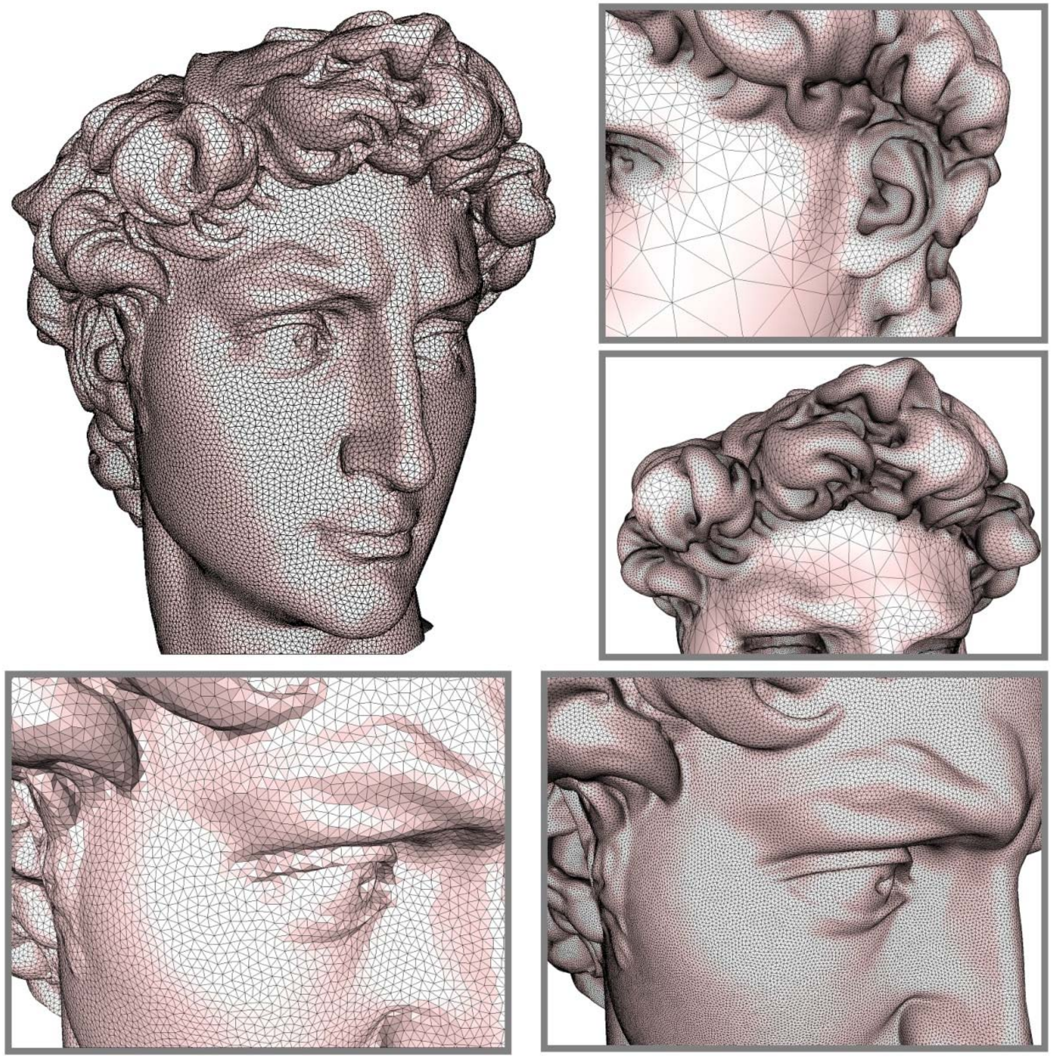
Remeshing examples



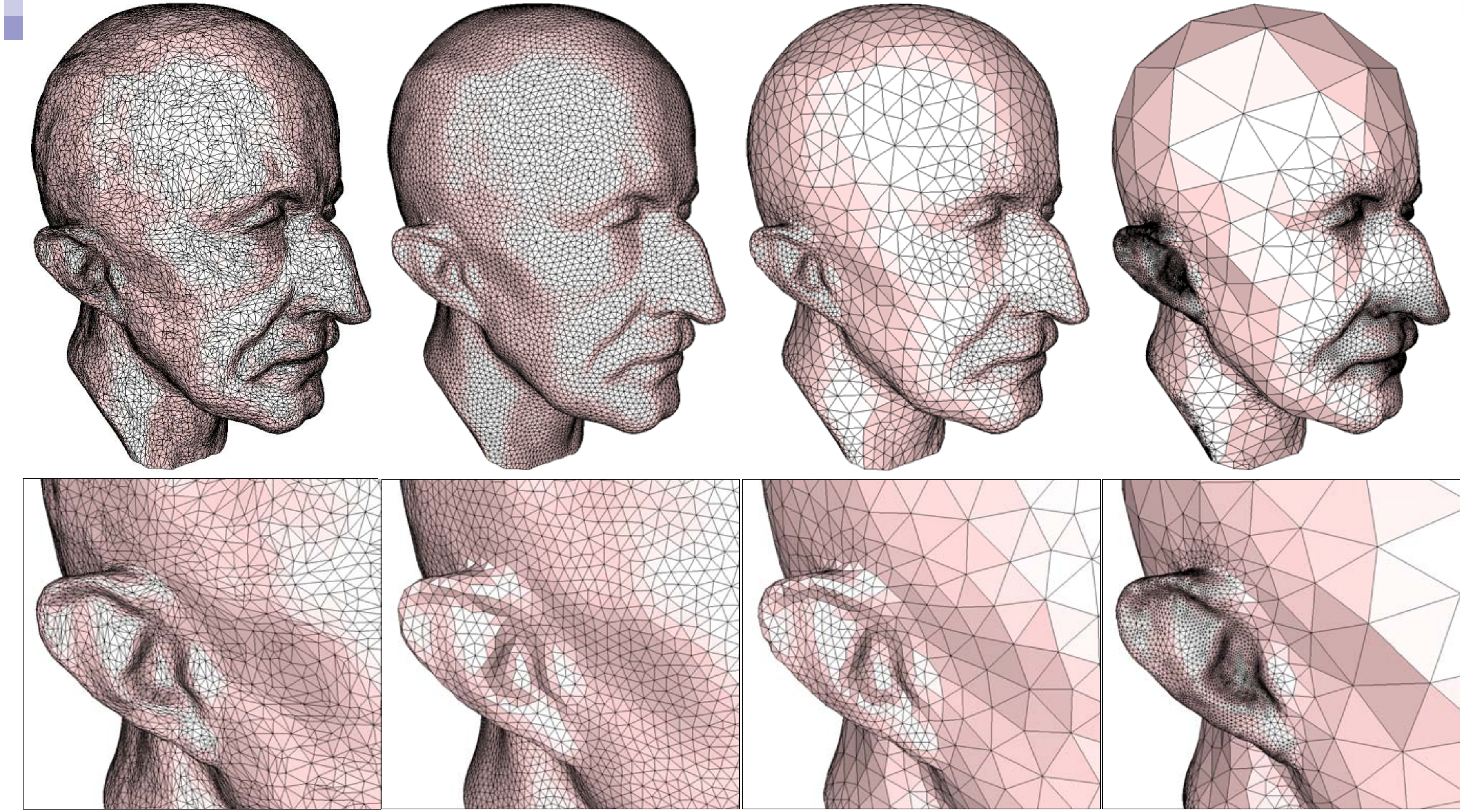
Remeshing



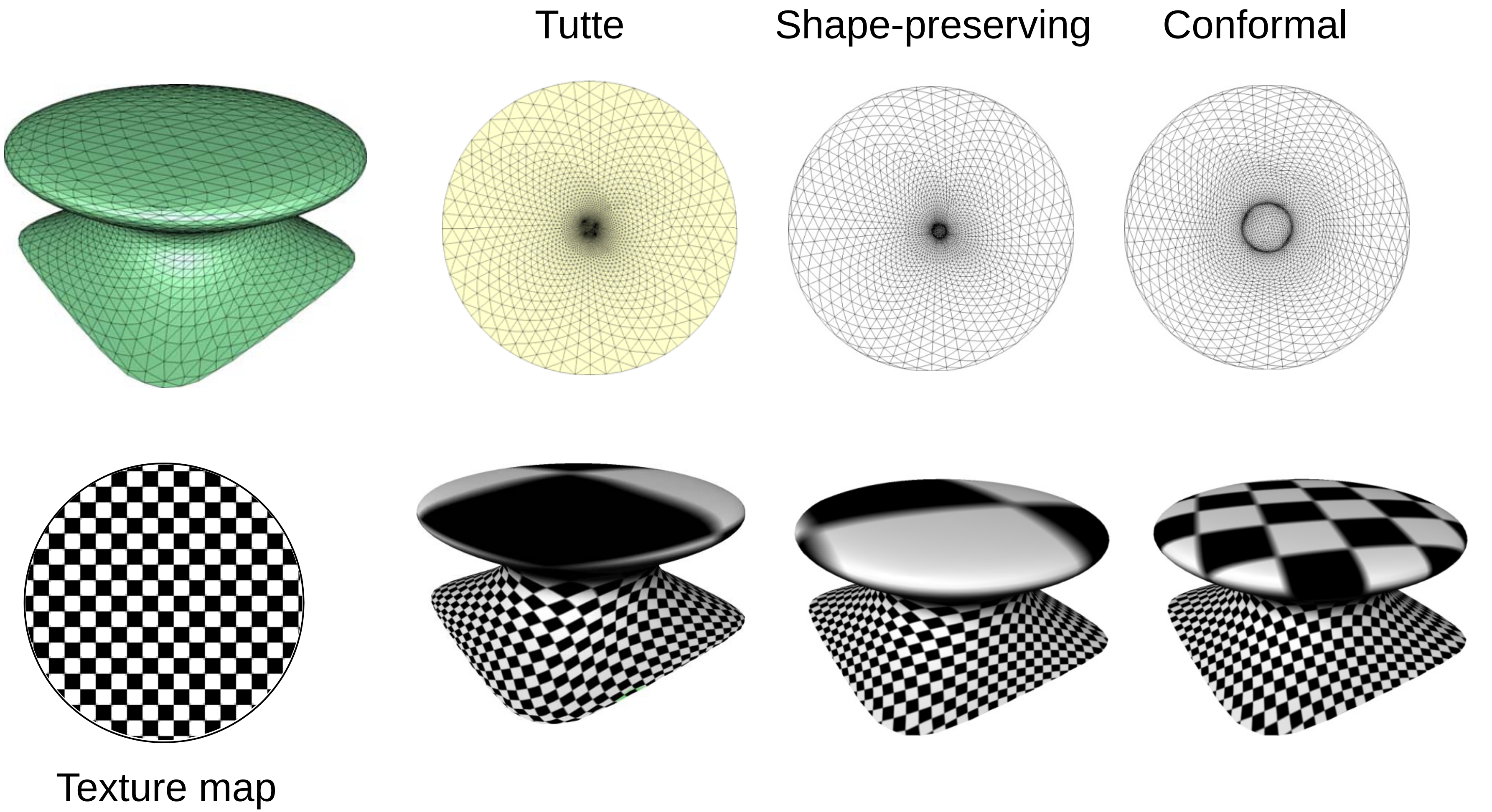
Remeshing



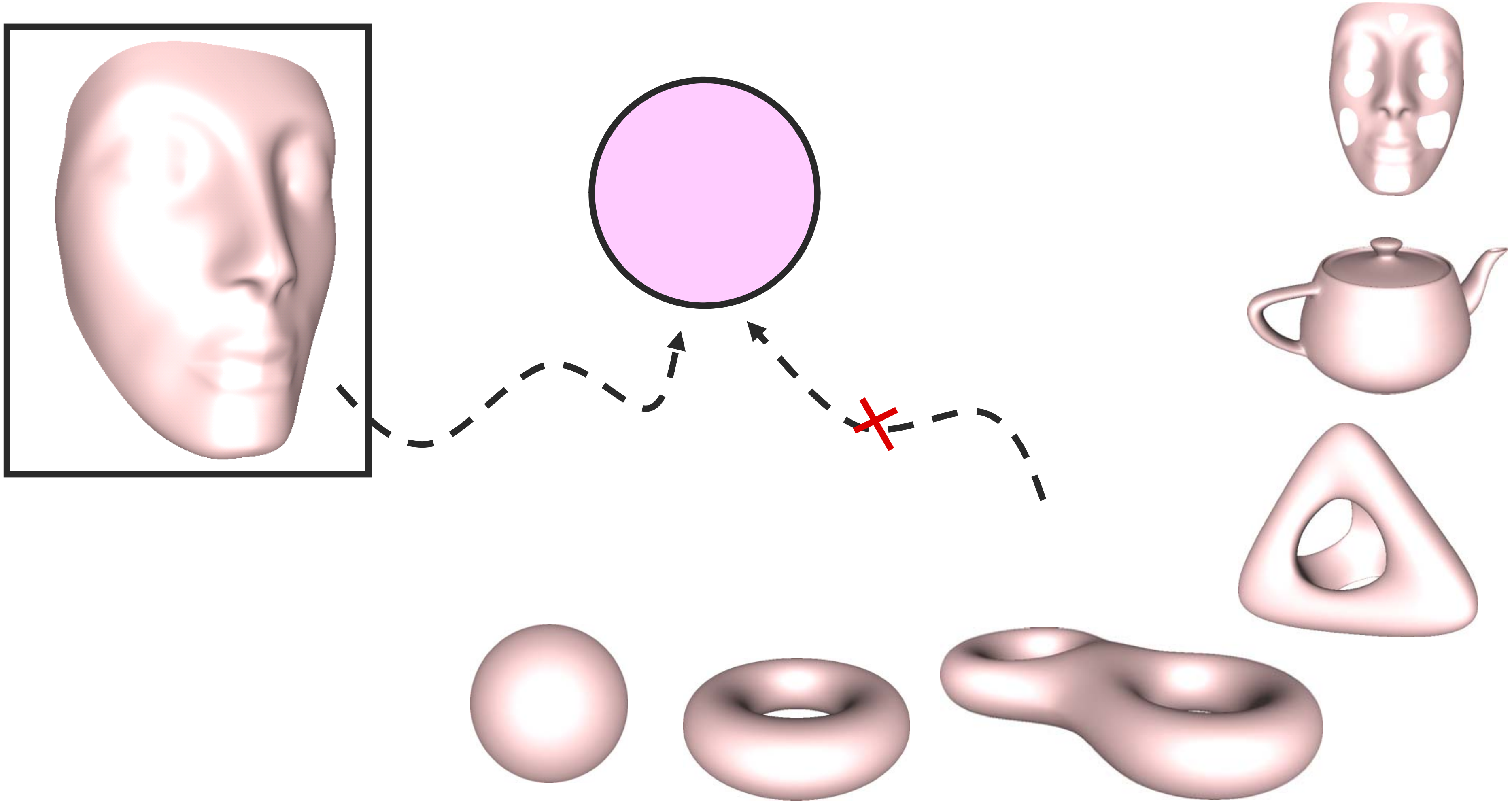
More remeshing examples



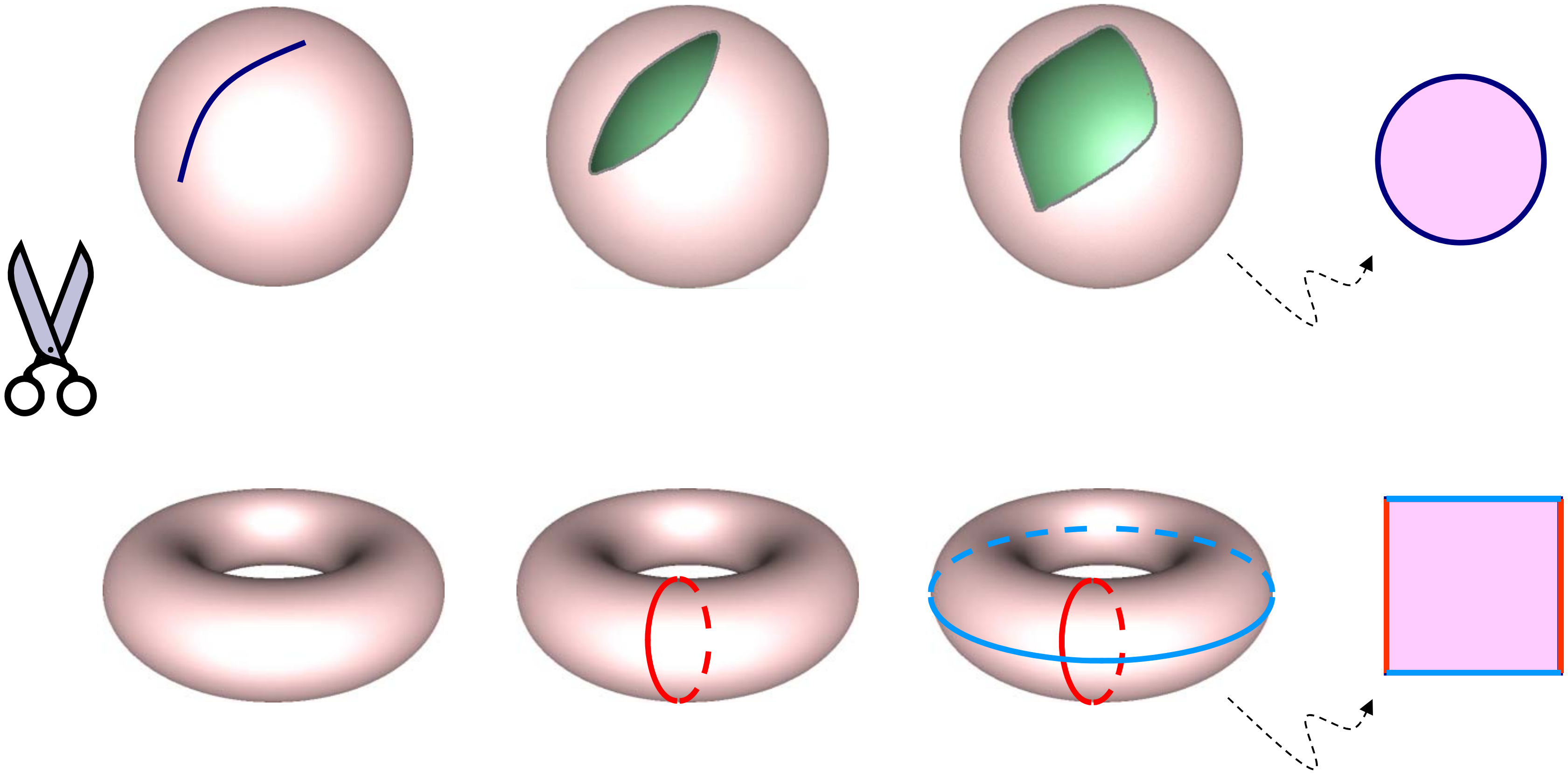
Conformal parametrization



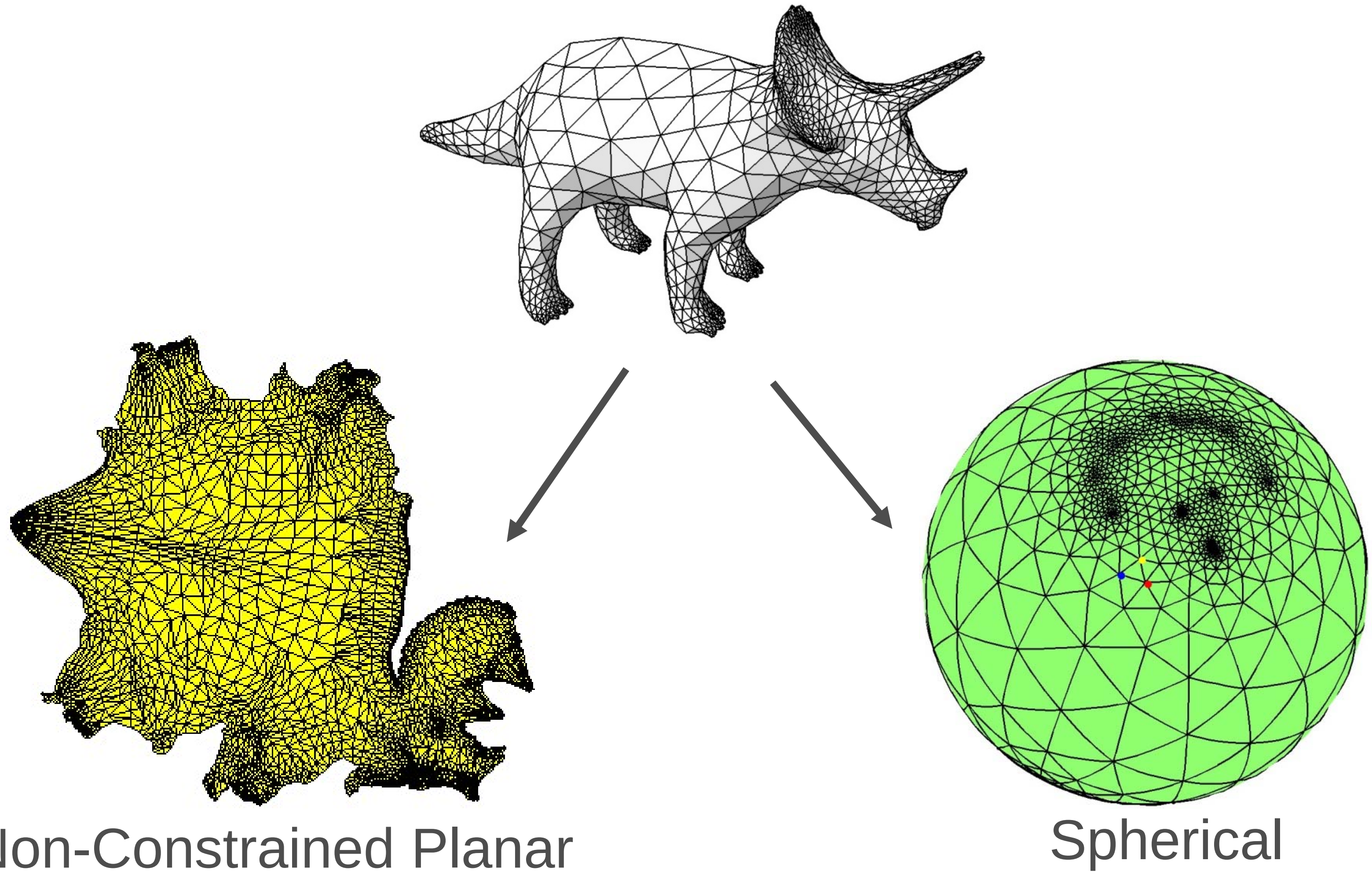
Non-simple domains



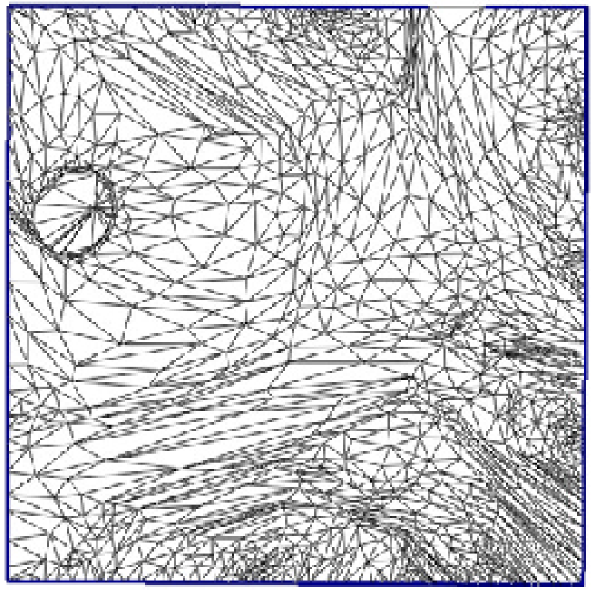
Cutting



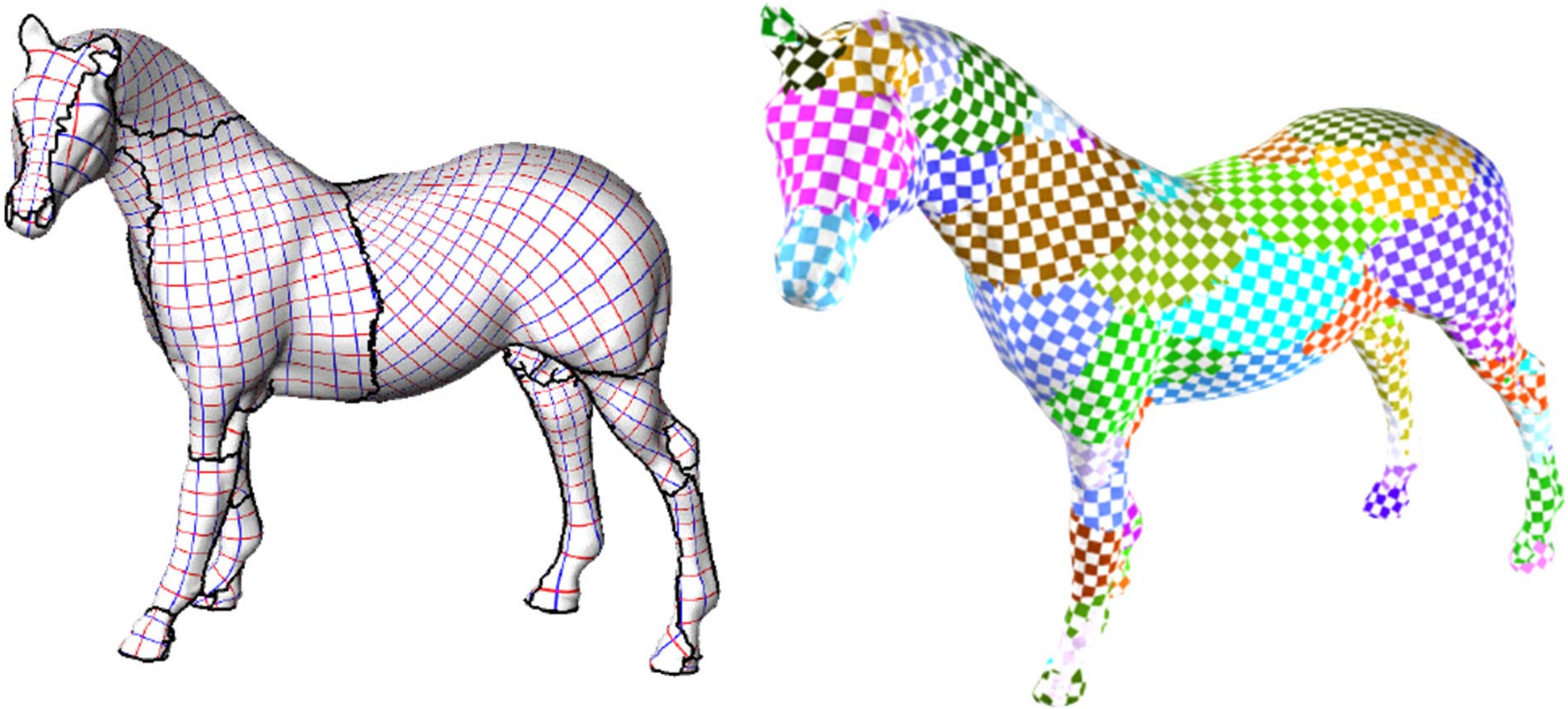
Parameterization of closed genus-0 triangle meshes



Introducing seams (cuts)



Partition

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CAMEL
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CAMEL
SURGEON GENERAL'S WARNING:
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Result in Fetal Injury, Premature
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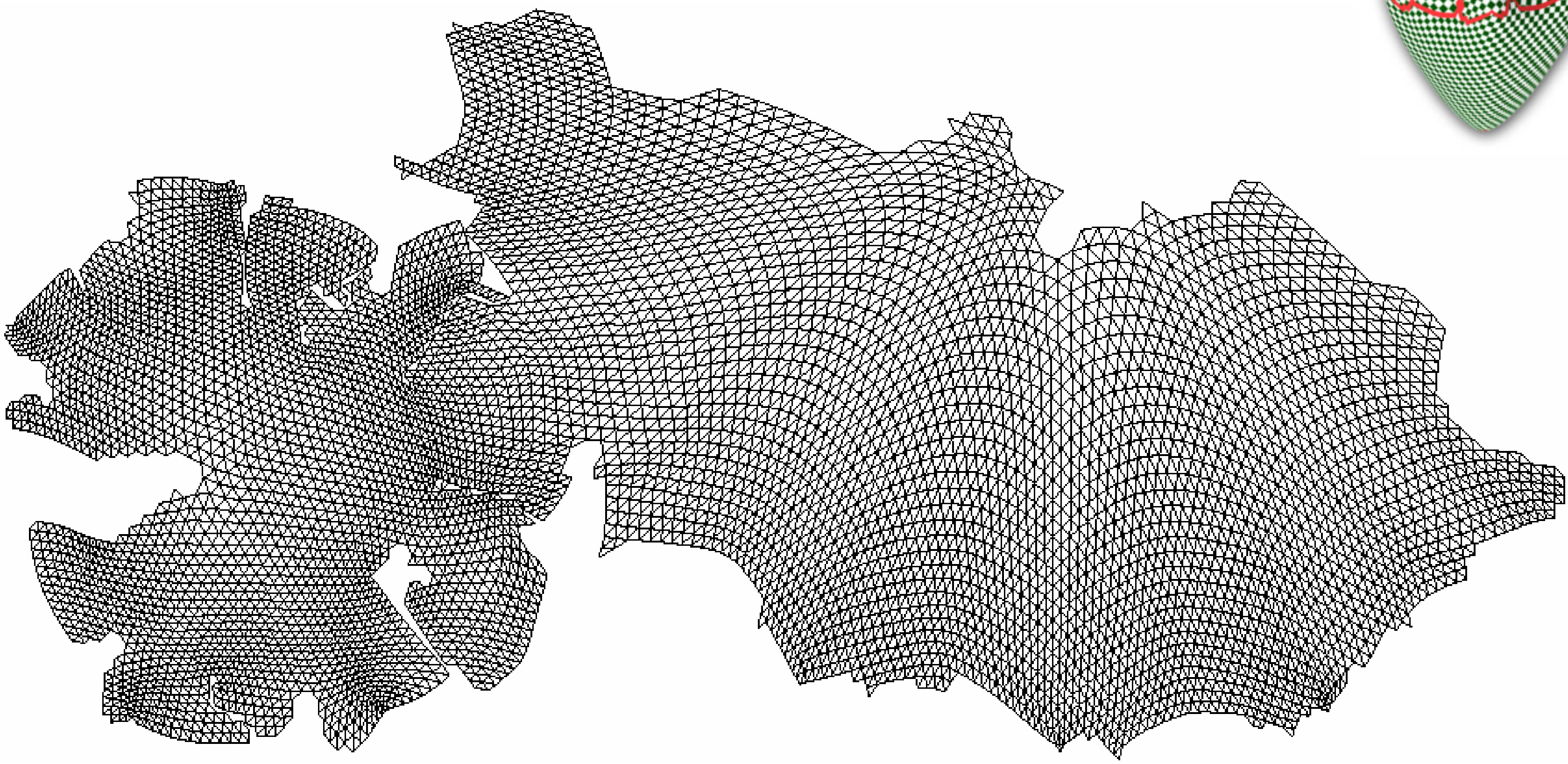
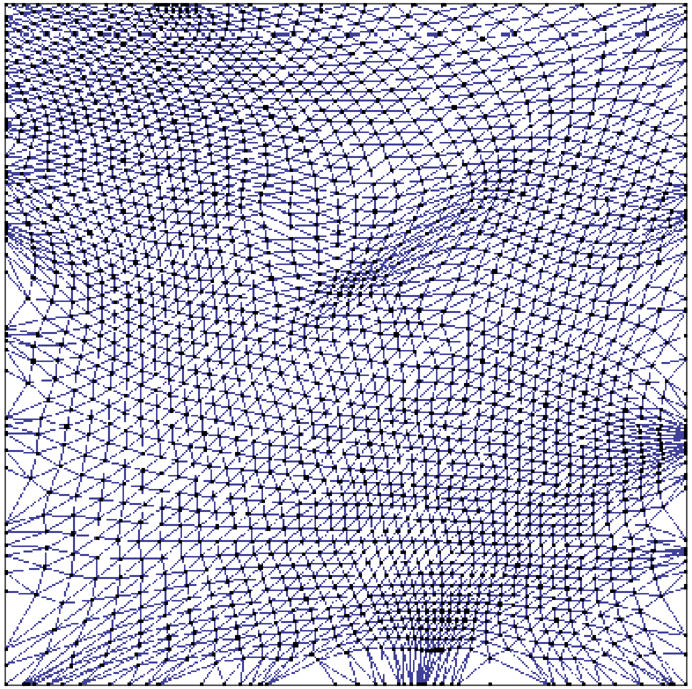
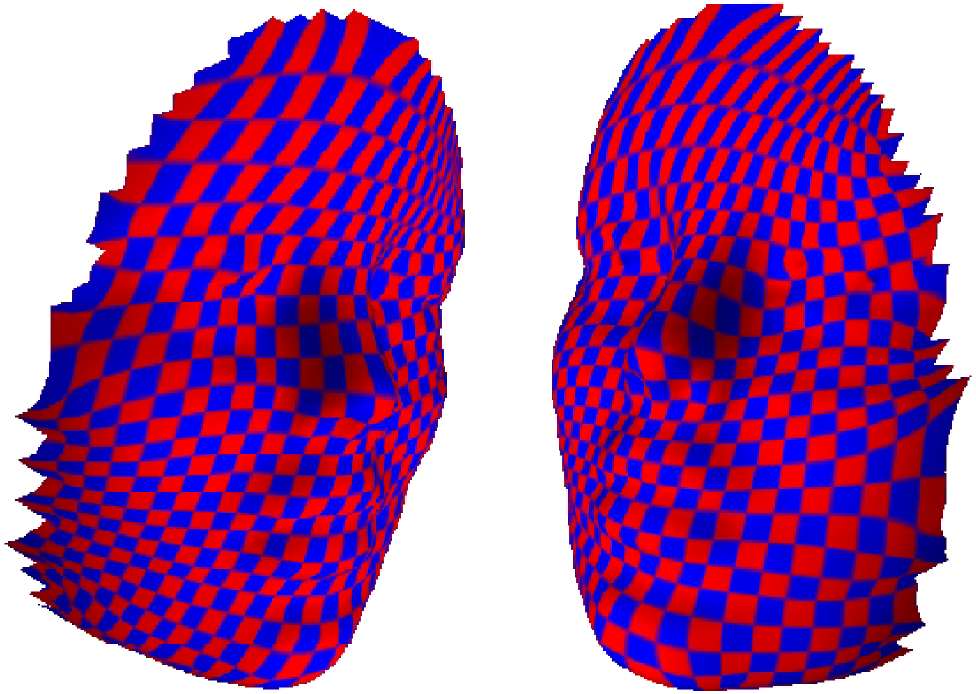
Introducing seams (cuts)

Introducing seams (cuts)

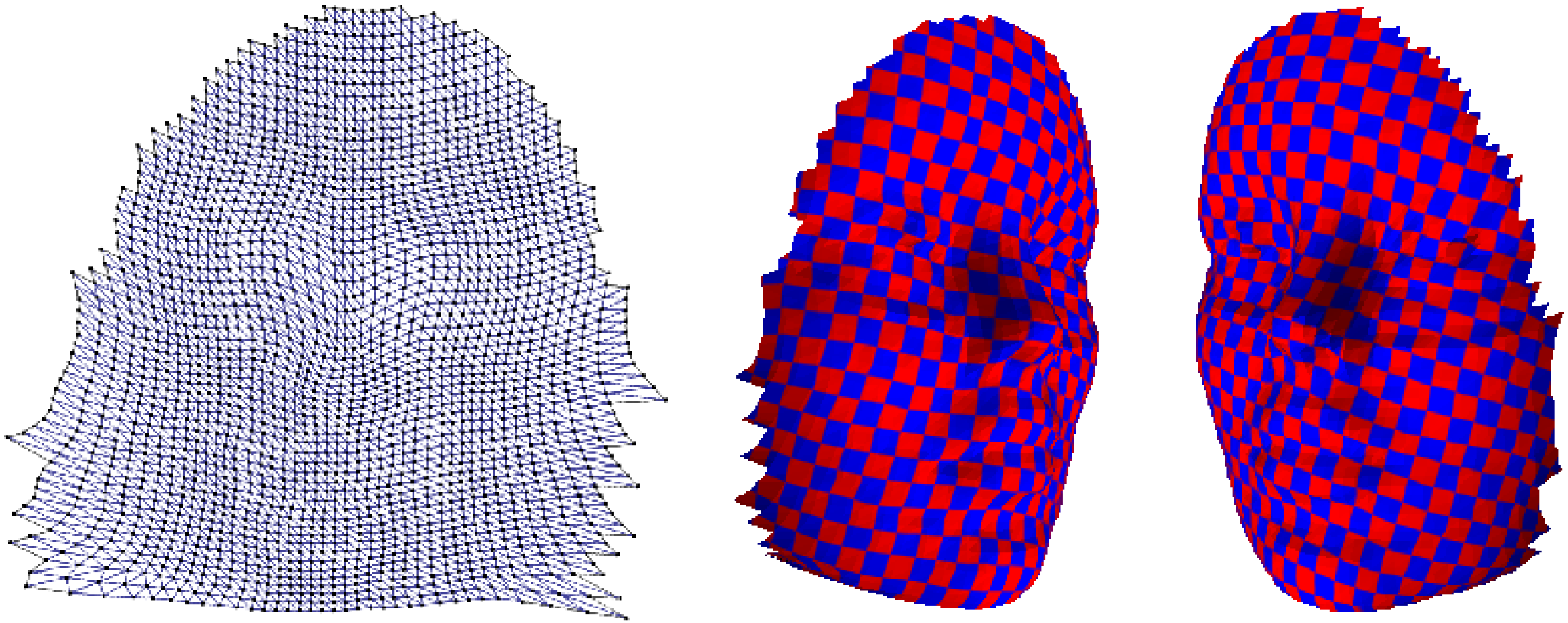


Bad parameterization





Better…(free boundary)



Partition – problems

- Discontinuity of parameterization
- Visible artifacts in texture mapping
- Require special treatment
  - Vertices along seams have several (u,v) coordinates
  - Problems in mip-mapping

Make seams short and hide them

Summary

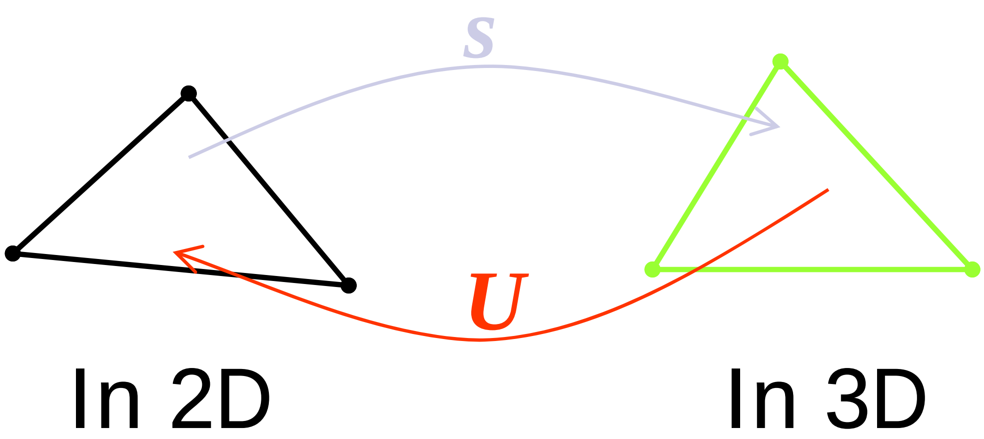
- “Good” parameterization = non-distorting
  - Angles and area preservation
  - Continuous param. of complex surfaces cannot avoid distortion.

- “Good” partition/cut:
  - Large patches, minimize seam length
  - Align seams with features (=hide them)

Mesh parameterization

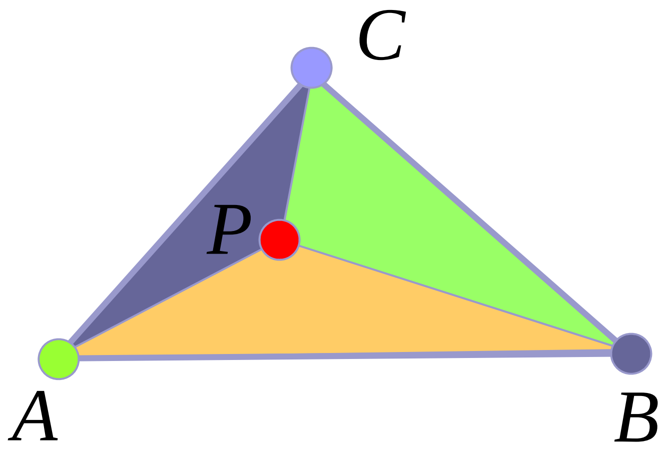
s and *U* are piecewise-linear

Linear inside each mesh triangle



A mapping between two triangles is a *unique affine* mapping

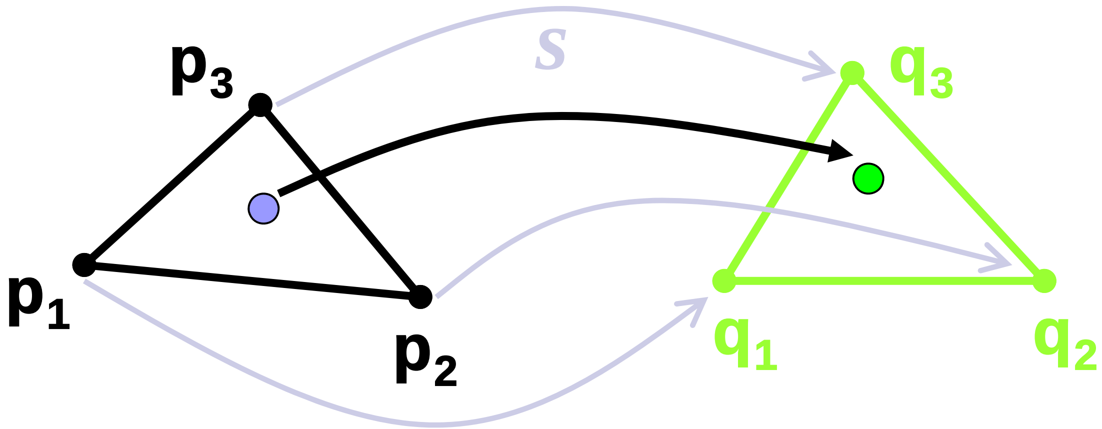
Barycentric coordinates



$$\vec{P} = \frac{\langle P,\ B,\ C \rangle}{\langle A,\ B,\ C \rangle}\vec{A} + \frac{\langle P,\ C,\ A \rangle}{\langle A,\ B,\ C \rangle}\vec{B} + \frac{\langle P,\ A,\ B \rangle}{\langle A,\ B,\ C \rangle}\vec{C}$$

$\langle \cdot,\ \cdot,\ \cdot \rangle$ denotes the (signed) area of the triangle

Mapping triangle to triangle

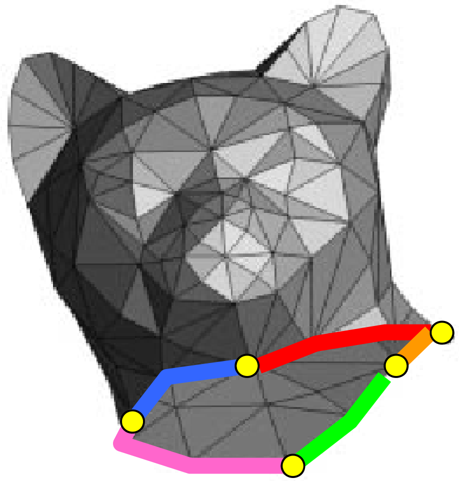


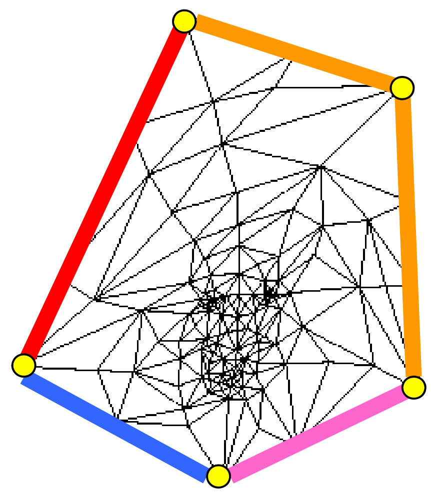
$$\mathbf{s}(\mathbf{p}) = \frac{\langle \mathbf{p}, p_2, p_3 \rangle}{\langle p_1, p_2, p_3 \rangle} q_1 + \frac{\langle \mathbf{p}, p_3, p_1 \rangle}{\langle p_1, p_2, p_3 \rangle} q_2 + \frac{\langle \mathbf{p}, p_1, p_2 \rangle}{\langle p_1, p_2, p_3 \rangle} q_3$$

Some techniques

Convex mapping (Tutte, Floater)

- Works for meshes equivalent to a disk
- First, we map the boundary to a convex polygon
- Then we find the inner vertices positions





$\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ – inner vertices; $\boldsymbol{v}_n, \boldsymbol{v}_{n+1}, \ldots, \boldsymbol{v}_N$ – boundary vertices

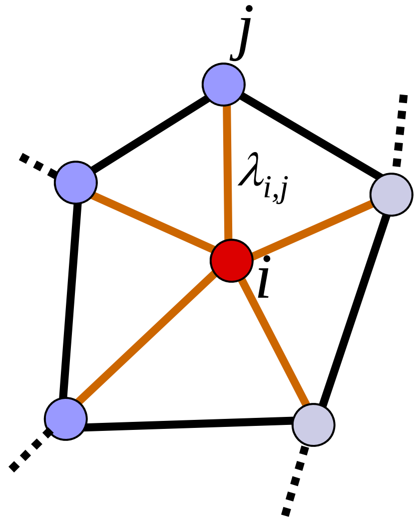
Inner vertices

- We constrain each inner vertex to be a weighted average of its neighbors:

$$\boxed{\boldsymbol{v}_i = \sum_{j \in N(i)} \lambda_{i,j} \boldsymbol{v}_j , \quad i = 1, 2, \ldots, n}$$

$$\lambda_{i,j} = \begin{cases} 0 & i, j \text{ are not neighbors} \\ > 0 & (i, j) \in E \text{ (neighbours)} \end{cases}$$

$$\sum_{j \in N(i)} \lambda_{i,j} = 1$$



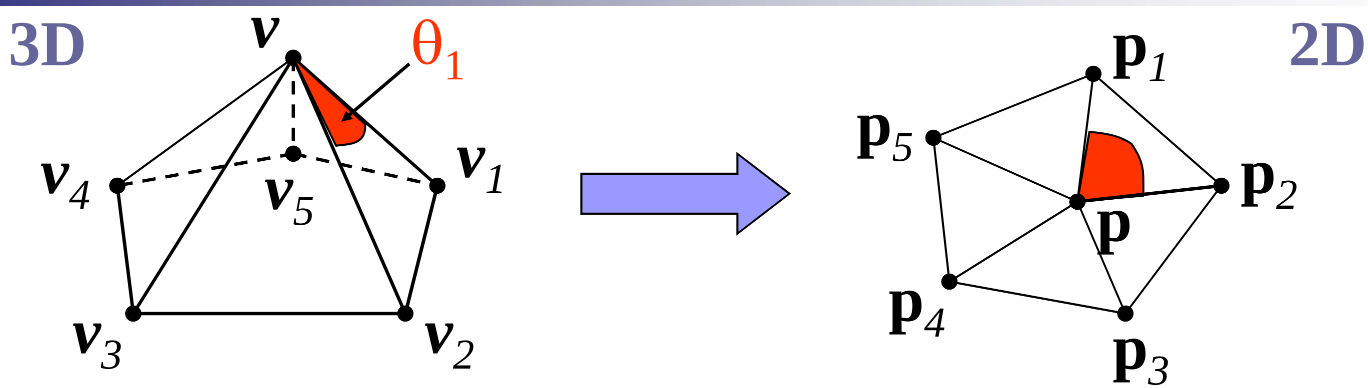
Linear system of equations

$$\boldsymbol{v}_i - \sum_{j \in N(i)} \lambda_{i,j} \boldsymbol{v}_j = 0, \quad i = 1, 2, \ldots, n$$

$$\boldsymbol{v}_i - \sum_{j \in N(i) \setminus B} \lambda_{i,j} \boldsymbol{v}_j = \sum_{k \in N(i) \bigcap B} \lambda_{i,k} \boldsymbol{v}_k, \quad i = 1, 2, \ldots, n$$

$$\begin{pmatrix} 1 & & -\lambda_{1,j_1} & & -\lambda_{1,j_{d1}} \\ & 1 & & & \\ & & 1 & & \\ -\lambda_{4,j_1} & & & \ddots & \\ & & -\lambda_{n,j_5} & & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \\ \\ \sigma_n \end{pmatrix}$$

Shape preserving weights



To compute $\lambda_1, \ldots, \lambda_5$, a local embedding of the patch is found:

1) $\| \mathbf{p}_i - \mathbf{p} \| = \| \mathbf{x}_i - \mathbf{x} \|$

2) $angle(\mathbf{p}_i, \mathbf{p}, \mathbf{p}_{i+1}) = (2\pi / \Sigma\theta_i)\ angle(\mathbf{v}_i, \mathbf{v}, \mathbf{v}_{i+1})$

$$\exists\ \lambda_i\ ,\ \begin{cases} \mathbf{p} = \Sigma\ \lambda_i\ \mathbf{p}_i \\ \lambda_i > 0 \\ \Sigma\ \lambda_i = 1 \end{cases}$$

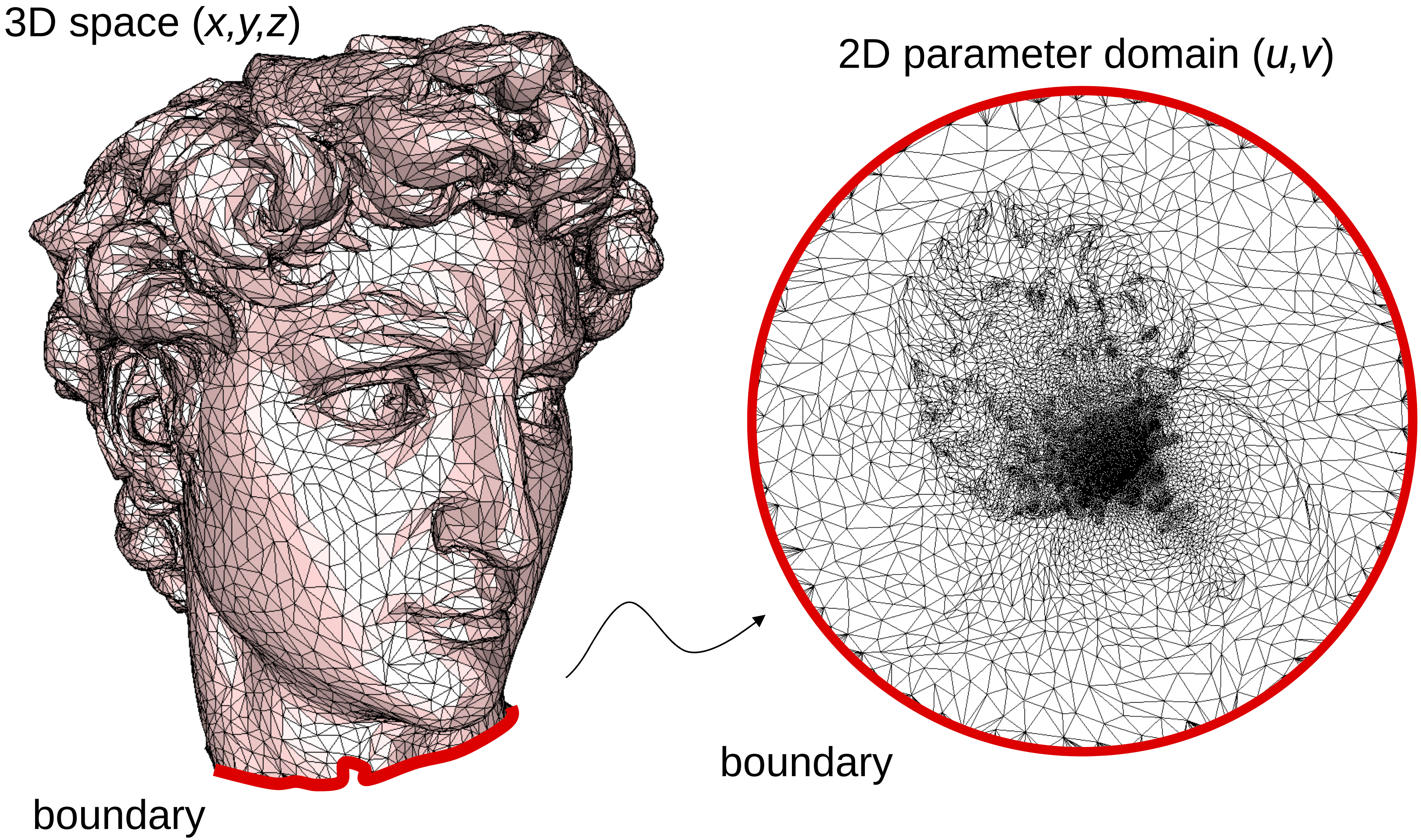
$\Rightarrow$ use these λ as edge weights.

Linear system of equations

- A unique solution always exists
- Important: the solution is legal (bijective)

- The system is sparse, thus fast numerical solution is possible
- Numerical problems (because the vertices in the middle might get very dense…)

2D parameterization



Harmonic mapping

- Another way to find inner vertices
- Strives to preserve angles (conformal)
- We treat the mesh as a system of springs.
- Define spring energy:

$$\boxed{E_{harm} = \frac{1}{2} \sum_{(i,j)\in E} k_{i,j} \left\| \boldsymbol{v}_i - \boldsymbol{v}_j \right\|^2}$$

where $\boldsymbol{v}_i$ are the flat position (remember that the boundary vertices $\boldsymbol{v}_n, \boldsymbol{v}_{n+1}, \ldots, \boldsymbol{v}_N$ are constrained).

Energy minimization – least squares

- We want to find such flat positions that the energy is as small as possible.
- Solve the linear least squares problem!

$$\boldsymbol{v}_i = (x_i, y_i)$$

$$E_{harm}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{1}{2} \sum_{(i,j)\in E} k_{i,j} \left\| \boldsymbol{v}_i - \boldsymbol{v}_j \right\|^2 =$$

$$= \frac{1}{2} \sum_{(i,j)\in E} k_{i,j} \left((x_i - x_j)^2 + (y_i - y_j)^2 \right).$$

E_{harm} is function of 2n variables

Energy minimization – least squares

- To find minimum: $\nabla E_{harm} = 0$

$$\frac{\partial}{\partial x_i} E_{harm} = \frac{1}{2} \sum_{j \in N(i)} 2 k_{i,j} (x_i - x_j) = 0$$

$$\frac{\partial}{\partial y_i} E_{harm} = \frac{1}{2} \sum_{j \in N(i)} 2 k_{i,j} (y_i - y_j) = 0$$

- Again, $x_{n+1}, \ldots, x_N$ and $y_{n+1}, \ldots, y_N$ are constrained.

Energy minimization – least squares

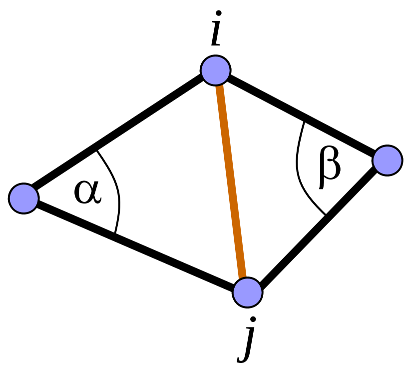
- To find minimum: $\nabla E_{harm} = 0$

$$\boxed{\begin{aligned}\sum_{j\in N(i)} k_{i,j}(x_i - x_j) = 0, \quad i = 1, 2, \dots, n \\ \sum_{j\in N(i)} k_{i,j}(y_i - y_j) = 0, \quad i = 1, 2, \dots, n\end{aligned}}$$

- Again, $x_{n+1}, \dots, x_N$ and $y_{n+1}, \dots, y_N$ are constrained.

The spring constants $k_{i,j}$

- The weights $k_{i,j}$ are chosen to minimize angles distortion:
  - Look at the edge (i, j) in the 3D mesh
  - Set the weight $\boxed{k_{i,j} = \cot \alpha + \cot \beta}$



3D

Discussion

- The results of harmonic mapping are better than those of convex mapping (local area and angles preservation).
- But: the mapping is not always legal (the weights can be negative for badly-shaped triangles…)
- Both mappings have the problem of fixed boundary – it constrains the minimization and causes distortion.
- There are more advanced methods that do not require boundary conditions.

Convex weights for inner vertices

$$\mathbf{v}_i = \sum_{(i,j)\in N(i)} w_{ij}\mathbf{v}_j \quad \text{s.t.} \quad \sum_{(i,j)\in N(i)} w_{ij} = 1 \text{ and } w_{ij} \geq 0$$

- If the weights are convex, the solution is always valid (no self-intersections) [Floater 97]
- The cotangent weight in Harmonic Mapping can be negative $\Rightarrow$ sometimes there are triangle flips
- In [Floater 2003] new *convex* weights are proposed that approximate harmonic mapping

Angle-based Flattening (ABF)

[Sheffer and de Sturler 2001]

- Angle-preserving parameterization
- The energy functional is formulated using the flat mesh angles only!
- Allows free boundary

Angle-based Flattening (ABF)

[Sheffer and de Sturler 2001]

- The goal: minimize the difference

$$\sum_{i=1}^{N}\left(\alpha_i - \beta_i\right)^2$$

where β_i are angles of original (3D) mesh and α_i are the unknowns (the flat mesh)

The angles equations (constraints)

All angles are positive:

$$\alpha_i > 0 \quad (1)$$

Angles around an inner vertex in 2D sum up to 2π

$$\sum_{j \text{ around } i} \alpha_j = 2\pi \quad (2)$$

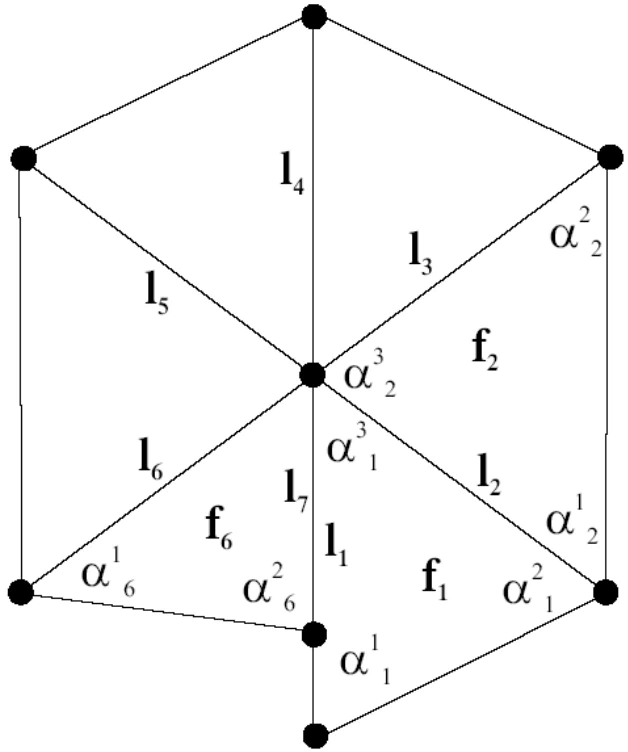
Angles in a triangle sum up to π

$$a_{i_1} + a_{i_2} + a_{i_3} = \pi \quad (3)$$

The angles equations (constraints)

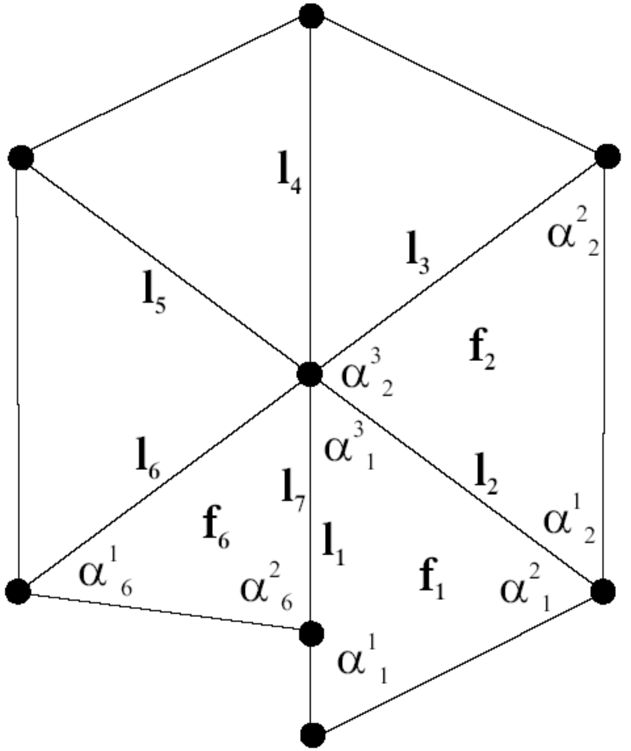
- Finally, something like the sine theorem must hold:

$$(4)\quad \frac{\prod_{j=1}^{N(i)} \sin\alpha_j}{\prod_{j=1}^{N(i)} \sin\tilde{\alpha}_j} = 1$$



The angles equations (constraints)

- Finally, something like the sine theorem must hold:



The final optimization:

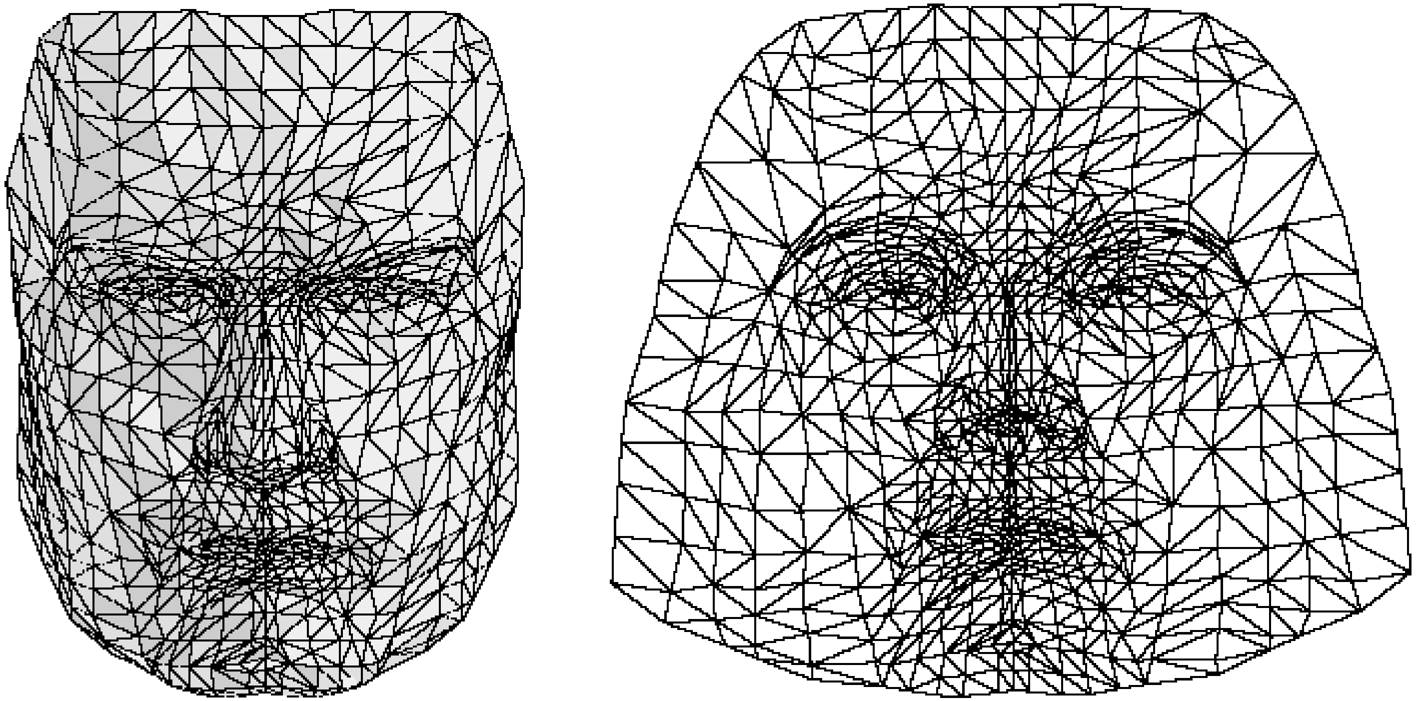
- We minimize

$$\sum_{i=1}^{N}\left(\alpha_i - \beta_i\right)^2$$

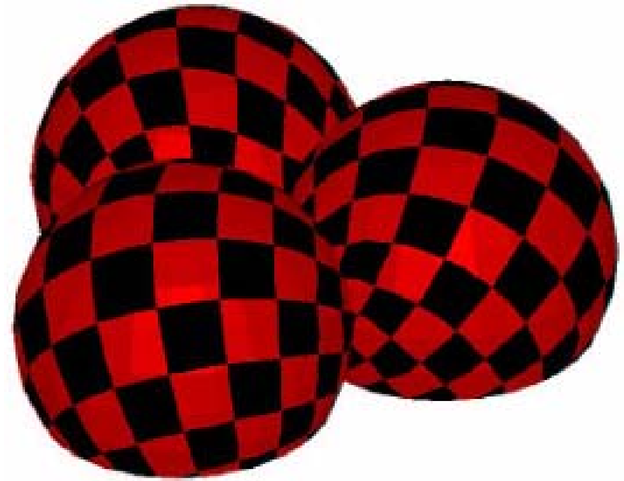
under the 4 constraints

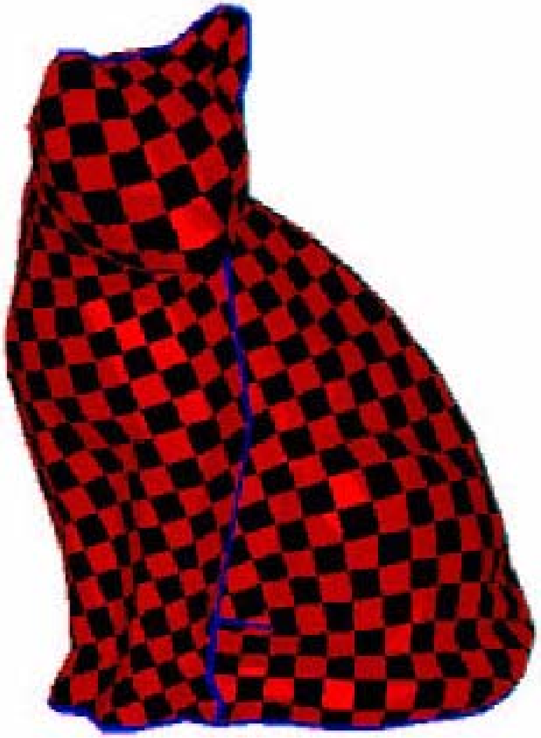
- It’s enough to fix one triangle in the plane to define the whole flat mesh

Results

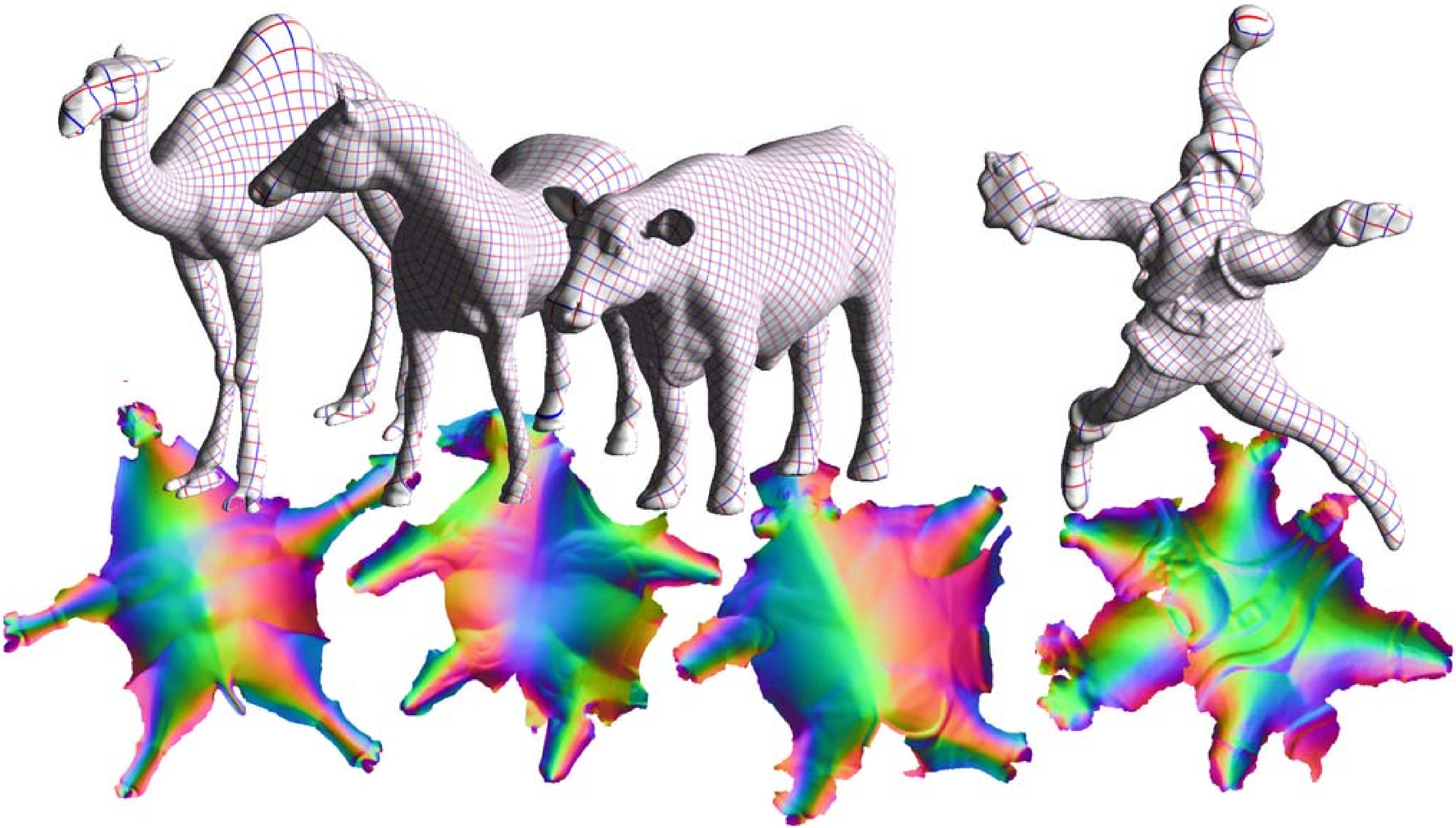


Results

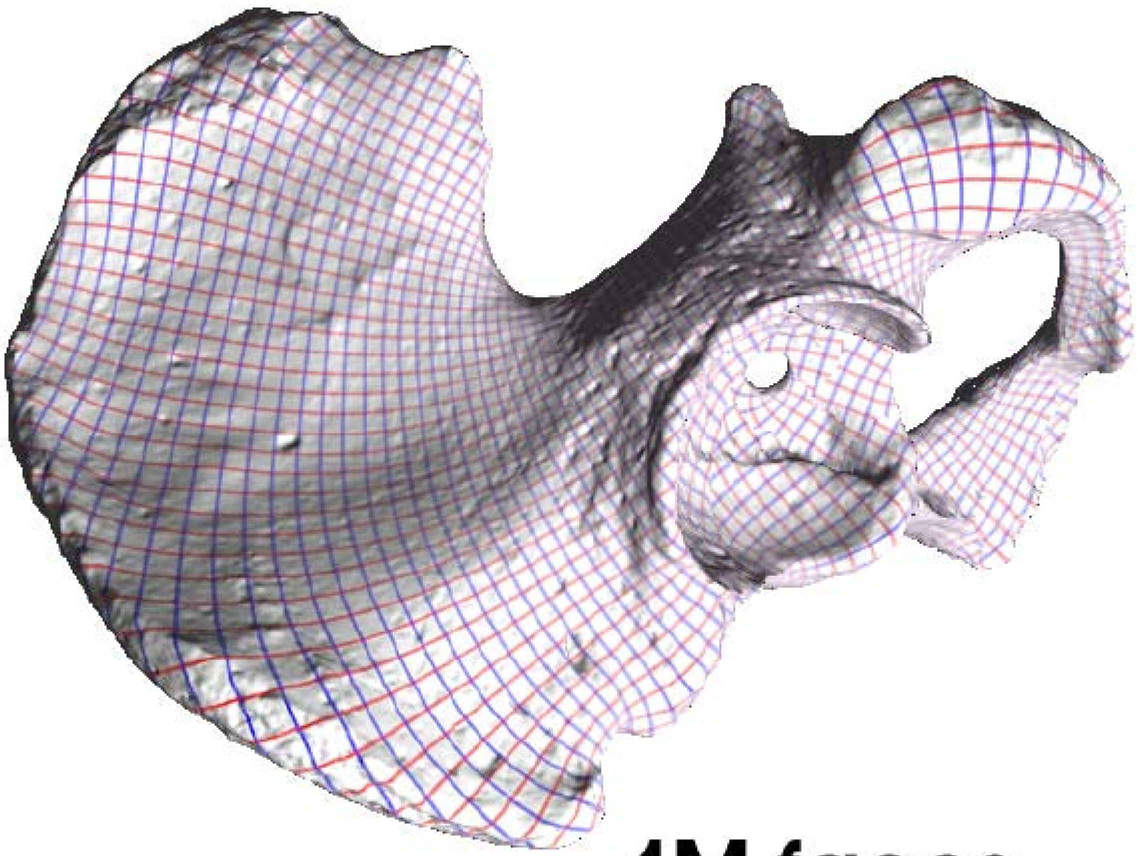




Results



Results



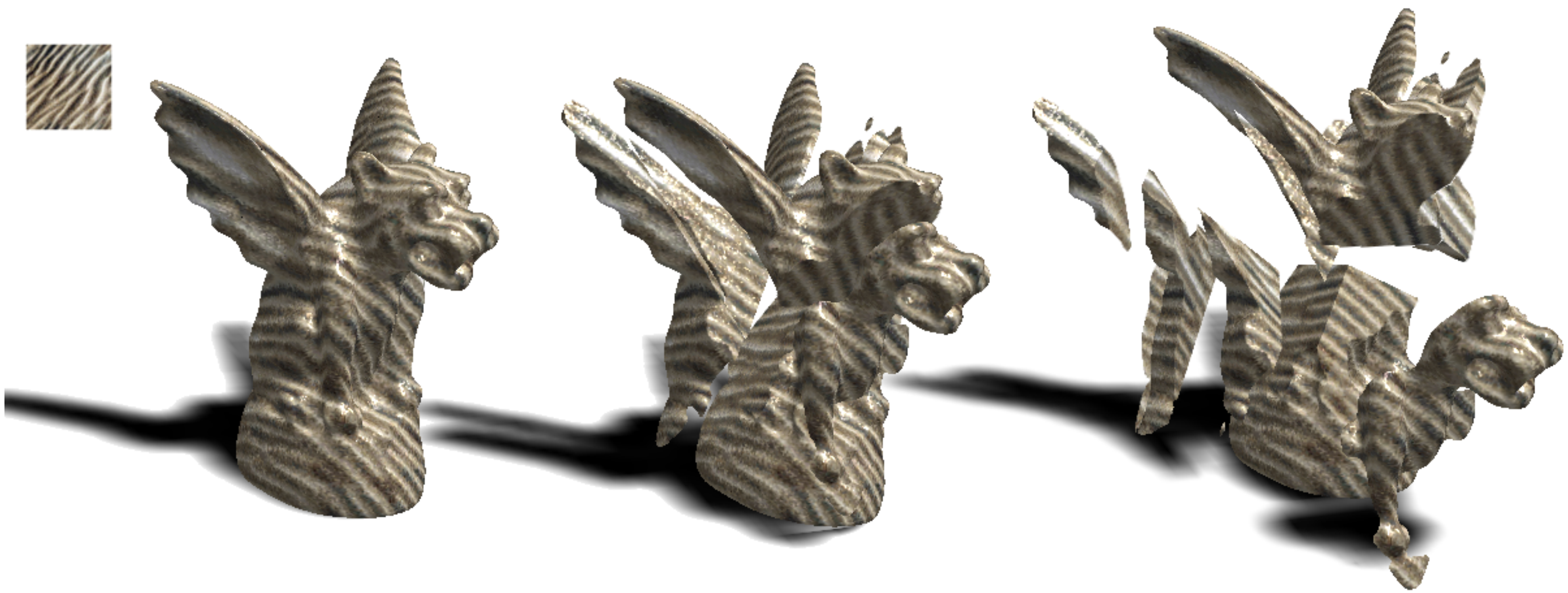
Discussion

- Pros:
  - Angle preserving
  - Always valid (at least internally)
  - No rigid boundary constraints
- Cons:
  - Non-linear optimization
    - Expensive (but now a multi-grid method exists)
  - Building the mesh from angles can be unstable

Solid Textures
(Peachey 1985, Perlin 1985)

- Problem: mapping a 2D image/function onto the surface of a general 3D object is a difficult problem:
  - Distortion
  - Discontinuities
- Idea: use a texture function defined over a 3D domain - the 3D space containing the object
  - Texture function can be digitized or procedural

Solid Textures



Thanks