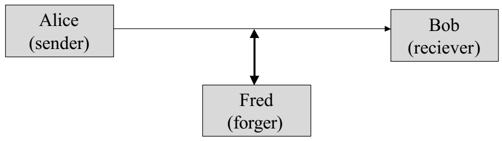
Introduction to Modern Cryptography

Lecture 6

1. A Clarification regarding CBC MACs.
2. Chinease Remainder Theorem (at long last).
3. Testing Primitive elements in Z_p
4. Primality Testing.
5. Integer Multiplication & Factoring as a One Way Function.

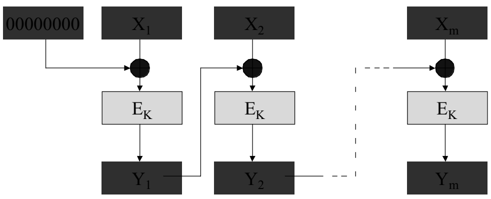
Reminder: MACs

Ensure integrity of messages, even in presence of an active adversary who sends own messages.



Remark: Authentication is orthogonal to secrecy, yet systems often required to provide both.

Reminder: CBC MAC_K



$\text{CBC-MAC}_K(X_1,X_2,\ldots,X_m)= Y_m$

Clarification: Security of CBC MAC

Claim [Bellaire,Kilian,Rogaway]:

If $E:\{0,1\}^k \times \{0,1\}^n \dashrightarrow \{0,1\}^n$ is a pseudo random function, then CBC MACis resilient to adaptive existential forgery.

Proof of security applies only to fixed number of blocks m (e.g. m=17 or $m=n^2+51$).
Proof is inapplicable to variable length m (as discussed in Problem Set II).

Adaptive Existential Forgery

1. Forger picks $message_1$, gets $MAC_K(message_1)$
2. Forger picks $message_2$, gets $MAC_K(message_2)$

.
.   adaptive

3. Forger picks $message_s$, gets $MAC_K(message_s)$

Now forger should come up with any new pair
new_message, MAC_K(new_message)

existential

The Chinese Remainder Theorem (CRT)

Testing Primitive Element mod p

Let p be a prime number so that the prime factorization of p-1 is known:

$p-1 = q_1^{e1} q_2^{e2} \dots q_k^{ek}$ ($q_1, q_2, \dots, q_k$ primes).

Theorem: $g \in Z_p$ **is a primitive element in Z_p iff**
$g^{(p-1)/q1}$, $g^{(p-1)/q2}$, ... , $g^{(p-1)/qk}$ are all **$\neq$ 1 mod p**

Algorithm: **Efficiently compute all k powers.**

Caveat: Requires factorization of p-1.

Testing Primitive Element mod p

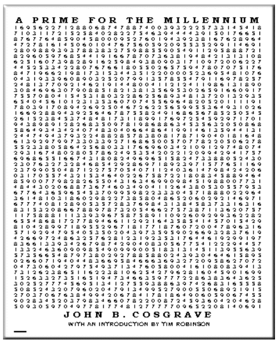
[redacted]

Testing Primitive Element (cont.)

[redacted]

So far, 233926 looks like a good candidate (it passed all five tests it went through). However, we cannot know for sure without factoring the remaining c-55-1 (which is not a prime).

Primality Testing



A prime number with 2000 digit (40-by-50)

from John Cosgrave, Math Dept, St. Patrick's College, Dublin, IRELAND.

http://www.spd.dcu.ie/johnbcos/

Primality Testing

Input: A positive integer M, $2^{n-1}<M<2^n$

Decision Problem: Is M a composite number ?

Decision problem is in NP (guess & verify).

Search Problem: Find prime factors of M.

Factoring integers deterministically is believed to be computationally infeasible.

Primality Testing

Question: Is there a better way to solve the decision problem (test if M is composite) than by solving the search problem (factoring M)?

Basic Idea [Solovay-Strassen, 1977]:
To show that M is composite, enough to find evidence that M does not behave like a prime. Such evidence need not include any prime factor of M.

Primality Testing

Evidence that M is non prime may come from Fermat's little theorem:
Any $1< a < M$ satisfying $a^{M-1} \neq 1$ supplies concrete evidence that M is non prime (but no factorization !)

Example: [redacted] M is composite

Will "Fermat test" always find such evidence ?

Primality Testing

There are some M where Fermat test fails !

Example: [redacted]

Well, maybe M is prime after all ?

[redacted]

End of story regarding M…

Carmichael Numbers

Composites M where Fermat test fails ($a^{M-1} = 1$) for most a, $1 < a < M-1$.

Theorem: M is a Carmichael number iff $M=p_1p_2p_3\dots p_k$ ($k>2$), all p_i are distinct primes, and every p_i satisfies p_i-1 divides M-1.

Example [redacted]

Carmichael numbers: Rare, still infinitely many.

Evidence that M is non prime

An integer a, $1 < a < M$ such that either

1. gcd(a , M) > 1 (non trivial factor).
2. $a^{M-1} \neq 1 \bmod M$ (Fermat test) .
3. $a^2 = 1 \bmod M$ but $a \neq M - 1$??????

Such integer a will be called a witness for M being composite.

Evidence that M is non prime

A witness a, $1 < a < M$ such that either

1. gcd(a , M) > 1 implies M has non trivial factors .
2. $a^{M-1} \neq 1 \bmod M$ implies the size of the multiplicative group Z_M* is smaller than M-1.
3. $a^2 = 1 \bmod M$ but $a \neq M - 1$ implies 1 has more than two square roots in Z_M*.

Back to our favorite M=225593397919

Being a Carmichael number, we won't easily find a witness that is either a non trivial factor or flunks the Fermat test.
Denote M-1=2r. So $b^{M-1} = (b^r)^2 = 1 \bmod M$.
If $b^r \neq M - 1 \bmod M$, then $a=b^r$ is a witness of type (3).

Gotcha !
In both cases
$a^2 = 1$ but $a \neq M - 1$.

[redacted]

Pushing this Idea Further (General M)

Let $M-1=2^k r$ where r is odd.
Then $b^{M-1} = (\ldots((b^r)^2)\ldots)^2$ (k squaring ops).

If $b^{M-1} \neq 1 \bmod M$, we're all set. Otherwise,
let $a_0 = b^r$, $a_1 = (a_0)^2$, $a_2 = (a_1)^2, \ldots, a_k = (a_{k-1})^2$.
Then $a_k = b^{M-1} = 1 \bmod M$.
Let j be the smallest index with $a_j = 1 \bmod M$.
If $0 < j$ and $a_{j-1} \neq M-1$ then M is composite.

Evidence that M is Composite

Let $M-1=2^k r$ where r is odd.
Pick $1 < b < M$.
Compute mod M

$a_0 = b^r$, $a_1 = (a_0)^2$, $a_2 = (a_1)^2, \ldots, a_k = (a_{k-1})^2$.

1. If $a_k \neq 1$ then M is composite.

Let j be the smallest index with $a_j = 1 \bmod M$.

2. If $0 < j$ and $a_{j-1} \neq M-1$ then M is composite.

Call b satisfying (1) or (2) a smart witness.

Miller Theorem (1977)

Let $M=2^k r+1$ where r is odd.
If M is composite then there is* a small smart witness b
(small means $b < (\log M)^2$.

* Assuming a (yet) unproven number theoretic statement: The extended Riemann hypothesis

Rabin Theorem (1980)

Let $M=2^k r+1$ where r is odd.
If M is composite then at least $3M/4$ of all b in the range $1 < b < M$ are smart witnesses.

No assumption required, and proof employs only elemetrary tools.

Miller-Rabin Primality Testing

Input: Odd integer M ($2^{n-1} < M < 2^n$).
Repeat 100 times:

Pick b at random ($1 < b < M$).
Check if b is a smart witness (poly(n) time).

If one or more b is a smart witness, output "M is composite".
Otherwise output "M is prime".

Miller-Rabin Primality Testing

Properties of Algorithm:

- Randomized (uses coin flips to pick b's).
- Run time - polynomial in $n = \log M$.
- If M is prime the algorithm always outputs "M is prime".
- If M is composite the algorithm may err. However to err, all choices of b should give non-witnesses, so

Probability of error $< (0.25)^{100} \lll 1$.

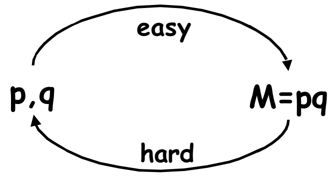
Primality Testing

In terms of complexity classes, this algorithm (and its predecessor, Solovay-Strassen algorithm) imply

Composites $\in$ **RP**

RP=Random Poly Time, one sided error.
Easy fact: RP is contained in NP.

Integer Multiplication & Factoring as a One Way Function.

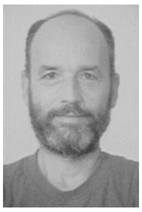


Q.: Can a public key system be based on this observation ?????

Next Lecture (2002)

A.: RSA public key cryptosystem


Rivest


Shamir


Adelman