Algorithmic Methods	1/11/2010

Lecture Notes 3: Duality

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1 Introduction

In this lecture we will present the dual concept, Farkas's Lema and their relation to the field of game theory.

2 Duality

The general case:

primal	dual
$a_i\text{x} = b_i$	$y_i \lesseqgtr 0$
$a_i x \geq b_i$	$y_i \geq 0$
$x_j \lesseqgtr 0$	$y^t A_j = c_j$
$x_j \geq 0$	$y^t A_j \leq c_j$
min $c^t x$	max $y^t b$

(A_j describs column j, a_j describes row j)

Claim: If x∈ Primal and y ∈ Dual than $c^t x \geq y^t b$

Proof: min $c^t x, Ax = b, A^{'}x \geq b^{'}$ where A is of size lxn and $A^{'}$ is of size $(m-l)$xn. In addition $x_1, ..., x_k \geq 0$ and $x_{k+1}, ..., x_n \lesseqgtr 0$. We have:

$$\begin{aligned} c^t x &= \sum_{j=1}^{k} c_j x_j + \sum_{j=k+1}^{n} c_j x_j \geq \sum_{j=1}^{k} y^t A_j x_j + \sum_{j=k+1}^{n} y^t A_j x_j \\ &= y^t A x = \sum_{i=1}^{m} y_i a_i x \geq \sum_{i=1}^{l} y_i b_i + \sum_{i=l+1}^{m} y_i b_i = y^t b \end{aligned}$$

Claim: If x is an optimal solution to the primal system then there exists an optimal solution to the dual system y s.t. $c^t x = y^t b$.

Proof: The proof is given only for LPS (Ax=b, x≥0, min c^tx). We solve using the Simplex algorithm and find an optimal solution. For this solution:
Let B denote the base, x $= A_B^{-1} b$, particularly $c^t - c_B^t A_B^{-1} A \geq 0$ and equals $c_B^t A_B^{-1} b$ (Actually, we showed that $c_N^t - c_B^t A_B^{-1} A_N \geq 0$ but this immediately implies that $c^t - c_B^t A_B^{-1} A \geq 0$). We claim that $y^t = c_B^t A_B^{-1}$ is an optimal solution to the dual system and $c^t - y^t A \geq 0$,

$y^t b = c^t x$. In addition the dual system of LPS is valid. meaning:
max $y^t b$, $y^t A \leq c_j$, y $\substack{<\\=\\>}$ 0.
This optimal solution is also equal to the optimal solution to the primal system and therefore it is optimal and feasible.

Remarks:

- (1) - Proof for non LPS is done using reduction.
- (2) - dual of dual is primal.

Possible Cases:

- (1) - Both dual and primal are feasible, both have an optimum and therefore the optimum values are equal.
- (2) - One is empty and the other is unbounded.
- (3) - Neither is feasible.

Claim: Complementary Slackness
For optimal solutions:

$$y_i(a_i x - b_i) = 0; (c_j - y_t A_j)x_j = 0.$$

Proof:
Recall the proof for the weaker claim. For optimal solutions the inequalities becomes equalities:

$$\begin{aligned} c^t x &= \sum_{j=1}^{k} c_j x_j + \sum_{j=k+1}^{n} c_j x_j = \sum_{j=1}^{k} y^t A_j x_j + \sum_{j=k+1}^{n} y^t A_j x_j \\ &= y^t A x = \sum_{i=1}^{m} y_i a_i x = \sum_{i=1}^{l} y_i b_i + \sum_{i=l+1}^{m} y_i b_i = y^t b. \end{aligned}$$

and the claim is immediately derived.

3 Financial Meaning for Dual Variables (Production Problems)

Assuming all constraints are positive. A factory which manufactures furniture from base materials. we have the following data:

	table	chair	couch
wood	8	3	1
metal	2	1	5
fabric	0	2	8

The Dual Case:

$$\text{Objective function: max } 80x_1 + 30x_2 + 200x_3$$
$$c^T = (100\ 90\ 84)$$
$$x_1, x_2, x_3 \geq 0$$

The Primal Case:

$$G \leq u_1(8x_1 + 3x_2 + x_3) + u_2(2x_1 + x_2 + 5x_3) + u_3(2x_2 + 8x_3) \leq 100u_1 + 90u_2 + 84u_3$$

Question: Would purchasing more from a certain material significantly increase profits? We look for changes in the objective function. We increase the quantities of products u_1, u_2, u_3 by $\epsilon_1, \epsilon_2, \epsilon_3$ respectively. We have: $\Delta G = \epsilon_1 \cdot u_1 + \epsilon_2 \cdot u_2 + \epsilon_3 \cdot u_3$

4 Farkas's Lemma:

Let $b, a_1, ..., a_n \in \mathbb{R}^m$ therefore exists $\lambda_i \geq 0$ s.t.:

$$b = \Sigma\lambda_i a_i \Leftrightarrow (\forall y \text{if} \forall i \quad ya_i \geq 0 \Rightarrow yb \geq 0)$$

Proof:

($\Rightarrow$) Assuming that there exist $\lambda_i \geq 0$ s.t. b $= \Sigma\lambda_i a_i$. We have

$$yb = y\sum \lambda_i a_i = \sum \lambda_i y a_i \geq 0.$$

($\Leftarrow$) Assuming that $\forall y$ if $ya_i \geq 0 \Rightarrow yb \geq 0$. Define a matrix A$=(a_1, ..., a_n)$. We solve:

$$\text{A}\lambda = b$$
$$\lambda \geq 0$$
$$max 0 \cdot \lambda \equiv 0$$

Now, looking at the dua systeml and checking what is the primal system:

$$\text{min} y_t b$$
$$y_t A \geq 0$$
$$\text{y} \lesseqqgtr 0$$

According to the assumption, the objective function ≥ 0. The solution $y = 0$ is feasible and equals 0 and therefore is optimal. We conclude that there exists a feasible solution for the dual system.

5 A reduction from the problem of finding an optimal solution to the problem of finding a feasible solution:

We show that if there exists a black box which solves the feasibility problem for a linear programming system then we can also find an optimum for ther system.
Given a linear programming system and objective function min $c^t x$. wel add the dual $y^t A \leq c$ and the constraint $c^t x = y^t b$ and get a new system.

Claim:
There is a feasible solution to the new system $\Leftrightarrow$ There is an optimal solution the the original system. Moreover, each solution to the new system gives an optimal solution to the original system.

6 Application: Game Theory, Zero-Sum Games:

Two players are playing a zero-sum game. Player I chooses a column and player II chooses a row (In the general form the size of the matrix can be any $m \times n$). a_{ij} represents the amount that player II pays player I if player I chooses j and player II chooses i.
For Example:
The table below describes the chances of each player choosing their respective column/row. Each element a_{ij} equals the amount player II has to pay player I.

II/I	q	1-q
p	4	-1
1-p	1	2

For the above example Player I can guarentee making a profit of at least 1 by choosing the column 1. Let $\alpha = max_j min_i a_{ij}$
Player II can guarantee losing less than 2 by choosing the second row. Let $\beta = min_i max_j a_{ij}$

Another strategy that each player can take is choosing i (player I) and j (player II) with different probabilities, to ensure maximum (player I) or minimum (player II) payment. Let q be the probability player I chooses the first column and 1-q the probability she chooses the second column. Accordingly, let p be the probability player II chooses the first row and 1-p the probability she chooses the second row. The expected payoff for player I will be:
Player II chooses first row :

$$4q\text{-}(1\text{-}q) = 5q\text{-}1$$

Player II chooses second row :

$$q + 2(1\text{-}q) = 2\text{-}q$$

We choose q by the equation:

$$5q\text{-}1 = 2\text{-}q \Rightarrow q = 0.5$$

If player I chooses a column with probability 0.5 he guarantees himself at least 1.5 (the expectation value) regardless of player II's choice .
We calculate Player II's strategy in the same way:
Player I chooses first column :

$$4p+1-p = 1+3p$$

Player I chooses second Column :

$$-p+2(1-p) = 2-3p$$

We choose p by the equation:

$$1+3p = 2-3p \Rightarrow p=\frac{1}{6}$$

If player II choosea rows 1,2 with probability $\frac{1}{6}$, $\frac{5}{6}$ respectfuly, he guarantees himself maximum pay of 1.5 (the expectation value) regardless of player I's choice.

Consider a general case where we have a matrix A. player I's variables are $x_1, ..., x_n$ where x_i is the probability of choosing column i. Player II's variables are $y_1, ..., y_m$ where y_j is the probability of choosing row j.
For player I:

$$\begin{gathered} \max \alpha \\ Ax \geq \alpha \cdot e_m \\ x \geq 0 \\ e_n^t \cdot x = 1 \\ \alpha \lesseqgtr 0 \end{gathered}$$

For player II:

$$\begin{gathered} \min \beta \\ y^t A \leq \beta \cdot e_n^t \\ y \geq 0 \\ e_m^t \cdot y = 1 \\ \beta \lesseqgtr 0 \end{gathered}$$

We show that one is dual to the other:

$$\begin{gathered} \min\ (-\alpha) \\ Ax - \alpha \cdot e_m \geq 0 \\ -e_n^t \cdot x = -1 \\ x \geq 0 \end{gathered}$$

$$\begin{gathered} \max\ (-\beta) \\ y^t A - \beta e_n^t \leq 0 \\ -e_m^t \cdot y = -1 \\ y \geq 0 \end{gathered}$$

This can be easily seen by looking at the Matrix:

$$
\begin{array}{cccc}
(c/b) & (0\quad 0\quad 0\quad 0\quad 0\quad 0) & -1 \\
\begin{pmatrix} 0 \\ . \\ . \\ . \\ . \\ 0 \end{pmatrix} & \left(\quad\quad\quad \text{A} \quad\quad\quad \right) & \begin{pmatrix} -1 \\ . \\ . \\ . \\ . \\ -1 \end{pmatrix} \\
\text{(-1)} & \text{(-1 \ -1 \ -1 \ -1 \ -1)} & 0
\end{array}
$$

and both the primal and the dual are feasible.